We will now be concerned with defining the integral of continuous $W$-valued volume forms with quasi-compact support (recall that a topological space is called quasi-compact if every open cover of it has a finite subcover). Intuitively, the integral of such a form is the sum of all its values (which are elements of $W$). More precisely, one should think of Riemann sums; the volume form itself tells us how to measure the size of "small cubes". But the terms "cube" and "block" do not make much sense in $X$. So first we explain what we want to do locally; after that we set up an administration system to make sure that everything gets counted exactly once. I should also admit that the usual definition of integration of volume forms is in terms of paritions of unity. In this course I want to show that there is another definition that is closer to the way one actually computes integrals, and which does not need that the manifold is separated. Of course, both definitions give the same result for separated manifolds. We will probably need the usual definition do prove some general results on integration (such as Stokes's theorem).

So let $X$ be a $C^k$-manifold, with $k \geq 1$. Let $W$ be a finite dimensional $\mathbb{R}$-vector space and $v$ a $W$-valued volume form on $X$. Let $\phi\colon U \to X$ be a chart, with $U \subset \mathbb{R}^n$ open. For all $u$ in $U$, $\mathrm{T}_\phi(u)$ is an isomorphism from $\mathrm{T}_U(u)$ to $\mathrm{T}_X(\phi(u))$. This gives us isomorphisms from $(W \otimes_{\mathbb{R}} \mathrm{Vol}_U)(u)$ to $(W \otimes_{\mathbb{R}} \mathrm{Vol}_X)(\phi(u))$, and hence a $W$-valued volume form $\phi^*v$ on $U$. We have, uniquely, $\phi^*v = w|dx_1 \cdots dx_n|$, with $w\colon U \to W$ a $C^{k-1}$-map. Let $V \subset U$ be a measurable subset (in the sense of Lebesgue) of $U$ whose closure $\overline{U}$ in $\mathbb{R}^n$ is compact and contained in $U$ (for example, a bounded closed subset of $\mathbb{R}^n$ that is contained in $U$, or an open subset of such a closed subset). Then we define:

$$\int_V \phi^*v := \int_V w,$$

where the last integral is in the sense of Lebesgue. To make things a bit more concrete: if $(w_1, \ldots, w_d)$ is a basis for $W$, and $w = \sum_i f_i w_i$, then $\int_V w = \sum_i (\int_V f_i) w_i$. We are now ready to define the integral on $X$ itself.

**3.5.7 Construction.** Let $X$ be a $C^k$-manifold with $k \geq 1$. Let $W$ be a finite dimensional $\mathbb{R}$-vector space. Let $v$ be a $W$-valued volume form on $X$ with quasi-compact support. Suppose that we have an integer $m \geq 0$, charts $\phi_i\colon U_i \to X$, $1 \leq i \leq m$, $V_i \subset U_i$ measurable and with $\overline{V_i}$ compact and contained in $U_i$, such that $\mathrm{Supp}(v) \subset \cup_i \phi_i V_i$. Then we define the integral of $v$ over $X$, with respect to these data, to be:

$$\int_X v := \sum_{r=1}^{m} (-1)^{r+1} \sum_{i_1 < \cdots < i_r} \int_{V_{i_1,\ldots,i_r}} \phi_{i_1}^* v,$$

where $V_{i_1,\ldots,i_r} := \phi_{i_1}^{-1} \cap_{j=1}^{r} \phi_{i_j} V_{i_j}$.

