5 The exact sequence of Mayer–Vietoris

Let X be a Hausdorff paracompact C^∞-manifold. Suppose that X is the union of two open subsets U and V:

$$X = U \cup V.$$

Then we have a diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega(X) & \xrightarrow{i} & \Omega(U)\oplus\Omega(V) & \xrightarrow{r} & \Omega(U\cap V) & \longrightarrow & 0 \\ & & \omega & \mapsto & (\omega|_U, \omega|_V) & & & & \\ & & & & (\alpha,\beta) & \mapsto & \alpha|_{U\cap V} - \beta|_{U\cap V} & & \end{array} \tag{5.1}$$

It is clear that this diagram is exact at $\Omega(X)$ and at $\Omega(U)\oplus\Omega(V)$, and that the maps i and r commute with the differentials d on each of the terms. In order to obtain the long exact sequence of Mayer–Vietoris, we need that r is surjective. It is for proving this surjectivity that the hypothesis that X is paracompact and Hausdorff is used. We need a so-called *partition of unity* subject to the cover of X by U and V: elements f and g in $C^\infty(X)$ such that $1 = f + g$ and $\mathrm{Supp}(f) \subset U$ and $\mathrm{Supp}(g) \subset V$. It is a bit technical to show, but on our paracompact Hausdorff C^∞-manifold partitions of unity subject to any open cover do exist (see for example Definition 7.9, Proposition 7.10 and Theorem 7.8 of my syllabus "Géométrie Variable", out of which this course is more or less extracted, except this section and the next). In this course, we will admit the existence of f and g above.

5.2 Proposition. *The map r in (5.1) is surjective.*

Proof. Let f and g be as above. Let ω be in $\Omega(U\cap V)$. Then $g = 0$ on $X - V$ and $f = 0$ on $X - U$. It follows that $g\omega$ can be extended by zero to U because $g\omega$ is a section of Ω on the open part $U\cap V$ of U with support contained in that open part. Explicitly: there is a unique α in $\Omega(U)$ such that $\alpha|_{U\cap V} = g\omega$ and $\alpha(x) = 0$ for all x in $U - U\cap V$. Similarly, $f\omega$ can be extended by zero to an element β of $\Omega(V)$. But then $r((\alpha, -\beta)) = \omega$. $\square$

5.3 Remark. We note that the map $\omega \mapsto (g\omega, -f\omega)$ from $\Omega(U\cap V)$ to $\Omega(U)\oplus\Omega(V)$ does not necessarily commute with d, because df and dg are not necessarily zero. Hence our splitting of the exact sequence (5.1) of vector spaces is not necessarily a splitting of an exact sequence of complexes.

5.4 Proposition. *Let (A,d), (B,d) and (C,d) be complexes (of $\mathbb{R}$-vector spaces, say). Let:*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be a short exact sequence of complexes. Then we get maps $\partial^i\colon \mathrm{H}^i(C) \to \mathrm{H}^{i+1}(A)$, for all i in $\mathbb{Z}$, characterised by the following property.

> *Let $c \in C^i$ with $dc = 0$, let $b \in B^i$ with $gb = c$, then $g(db) = dc = 0$, hence there is a unique $a \in A^{i+1}$ with $fa = db$. As $f(da) = d(fa) = d^2b = 0$, and f is injective, we have $da = 0$, and a defines an element of $\mathrm{H}^{i+1}(A)$. This element is the image under ∂ of the element in $\mathrm{H}^i(C)$ given by c.*

Together with the morphisms $f\colon \mathrm{H}(A) \to \mathrm{H}(B)$ and $g\colon \mathrm{H}(B) \to \mathrm{H}(C)$ induced by f and g, the boundary maps ∂ give a long exact sequence:

$$\cdots \to \mathrm{H}^{i-1}(B) \to \mathrm{H}^{i-1}(C) \to \mathrm{H}^{i}(A) \to \mathrm{H}^{i}(B) \to \mathrm{H}^{i}(C) \to \mathrm{H}^{i+1}(A) \to \cdots$$

