4 De Rham cohomology

4.1 Definition. *A differential graded-commutative $\mathbb{R}$-algebra is an $\mathbb{R}$-algebra A with a grading $A = \oplus_{i\in\mathbb{Z}} A_i$, which is graded-commutative (i.e., for x in A_i and y in A_j one has $yx = (-1)^{ij}xy$), and which is equipped with a differential d of degree one (i.e., an $\mathbb{R}$-linear map d from A to A such that $d^2 = 0$, $d(A_i) \subset A_{i+1}$ and $d(xy) = (dx)y + (-1)^i x dy$ for x in A_i and y in A).*

We have a contravariant functor $X \mapsto (\Omega_X(X), d)$ from the category of C^∞-manifolds to that of differential graded-commutative $\mathbb{R}$-algebras. Note that the vector spaces $\Omega_X(X)$ tend to be very big (the typical dimension is $|\mathbb{R}|$). Although this functor does transform morphisms of manifolds into $\mathbb{R}$-linear maps, it does not really simplify the study of the category of C^∞-manifolds. For example, a compact manifold X can be reconstructed from $\Omega_X^0(X)$ alone (the points of X correspond to the maximal ideals, the topology is the Zariski topology, etc.). But, if one composes the functor $X \mapsto (\Omega_X(X), d)$ with the functor that takes homology of differential graded-commutative algebras, then a miracle happens. This composed functor, called de Rham cohomology, has reasonable finiteness properties, and, most importantly, is homotopy invariant. Before proving that, let us write down the definitions in detail.

4.2 Lemma. *Let A be a differential graded-commutative $\mathbb{R}$-algebra. Then its homology $\mathrm{H}(A)$, defined as $\ker(d)/\mathrm{im}(d)$, has an induced structure of graded-commutative $\mathbb{R}$-algebra.*

Proof. Details are left to the reader. Show that $\ker(d)$ is a graded-commutative subalgebra of A, in which $\mathrm{im}(d)$ is a homogeneous ideal. □

4.3 Definition. *Let X be a paracompact C^∞-manifold. Then the de Rham cohomology of X is the graded-commutative $\mathbb{R}$-algebra $\mathrm{H}_{\mathrm{dR}}(X) := \mathrm{H}(\Omega_X(X), d)$. Hence one has:*

$$\mathrm{H}^i_{\mathrm{dR}}(X) = \frac{\ker(d\colon \Omega^i_X(X) \to \Omega^{i+1}_X(X))}{\mathrm{im}(d\colon \Omega^{i-1}_X(X) \to \Omega^i_X(X))}.$$

As explained above, $\mathrm{H}_{\mathrm{dR}}(\cdot)$ is a contravariant functor from the category of manifolds to that of graded-commutative $\mathbb{R}$-algebras. For $f\colon X \to Y$ we will write f^* for the morphism $\mathrm{H}_{\mathrm{dR}}(f)$ from $\mathrm{H}_{\mathrm{dR}}(Y)$ to $\mathrm{H}_{\mathrm{dR}}(X)$.

4.4 Remark. Of course we could define de Rham cohomology for arbitrary C^∞-manifolds (i.e., not necessarily paracompact), but that would not give the results we want. In the general case one should consider the so-called *hypercohomology* $\mathbb{H}_{\mathrm{d}R}(X) := \mathbb{H}(\Omega_X, d)$ of the complex of sheaves Ω_X.

4.5 Theorem. *The de Rham cohomology of paracompact C^∞-manifolds is homotopy invariant, i.e., if f_0 and f_1 from X to Y are homotopic, then the two maps f_0^* and f_1^* from $\mathrm{H}_{\mathrm{dR}}(Y)$ to $\mathrm{H}_{\mathrm{dR}}(X)$ are equal.*