The sum over $r$, and the signs in it, are there to make that we do not count the intersections twice, etc.; it is called the inclusion exclusion principle, easily understood in terms of the characteristic functions of the $V_i$. Note that the $\phi_{i_1}^{-1} V_{i_j} \cap U_{i_1}$ are measurable subsets of $U_{i_1}$, hence that $V_{i_1,\ldots,i_r}$ is indeed a measurable subset of $U_{i_1}$. Let us show how charts $\phi_i$ and measurable subsets $V_i$ of $U_i$ with $\mathrm{Supp}(v) \subset \cup_i \phi_i V_i$ can be obtained. For each $x$ in $\mathrm{Supp}(v)$, choose a chart $\phi_x\colon U_x \to X$ and a measurable compact neighborhood $V_x$ of $\phi_x^{-1} x$ in $U_x$. Since $\mathrm{Supp}(v)$ is quasi-compact, it is covered by a finite number of the $V_x$. Numbering those $x$ gives the desired charts and measurable subsets. Of course, in practice one usually tries to take the $V_i$ disjoint, or at least such that the $V_i \cap V_j$ have measure zero for $i \neq j$.

**3.5.8 Proposition.** *The integral of $v$ as defined in Construction 3.5.7 does not depend on the choice of the charts $\phi_i$ and the sets $V_i$.*

**Proof.** Suppose we have two sets of data: $m$, $m'$, $\phi_i$, $\phi'_j$, etc. Then we construct two new sets of data as follows: $m'' := mm'$, $U_{i,j} := \phi_i^{-1}(\phi_i U_i \cap \phi'_j U'_j)$, $\phi_{i,j} := \phi_i|_{U_{i,j}}$, $U'_{i,j} := (\phi'_i)^{-1}(\phi_i U_i \cap \phi'_j U'_j)$, $\phi'_{i,j} := \phi'_j|_{U'_{i,j}}$, $V_{i,j} := \phi_i^{-1}(\phi_i V_i \cap \phi'_j V'_j)$, $V'_{i,j} := (\phi'_i)^{-1}(\phi_i V_i \cap \phi'_j V'_j)$. Let us first argue that these new two sets of data give the same integral. For that purpose, consider a pair $(i,j)$. Let $U := U_{i,j}$ and $U' := U'_{i,j}$. Then $f := {\phi'_j}^{-1}\phi_i$ defines an isomorphism from $U$ to $U'$, such that $V := V_{i,j}$ has image $V' := V'_{i,j}$. Let us write $\phi_i^* v = v|dx_1 \cdots dx_n|$ and ${\phi'_j}^* v = v'|dx'_1 \cdots dx'_n|$. The partial derivatives $\partial/\partial x_i$, $1 \leq i \leq n$ form a basis of $\mathrm{T}_U$, and likewise for $\partial/\partial x'_j$ for $\mathrm{T}_{U'}$. Written in this basis, the tangent map $\mathrm{T}_f$ is given by the matrix whose $(i,j)$th coefficient is $\partial f_j/\partial x_i$, where $f$ is written $(f_1, \ldots, f_n)$. Note that $|dx_1 \cdots dx_n|$ is a basis for $\mathrm{Vol}_U$, and that $|dx'_1 \cdots dx'_n|$ is one for $\mathrm{Vol}_{U'}$. Let $u$ be in $U$ and put $u' := f(u)$. Then $\mathrm{T}_f(u)$ induces an isomorphism from $\mathrm{Vol}_U(u)$ to $\mathrm{Vol}_{U'}(u')$. Using the defitions of $\mathrm{Vol}_U(u)$ and $\mathrm{Vol}_{U'}(u')$, one sees that under this isomorphism $|dx'_1 \cdots dx'_n|$ is mapped to $|\det(\mathrm{T}_f(u)|\,|dx_1 \cdots dx_n|$. By construction, this isomorphism sends $v'(u')|dx'_1 \cdots dx'_n|$ to $v(u)|dx_1 \cdots dx_n|$, hence we get $v'(u')|\det(\mathrm{T}_f(u))| = v(u)$. The "change of variables formula" from vector calculus says:

$$\int_{V'} v' = \int_{V'} v'|dx'_1 \cdots dx'_n| = \int_V (v' \circ f)|df_1 \cdots df_n| = \int_V (v' \circ f)|\det(\mathrm{T}_f)|.$$

So, from what we have just seen, it follows that $\int_{V'} v' = \int_V v$. This equality is also valid for $V := V_{(i_1,j_1),\ldots,(i_r,j_r)}$ and $V' := V'_{(i_1,j_1),\ldots,(i_r,j_r)}$. That means that indeed our two new sets of data for integration give the same result.