Proof. This is a long but simple verification that we will not write out. Any book on homological algebra or on algebraic topology should contain this statement. A lemma often used as ingredient for this proof is the so-called "Snake Lemma". This lemma is then be applied to the commutative diagram with exact rows:

$$\begin{array}{ccccccccc} & & A^i/dA^{i-1} & \longrightarrow & B^i/dB^{i-1} & \longrightarrow & C^i/dC^{i-1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (A^{i+1})^{d=0} & \longrightarrow & (B^{i+1})^{d=0} & \longrightarrow & (C^{i+1})^{d=0}, & & \end{array}$$

where the notation $(A^{i+1})^{d=0}$ stands for $\ker(d\colon A^{i+1} \to A^{i+2})$. $\square$

5.5 Theorem. (Mayer–Vietoris) *Let X be a paracompact Hausdorff C^∞-manifold, covered by two open subsets U and V. Then the diagram in (5.1) gives, via Proposition 5.4, a long exact sequence:*

$$\cdots \to \mathrm{H}^{i-1}_{\mathrm{dR}}(U) \oplus \mathrm{H}^{i-1}_{\mathrm{dR}}(V) \to \mathrm{H}^{i-1}_{\mathrm{dR}}(U \cap V) \to \mathrm{H}^{i}_{\mathrm{dR}}(X) \to \mathrm{H}^{i}_{\mathrm{dR}}(U) \oplus \mathrm{H}^{i}_{\mathrm{dR}}(V) \to \mathrm{H}^{i}_{\mathrm{dR}}(U \cap V) \to \cdots$$

Proof. This is a direct application of Propositions 5.2 and 5.4. $\square$

Now we can combine our two tools (homotopy invariance and Mayer–Vietoris) in order to determine the de Rham cohomology for some manifolds, such as the spheres S^n, the projective spaces $\mathbb{P}^n(\mathbb{C})$ and $\mathbb{P}^n(\mathbb{R})$, compact surfaces and $\mathbb{C}$ minus a finite set of points.

5.6 Theorem. *Let $n \geq 1$. Then $\mathrm{H}^i_{\mathrm{dR}}(S^n)$ is zero if $i \notin \{0, n\}$, $\mathrm{H}^0_{\mathrm{dR}}(S^n) = \mathbb{R}$, and $\mathrm{H}^n_{\mathrm{dR}}(S^n)$ is one-dimensional, generated by any ω in $\Omega^n(S^n)$ with $\int_{S^n} \omega \neq 0$.*

Proof. Induction on n. Let $n \geq 1$. First we prove the statements concerning the dimension of the $\mathrm{H}^i_{\mathrm{dR}}(S^n)$. On S^n we have the two points $N := (0, \ldots, 0, 1)$ (the North pole) and $S := (0, \ldots, 0, -1)$ (the South pole). We let $U := S^n - \{S\}$ and $V := S^n - \{N\}$. Then U and V are isomorphic (as C^∞-manifolds) to $\mathbb{R}^n$, and $U \cap V$ is homeomorphic to $S^{n-1} \times \mathbb{R}$, hence homotopic to S^{n-1}. We note that S^0 is the disjoint union of two points. As S^n is connected, we have $\mathrm{H}^0_{\mathrm{dR}}(S^n) = \mathbb{R}$, by construction. For $2 \leq i < n$ both terms surrounding $\mathrm{H}^i_{\mathrm{dR}}(S^n)$ in the Mayer–Vietoris sequence are zero, hence $\mathrm{H}^i_{\mathrm{dR}}(S^n) = 0$. For $i = 1$ we have the exact sequence:

$$\mathrm{H}^0_{\mathrm{dR}}(U) \oplus \mathrm{H}^0_{\mathrm{dR}}(V) \longrightarrow \mathrm{H}^0_{\mathrm{dR}}(U \cap V) \longrightarrow \mathrm{H}^1_{\mathrm{dR}}(S^n) \longrightarrow \mathrm{H}^1_{\mathrm{dR}}(U) \oplus \mathrm{H}^1_{\mathrm{dR}}(V),$$

which is the sequence:

$$\begin{array}{ccccccc} \mathbb{R} \oplus \mathbb{R} & \longrightarrow & \mathbb{R} & \longrightarrow & \mathrm{H}^1_{\mathrm{dR}}(S^n) & \longrightarrow & 0 \\ (x, y) & \mapsto & x - y & & & & \end{array}$$