Proof. By definition of homotopy, we have a morphism of C^∞-manifolds $F\colon X \times I \to Y$ with I an open interval containing 0 and 1, such that $F|_{X\times\{0\}} = f_0$ and $F|_{X\times\{1\}} = f_1$. (It would be enough to have F a C^1-morphism such that its restrictions to all $X \times i$ with i in I are C^k, but let us not bother.) We will construct a homotopy from 0 to $f_0^* - f_1^*$, i.e., a sequence of $\mathbb{R}$-linear maps $K^i\colon \Omega^i_Y(Y) \to \Omega^{i-1}_X(X)$ such that $f_0^* - f_1^* = dK + Kd$. In order to construct this K, we need to consider $\Omega_{X\times I}$ more closely. Let p_X and p_I denote the projections from $X \times I$ to X and

I, respectively. (More generally, one should consider the vector bundle of differential forms on a product of two manifolds.) At a point (x,i) of $X\times I$, the tangent space $\mathrm{T}_{X\times I}(x,i)$ is canonically isomorphic to the direct sum $\mathrm{T}_X(x)\oplus\mathrm{T}_I(i)$. On the level of vector bundles this gives us:

$$\mathrm{T}_{X\times I}=p_X^*\mathrm{T}_X\oplus p_I^*\mathrm{T}_I.$$

Dualizing this gives:

$$\Omega^1_{X\times I}=p_X^*\Omega^1_X\oplus p_I^*\Omega^1_I.$$

We suggest to the reader to prove that for a ring A and A-modules M_1 and M_2 one has

$$\Lambda^i(M_1\oplus M_2)=\bigoplus_{j+k=i}\Lambda^j(M_1)\otimes_A\Lambda^k(M_2).$$

In fact, $\Lambda(M_1\oplus M_2)$ and $\Lambda(M_1)\otimes_A\Lambda(M_2)$ (with its grading obtained by "convolution" are naturally isomorphic, as they have the same universal property: to give a morphism of graded A-algebras to a graded-commutative A-algebra B is to give morphisms of A-modules $\phi_1\colon M_1\to B_1$ and $\phi_2\colon M_2\to B_2$. It follows that we have a natural isomorphism:

$$\Omega^i_{X\times I}=p_X^*\Omega^i_X\oplus\left(p_X^*\Omega^{i-1}_X\otimes p_I^*\Omega^1_I\right).$$

In local coordinates, this means the following. Suppose that $x_1,\ldots,x_n$ are coordinates on $U\subset X$, and let t be the coordinate on I. Then $\Omega^i_{U\times I}$ has the basis:

$$\begin{cases} dx_{j_1}\cdots dx_{j_i}, & 1\leq j_1<\cdots<j_i\leq n,\\ dx_{j_1}\cdots dx_{j_{i-1}}dt, & 1\leq j_1<\cdots<j_{i-1}\leq n.\end{cases}$$

For ω in $\Omega^i_{X\times I}(X\times I)$ we can write uniquely $\omega=\omega_1+\omega_2\,dt$, with ω_1 in $p_X^*\Omega^i_X(X\times I)$ and ω_2 in $p_X^*\Omega^{i-1}_X(X\times I)$. This decomposition of the $\Omega^i_{X\times I}$ also induces a decomposition of the differential $d_{X\times I}$ on $\Omega_{X\times I}$: we have, uniquely, $d_{X\times I}=d_X+d_I$, with d_X increasing the degree with respect to X by one, and d_I the same for I. (For an arbitrary product, this displays the complex $\Omega_{X\times Y}$ as the total complex associated to the double complex $p_X^*\Omega_X\otimes p_Y^*\Omega_Y$.)

We can now define our homotopy operators K_1^i from $\Omega^i_{X\times I}(X\times I)$ to $\Omega^{i-1}_X(X)$; the operator K we want to have will be $K_1\circ F^*$. With the notation as above ($\omega=\omega_1+\omega_2$) we define:

$$K_1^i\omega:=(-1)^i\int_0^1\omega_2\,dt. \tag{4.5.1}$$

Since the object to be integrated does not look like a function that one usually integrates, let us write it a bit more explicit. Let x be in X. Then $(K_1^i\omega)(x)=(-1)^i\int_0^1\omega_2(x,t)\,dt$, and $\omega_2(x,t)$ is an element of the vector space $\Omega^{i-1}_X(x)$ that does not depend on t. This means that the integral is exactly of the kind we considered in the previous section: we integrate a vector space valued volume form (namely: $\omega_2\,|dt|$) over a compact subset $[0,1]$ of the manifold I. Let i_0 and i_1 denote the inclusions of X into $X\times I$ that send x to $(x,0)$ and $(x,1)$, respectively. Then we claim that for all ω in $\Omega^i_{X\times I}(X\times I)$ we have:

$$(dK_1^i+K_1^{i+1}d)\omega=i_0^*\omega-i_1^*\omega. \tag{4.5.2}$$

To prove this identity, note that it is a local problem on X, and that both sides are additive in ω. Hence we may assume that $x_1,\ldots,x_n$ are local coordinates on X and that ω is of the form $f dx_1 \cdots dx_i$ or $g dx_1 \cdots dx_{i-1} dt$ with f and g in $C_{X\times I}(X \times I)$. Let us first consider the case $\omega = f dx_1 \cdots dx_i$. Then $K_1^i \omega = 0$ because $\omega_2 = 0$. On the other hand,

$$d\omega = df dx_1 \cdots dx_i = (d_X f) dx_1 \cdots dx_i + (d_I f) dx_1 \cdots dx_i.$$

It follows that:

$$K_1 d\omega = (-1)^{i+i+1} K_1((\partial f/\partial t) dx_1 \cdots dx_i dt) = -\left(\int_0^1 (\partial f/\partial t) dt\right) dx_1 \cdots dx_i = i_0^* \omega - i_1^* \omega.$$

One should note that the last identity is exactly the fundamental theorem of calculus. Let us now consider the second case: $\omega = g dx_1 \cdots dx_{i-1} dt$. Now we have: $\omega = \omega_2$, hence:

$$K_1^i \omega = (-1)^i \left(\int_0^1 g\, dt\right) dx_1 \cdots dx_{i-1} = (-1)^i G dx_1 \cdots dx_{i-1},$$

where G is the function on X defined by $G(x) = \int_0^1 g(x,t)\, dt$. It follows that:

$$dK_1^i \omega = (-1)^i dG dx_1 \cdots dx_{i-1}.$$

On the other hand, $d\omega = (d_X g) dx_1 \cdots dx_{i-1} dt$. Hence:

$$\begin{aligned}(-1)^{i+1} K_1^{i+1} d\omega &= \left(\int_0^1 (d_X g)\, dt\right) dx_1 \cdots dx_{i-1} = d\left(\int_0^1 g\, dt\right) dx_1 \cdots dx_{i-1} = \\ &= (dG) dx_1 \cdots dx_{i-1},\end{aligned}$$

where the middle equality is the theorem in calculus that says that the derivative with respect to parameters of an integral is the integral of the derivative. So we find that $(dK_1 + K_1 d)\omega = 0$. This is just what we need, since $i_0^*(dt) = d(i_0^* t) = d(0) = 0$, and also $i_1^*(dt) = 0$.

To finish the proof of the theorem, define $K^i := K_1^i \circ F^*$ and note that, for ω in $\Omega_Y^i(Y)$, $i_0^* F^* \omega = f_0^* \omega$ and $i_1^* F^* \omega = f_1^* \omega$. □

We can now compute the de Rham cohomology for some manifolds. The empty manifold gives the zero ring, and a one point manifold gives the $\mathbb{R}$-algebra $\mathbb{R}$ itself. Now suppose that X is a contractible C^k-manifold (i.e., there is a point x in X such that the constant map $f\colon X \to X$ that sends every element of X to x is homotopic to the identity morphism id_X of X. Then f^* induces the identity endomorphism of $\mathrm{H}_{\mathrm{dR}}(X)$. But we can write $f = i_x \circ p$, with i_x the inclusion of $\{x\}$ in X and p the unique map $X \to \{x\}$. But then we see that i_x^* and p^* are inverses, hence $\mathrm{H}_{\mathrm{dR}}(X) = \mathbb{R}$.