It remains now to be shown that two sets of data, one of which is a refinement of the other, give the same integral. In order to see this, let us reconsider what happens for just one set of data $m$, $\phi_i$ and $V_i$. Considering all possible intersections of the $V_i$ and their complements gives us $2^m$ subsets that partition $X$. All these subsets are contained in some $V_i$, except one: the complement of the union of the $V_i$. Note that on this last set $v$ is zero. So on each of our subsets we can integrate $v$, and the sum of these integrals equals the integral of $v$ over $X$ relative to the set of data $m$, $\phi_i$, $V_i$. (To see that, use that $f \mapsto \int_{\mathbb{R}^n} f$ is additive.) Of course, what we are doing here is the standard game with the boolean algebra generated by the characteristic functions of the $V_i$. Now suppose that we have a refinement $m'$, $\phi'_j$, $V'_j$. The $2^{m'}$ subsets of $X$ obtained from the $V'_j$ give a partition of $X$ that refines the partition obtained from the $V_i$. Then our claim is clear. □

Now that we know how to integrate volume forms (i.e., sections of $\mathrm{Vol}_X$), let us discuss the relation between volume forms and differential forms of top degree, i.e., sections of $\Omega_X^{\dim X}$.

**3.5.9 Definition.** *Let $X$ be a manifold and $L$ a rank one vector bundle on $X$. An orientation of $L$ is a collection of orientations of all $L(x)$, $x \in X$, which is locally constant. If $X$ is $C^k$ with $k \geq 1$, then an orientation of $X$ is an orientation of $\Omega_X^{\dim X}$.*

Some examples. The trivial line bundle $\mathbb{R} \times X$ has a standard orientation. The same is true for $\mathrm{Vol}_X$, but not always for $\Omega_X^{\dim X}$. If $\omega$ is a global section of $\Omega_X^{\dim X}$ such that $\omega(x) \neq 0$ for all $x$ in $X$, then $\omega$ is a basis for $\Omega_X^{\dim X}$, hence gives an isomorphism from $\mathbb{R} \times X$ to $\Omega_X^{\dim X}$, hence gives an orientation on $\Omega_X^{\dim X}$.

**3.5.10 Remark.** It is not true in general that all orientations on a line bunlde $L$ come from a trivialization of it (example: take $X$ to be two copies of $\mathbb{R}$, glued via the identity along $\mathbb{R} - \{0\}$, and take an ugly line bundle). If $X$ is paracompact, then all orientations come indeed from trivializations.

**3.5.11 Proposition.** *Let $X$ be a $C^k$-manifold with $k \geq 1$. An orientation on $X$ induces a unique isomorphism from $\mathrm{Vol}_X$ to $\Omega_X^{\dim x}$, such that, at each $x$ in $X$, it coincides with the map from $\Omega_X^{\dim x}(x)$ to $\mathrm{Vol}_X(x)$ that sends $l$ to $|l|$. Conversely, an isomorphism from $\mathrm{Vol}_X$ to $\Omega_X^{\dim x}$ with this property induces an orientation on $X$ and the two constructions are inverses.*

**Proof.** Let $x$ be in $X$, $V := \mathrm{T}_X(x)$, $n := \dim(V)$, $L := \Omega_X^{\dim x}(x) = (\Lambda^n(V))^\vee$, and finally $L' := \mathrm{Vol}_X(x) = |\Lambda^n(V)|^\vee$. Recall that we have the map $|\cdot|\colon L \to L'$ that sends $l$ to $|l|$. This map is, of course, not linear. We get a linear map as follows: choose $l$ in $L^+$ non-zero and send $\lambda l$, for $\lambda$ in $\mathbb{R}$, to $\lambda|l|$. Check that this map does not depend on the choice of $l$, that it coincides with $|\cdot|$ on $L^+$ and that it is the only linear map with that property. □