if $n \geq 2$, and the sequence:

$$\begin{array}{ccccccc} \mathbb{R} \oplus \mathbb{R} & \longrightarrow & \mathbb{R}^2 & \longrightarrow & \mathrm{H}^1_{\mathrm{dR}}(S^n) & \longrightarrow & 0 \\ (x, y) & \mapsto & (x - y, x - y) & & & & \end{array}$$

if $n=1$. It follows that $\mathrm{H}^i_{\mathrm{dR}}(S^n)=0$ for $0<i<n$, and that $\mathrm{H}^1_{\mathrm{dR}}(S^1)$ is one-dimensional. Suppose now that $n\geq 2$. Then the exact sequence:

$$\mathrm{H}^{n-1}_{\mathrm{dR}}(U)\oplus\mathrm{H}^{n-1}_{\mathrm{dR}}(U)\longrightarrow\mathrm{H}^{n-1}_{\mathrm{dR}}(U\cap V)\longrightarrow\mathrm{H}^n_{\mathrm{dR}}(S^n)\longrightarrow\mathrm{H}^n_{\mathrm{dR}}(U)\oplus\mathrm{H}^n_{\mathrm{dR}}(V)$$

reads:

$$0\longrightarrow\mathrm{H}^{n-1}_{\mathrm{dR}}(S^{n-1})\longrightarrow\mathrm{H}^n_{\mathrm{dR}}(S^n)\longrightarrow 0.$$

Hence all the dimensions are as claimed. In the exercises of last week it was shown that for all η in $\Omega^{n-1}(S^n)$ we have $\int_{S^n}d\eta=0$, hence $d\Omega^{n-1}(S^n)$ is in the kernel of $\int_{S^n}$. As both $\int_{S^n}\colon\Omega^n(S^n)\to\mathbb{R}$ as $\Omega^n(S^n)\to\mathrm{H}^n_{\mathrm{dR}}(S^n)$ are 1-dimensional quotients, they are equal. $\square$

5.7 Theorem. *Let $n\geq 0$. Then $\mathrm{H}^i_{\mathrm{dR}}(\mathbb{P}^n(\mathbb{C}))$ is of dimension one if i is even and $0\leq i\leq 2n$, and zero otherwise.*

Proof. Let $n\geq 0$. Let $P:=(0:0\cdots:1)$ in $\mathbb{P}^n(\mathbb{C})$. Let

$$U:=\mathbb{P}^n(\mathbb{C})-\{P\},\quad V:=\{(a_0:\cdots:a_{n-1}:1)\in\mathbb{P}^n(\mathbb{C})\mid |a_i|<1\text{ for all }i\}.$$

Then V is contractible hence homotopic to a point, and $V\cap U=V-\{P\}$ is homotopic to S^{2n-1}. The map:

$$F\colon U\times\mathbb{R}\to U,\quad((a_0:\cdots:a_{n-1}:a_n),t)\mapsto(a_0:\cdots:a_{n-1}:ta_n)$$

gives a homotopy from the identity on U to the projection to the subspace $\mathbb{P}^{n-1}(\mathbb{C})$ (the hyperplane at infinity) of $\mathbb{P}^n(\mathbb{C})$. Summarizing:

$$\mathrm{H}_{\mathrm{dR}}(U)=\mathrm{H}_{\mathrm{dR}}(\mathbb{P}^{n-1}(\mathbb{C})),\quad\mathrm{H}_{\mathrm{dR}}(V)=\mathrm{H}_{\mathrm{dR}}(\cdot),\quad\mathrm{H}_{\mathrm{dR}}(U\cap V)=\mathrm{H}_{\mathrm{dR}}(S^{2n-1}).$$

Hence the Mayer–Vietoris sequence reads:

$$\cdots\to\mathrm{H}^{i-1}_{\mathrm{dR}}(S^{2n-1})\to\mathrm{H}^i_{\mathrm{dR}}(\mathbb{P}^n(\mathbb{C}))\to\mathrm{H}^i_{\mathrm{dR}}(\cdot)\oplus\mathrm{H}^i_{\mathrm{dR}}(\mathbb{P}^{n-1}(\mathbb{C}))\to\mathrm{H}^i_{\mathrm{dR}}(S^{2n-1})\to\cdots$$