It is now clear that for $X$ an oriented $C^k$-manifold we can integrate sections of $\Omega_X^{\dim x}$ that have quasi-compact support, by using the isomorphism corresponding to the orientation to transform these sections in volume forms. The procedure to integrate a differential form of top degree $\omega$ with quasi-compact support is then the same as in Construction 3.5.7, except that one should take charts that are compatible with the orientation on $X$ and the standard orientation on $\mathbb{R}^n$. The standard orientation on $\mathbb{R}^n$ is the one such that $dx_1 \cdots dx_n$ (in this order!) is positive. Of course, if $W$ is an $\mathbb{R}$-vector space, then we can also integrate sections with quasi-compact support of $W \otimes_{\mathbb{R}} \Omega_X^{\dim x}$.

To finish this section, let us define a canonical volume form on a Riemannian manifold. So suppose that $X$ is a $C^k$-manifold, with $k \geq 1$, and that $\langle\cdot,\cdot\rangle$ is a metric on $\mathrm{T}_X$. Let $x$ be in $X$. Suppose that $v_1, \ldots, v_n$ is an orthonormal basis of $\mathrm{T}_X(x)$. Let $v(x)$ be the volume form on $\mathrm{T}_X(x)$ such that $(v(x))(v_1, \ldots, v_n) = 1$. One checks that this does not depend on the orthonormal basis chosen. Hence it defines a volume form $v$ on $X$.

To give some example where one uses this, note that a submanifold of a Riemannian manifold inherits the structure of Riemannian manifold. For example, consider the group $\mathrm{SO}_3(\mathbb{R})$ as a compact submanifold of $\mathbb{R}^9$ with its standard Riemannian metric. Then one can ask: what is the volume of $\mathrm{SO}_3(\mathbb{R})$?

## 3.6 Pullback of vector bundles and of differential forms

Let us first discuss pullback of vector bundles. Let $f\colon X \to Y$ be a morphism of manifolds, and let $E$ be a vector bundle on $Y$. Then we define a vector bundle $f^*E$ on $X$ by: $(f^*E)(x) := E(f(x))$ for all $x$ in $X$, and the local trivializations of $f^*E$ are induced by those of $E$. For $U \subset Y$ open, and $s$ in $E(U)$, we get an element $f^*s$ of $(f^*E)(f^{-1}U)$, defined by: $(f^*s)(x) := s(f(x))$ for all $x$ in $f^{-1}U$. (Note the special case $E = \mathbb{R} \times Y$, where $s$ is just a function and $f^*s = s \circ f$.) The sheaf of sections of $f^*E$ is the tensor product $C_X \otimes_{f^{-1}C_Y} f^{-1}E$, where $f^{-1}$ is pullback of sheaves. For an arbitrary $C_Y$-module $\mathcal{M}$, its pullback as a module is defined as $f^*\mathcal{M} := C_X \otimes_{f^{-1}C_Y} f^{-1}\mathcal{M}$. Hence on the side of locally free sheaves of modules this operation corresponds to the pullback of vector bundles.

Suppose now moreover that $F$ is a vector bundle on $X$, and that $g\colon F \to E$ is a morphism of vector bundles (see Definition 2.2.8). We claim that such a $g$ corresponds naturally to a morphism from $F$ to $f^*E$ of vector bundles on $X$. The proof is trivial, because, for all $x$ in $X$, $(f^*E)(x) = E(f(x))$.

In particular, the morphism $\mathrm{T}_f$ from $\mathrm{T}_X$ to $\mathrm{T}_Y$ corresponds to a morphism, also written $\mathrm{T}_f$, from $\mathrm{T}_X$ to $f^*\mathrm{T}_Y$. Dualizing gives us a morphism $f^*\colon f^*\Omega^1_Y \to \Omega^1_X$. Doing our tensor operations gives us $f^*\colon f^*\Omega_Y \to \Omega_X$. One easily verifies that this $f^*$ is a morphism of sheaves of graded algebras, and that for $U \subset Y$ open and $\omega$ in $\Omega_Y(U)$ one has $f^*d\omega = d(f^*\omega)$; it suffices to prove this for forms of degree one, where one has:

$$(f^*dg)\partial = (dg)(\mathrm{T}_f\partial) = (dg)(\partial \circ f^*) = (\partial \circ f^*)g = \partial(g \circ f) = (d(f^*g))\partial.$$

Suppose now that $\dim_X(x) = \dim_Y(f(x))$ for all $x$ in $X$. Then one has $f^*\colon f^*\mathrm{Vol}_Y \to \mathrm{Vol}_X$ and $f^*\colon \Omega_Y^{\dim_Y} \to \Omega_X^{\dim_X}$. If $f$ is an isomorphism from $X$ to an open subset $U$ of $Y$, and $v$ is in $\mathrm{Vol}_Y(Y)$ with quasi-compact support contained in $U$, then one has $\int_Y v = \int_X f^*v$.

## 3.7 Some exercises

Let $G$ be a Lie group, i.e., $G$ is a $C^\infty$-manifold, with a $C^\infty$-group structure. Let $e$ be its unit element. We consider the following group actions. The (left) action of $G$ on itself by left translations: for $x$ in $G$ we have $l_x\colon G \to G$ sending $y$ to $xy$. The (right) action of $G$ on itself by right translations: $r_x\colon y \mapsto yx$. The action of $G$ on itself by conjugation: $c_x\colon y \mapsto xyx^{-1}$. The action of $G \times G$ on $G$ by translations on both sides: $b_{x,y}\colon z \mapsto xzy^{-1}$. Let $l$ denote the morphism of groups from $G$ to $\mathrm{Aut}_{\mathbf{Man}}(G)$ given by the action by left translations. Similarly, we have the anti-morphism $r$ from $G$ to $\mathrm{Aut}_{\mathbf{Man}}(G)$ given by the right translations, the morphism $b$ from $G \times G$ to $\mathrm{Aut}_{\mathbf{Man}}(G)$ and the morphism $c$ from $G$ to the group $\mathrm{Aut}_{\mathrm{Lie}}(G)$ of automorphims of $G$ as Lie group.

We denote the tangent space $\mathrm{T}_G(e)$ by $\mathrm{Lie}(G)$ and by $L$. For every $x$ in $G$ we have the two isomorphisms $\mathrm{T}_{l_x}(e)$ and $\mathrm{T}_{r_x}(e)$ from $L$ to $\mathrm{T}_G(x)$. These two isomorphisms need not be the same. Show that in fact $\mathrm{T}_{r_x}(e)^{-1}\mathrm{T}_{l_x}(e)$ is the automorphism $\mathrm{T}_{c_x}(e)$ of $L$. (By the way, $L$ is called the Lie algebra of $G$; we will discuss the structure of Lie algebra on $L$ a bit further.) Show that $x \mapsto \mathrm{T}_{c_x}(e)$ defines an action of $G$ on $L$ by linear maps; this action is called the adjoint representation of $G$. Show that both $l$ and $r$ define isomorphisms, still denoted $l$ and $r$, from the trivial vector bundle $L \times G$ to $\mathrm{T}_G$. In particular, all Lie groups have a trivial tangent bundle, and are orientable.

Let us consider the vector space $\mathrm{T}_G(G)$ of vector fields on $G$. The group $G$ acts on it via $l$, $r$ and $c$; $G \times G$ acts via $b$. A vector field $\partial$ on $G$ is called left-invariant if it is invariant for the action given by $l$; similarly, it is called right-invariant if it is invariant under $r$, and bi-invariant if invariant under $b$. Explicitly: $\partial$ is left-invariant if and only if for all $x$ in $G$ one has $\partial(x) = \mathrm{T}_{l_x}(e)\partial(e)$. Show that $\partial \mapsto \partial(e)$ gives an isomorphism from the vector space of left-invariant vector fields on $G$ to $L$. Show the same with left replaced by right. Show that the space of bi-invariant forms is isomorphic in this way to the subspace of $L$ on which $G$ acts trivially via its adjoint representation.

Before we define the Lie algebra structure on $L$ we need a general result on derivations. Let $k$ be a field and $A$ a $k$-algebra. Then we have the $A$-module of $k$-derivations $\mathrm{Der}_k(A) := \mathrm{Der}_k(A, A)$. One verifies immediately that for $\partial_1$ and $\partial_2$ in $\mathrm{Der}_k(A)$, the commutator $[\partial_1, \partial_2] := \partial_1\partial_2 - \partial_2\partial_1$ is in $\mathrm{Der}_k(A)$. We apply this to vector fields. Let $X$ be a $C^k$-manifold (with $k \geq \infty$, remember?), $U \subset X$ open, $\partial_1$ and $\partial_2$ vector fields on $U$. Then we may view $\partial_1$ and $\partial_2$ as elements of $\mathrm{Der}_{\mathbb{R}}(C^k_U)$ (recall that we have a canonical isomorphism between $\mathrm{T}_X$ and $\mathbf{Der}_{\mathbb{R}}(C_X)$). Hence we get $[\partial_1, \partial_2]$

in $\mathrm{T}_X(U)$. (Assuming that $U$ is an open subset $\mathbb{R}^n$, compute explicitly what this operation looks like.)

We go back to our Lie group $G$. Show that for $\partial_1$ and $\partial_2$ two left-invariant vector fields on $G$, $[\partial_1, \partial_2]$ is left-invariant too. (Of course the same holds for right-invariance; one should in fact prove a lemma concerning $\partial_1$ and $\partial_2$ on a manifold $X$ that are invariant under an automorphism $\sigma$ of $X$.) Because the space of left-invariant vector fields is just $L$ (via $\partial \mapsto \partial(e)$), we get a map $[\cdot,\cdot]$ from $L \times L \to L$. This map is called the Lie bracket. Show that it is bilinear, alternating and that it satisfies Jacobi's identity:

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0.$$

A vector space $L$ with such an operation is called a Lie algebra. We will compute the Lie algebras of the Lie groups that we have seen in §1. We start with the group $G := \mathrm{GL}_n(\mathbb{R})$ (some $n \geq 0$). Since $G$ is an open subset of $\mathrm{M}_n(\mathbb{R})$, we identify the $\mathrm{T}_G(x)$ with $\mathrm{M}_n(\mathbb{R})$. Let $a$ be in $L = \mathrm{T}_e(G) = \mathrm{M}_n(\mathbb{R})$. We wish to describe explicitly the left-invariant vector field $\partial_a$ on $G$ such that $\partial_a(e) = a$. Verify that for $g$ in $G$ we have $\partial_a(g) = ga$. Now we compute $\partial_a x_{i,j}$, where the $x_{i,j}$ are the coordinate functions on $\mathrm{M}_n(\mathbb{R})$. For $g$ in $G$, $(\partial_a x_{i,j})(g)$ is by definition the derivative of $x_{i,j}$ at $g$ in the direction given by $\partial_a(g)$, i.e., in the direction $ga$. So we compute:

$$x_{i,j}(g+\varepsilon ga) = (g+\varepsilon ga)_{i,j} = g_{i,j} + \varepsilon (ga)_{i,j} = x_{i,j}(g) + \varepsilon \sum_k g_{i,k} a_{k,j}.$$

It follows that $\partial_a(x_{i,j}) = \sum_k x_{i,k} a_{k,j}$. Applying this formula twice gives:

$$(\partial_a \partial_b) x_{i,j} = \partial_a(\sum_k x_{i,k} b_{k,j}) = \sum_{k,k'} x_{i,k'} a_{k',k} b_{k,j} = \sum_{k'} x_{i,k'} (ab)_{k',j} = \partial_{ab} x_{i,j}.$$

From this we get:

$$[\partial_a, \partial_b] x_{i,j} = \partial_{[a,b]} x_{i,j}, \text{ and } [\partial_a, \partial_b] = \partial_{[a,b]},$$

since the $x_{i,j}$ are linearly independent over $\mathbb{R}$. So the Lie bracket for $\mathrm{GL}_n(\mathbb{R})$ is just the ordinary commutator of matrices. The reader should check that if we had used right-invariant vector fields to define the Lie bracket, we would have found the opposite result (use that $x \mapsto x^{-1}$ induces multiplication by $-1$ on $L$). Let us now reconsider the subgroups of $\mathrm{GL}_n(\mathbb{R})$ that we considered in §1. From the computations we did there, it follows that $\mathrm{Lie}(\mathrm{SL}_n(\mathbb{R}))$ is the subspace of $\mathrm{M}_n(\mathbb{R})$ consisting of the elements with trace zero, that $\mathrm{Lie}(\mathrm{SO}_n(\mathbb{R})) = \mathrm{M}_n(\mathbb{R})^-$, the space of anti-symmetric matrices, and that $\mathrm{Lie}(\mathrm{Sp}_{2n}(\mathbb{R}))$ is the space of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $c^t = c$, $b^t = b$ and $d = -a^t$. From the construction of the Lie bracket it follows that the Lie bracket for any of these subgroups is just the restriction of the one for $\mathrm{M}_n(\mathbb{R})$ in the first two cases and $\mathrm{M}_{2n}(\mathbb{R})$ in the last.

Let us now look at differential forms on Lie groups. Just as for vector fields, we have the notions of left-invariant, right-invariant and bi-invariant elements of $\Omega^i_G$. Of particular importance are bi-invariant differential forms of top degree, since those give us bi-invariant volume forms. Show, by pure thought, that $\mathrm{SO}_n(\mathbb{R})$ has such a non-zero bi-invariant form (use that $\mathrm{SO}_n(\mathbb{R})$ is compact and connected). Show that $\mathrm{O}_2(\mathbb{R})$ has a non-zero bi-invariant volume form, but not a non-zero bi-invariant 1-form. Show that $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{SL}_n(\mathbb{R})$ both have non-zero bi-invariant forms of top degree (use that the commutator subgroup of $\mathrm{SL}_n(\mathbb{R})$ is $\mathrm{SL}_n(\mathbb{R})$ itself).

Of particular fun should be the following exercise. Compute explicitly the bi-invariant volume form $v$ on $G := \mathrm{SO}_3(\mathbb{R})$ for which $G$ has volume one. Compute the distribution $g$, with respect to $v$, of the angles of rotation, say in the interval $[0, \pi]$, of the elements of $G$. More precisely, let $f$ be the function $G \to [-1, 1]$ that sends $x$ to $(\mathrm{tr}(x) - 1)/2$; determine the continuous function $g$ on $]-1, 1[$ such that for every continuous $h\colon [-1, 1] \to \mathbb{R}$ one has:

$$\int_G f^*(h)v = \int_{-1}^{1} gh.$$

In the same way, compute the distribution $g_2$ for the function $f_2$ from $G$ to $[-1, 1]$ that sends $x$ to $(\mathrm{tr}(x^2) - 1)/2$. Why is the result so remarkable? (Hint: it might be useful to use the following chart for $G$; let $U := \{x \in \mathbb{R}^3 \mid 0 < \|x\| < \pi\}$ and let $\psi\colon U \to G$ be the map that sends $a$ to the rotation of angle $|a|$ with (oriented) axis $\mathbb{R}a$.)