It follows that for $i\neq 2n$ the map $\mathrm{H}^i_{\mathrm{dR}}(\mathbb{P}^n(\mathbb{C}))\to\mathrm{H}^i_{\mathrm{dR}}(\mathbb{P}^{n-1}(\mathbb{C}))$ is an isomorphism, and that the map $\mathrm{H}^{2n-1}_{\mathrm{dR}}(S^{2n-1})\to\mathrm{H}^{2n}_{\mathrm{dR}}(\mathbb{P}^n(\mathbb{C}))$ is an isomorphism. $\square$

5.8 Theorem. *Let $n\geq 0$. Then $\mathrm{H}^i_{\mathrm{dR}}(\mathbb{P}^n(\mathbb{R}))$ is $\mathbb{R}$ if $i=0$, one-dimensional if $i=n$ and n is odd, and zero otherwise.*

Proof. Let $\iota\colon S^n\to S^n$ be the antipodal map. Then the obvious map $S^n\to\mathbb{P}^n(\mathbb{R})$ is the quotient for the action of the group $\langle\iota\rangle$. Hence the de Rham complex $(\Omega(\mathbb{P}^n(\mathbb{R})),d)$ is the subcomplex of ι-invariants of the de Rham complex $(\Omega(S^n),d)$ of S^n. The direct sum decomposition:

$$(\Omega(S^n),d)=(\Omega(S^n),d)^+\oplus(\Omega(S^n),d)^-,$$

shows that in the direct sum decomposition $\mathrm{H}_{\mathrm{dR}}(S^n)=\mathrm{H}_{\mathrm{dR}}(S^n)^+\oplus\mathrm{H}_{\mathrm{dR}}(S^n)^-$ we have:

$$\mathrm{H}_{\mathrm{dR}}(S^n)^+=\mathrm{H}((\Omega(S^n),d)^+),\quad\mathrm{H}_{\mathrm{dR}}(S^n)^-=\mathrm{H}((\Omega(S^n),d)^-).$$

It remains to show that $\iota^*:\mathrm{H}^n_{\mathrm{dR}}(S^n)\to\mathrm{H}^n_{\mathrm{dR}}(S^n)$ is multiplication by $(-1)^{n-1}$ if $n\geq 1$. This is left as an exercise to the reader. Of course, one can also obtain the result of this theorem using appropriate Mayer–Vietoris arguments. $\square$

Without proof we state two results that are easily proved using the techniques above.

5.9 Theorem. *Let $r \geq 1$ and $P_1, \ldots, P_r$ be r distinct points of S^2. Put $X := S^2 - \{P_1, \ldots, P_r\}$. Then $\mathrm{H}^0_{\mathrm{dR}}(X) = \mathbb{R}$, $\dim \mathrm{H}^1_{\mathrm{dR}}(X) = r - 1$, $\mathrm{H}^i_{\mathrm{dR}}(X) = 0$ for $i \geq 2$.*

5.10 Theorem. *Let X be a compact orientable connected C^∞-manifold of dimension two. The classification of such surfaces says that X can be obtained by adding a certain number of handles to S^2. This number of handles, g say, is called called the genus of X. One has: $\mathrm{H}^0_{\mathrm{dR}}(X) = \mathbb{R}$, $\dim \mathrm{H}^1_{\mathrm{dR}}(X) = 2g$, $\dim \mathrm{H}^2_{\mathrm{dR}}(X) = 1$, and $\mathrm{H}^i_{\mathrm{dR}}(X) = 0$ for $i > 2$.*

We end this section with some naive experiment that turns out to be very successful. We recall that the $\mathbb{C}$-algebra of meromorphic functions on $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ is the field of fractions $\mathbb{C}(t)$ of $\mathbb{C}[t]$.

Let $r \geq 0$ be an integer, and $a_1, \ldots, a_r$ be r distincs elements of $\mathbb{C}$. We want to consider $X := \mathbb{P}^1(\mathbb{C}) - \{\infty, a_1, \ldots, a_r\}$ as a *complex algebraic* variety, which, for us, at this moment, means that we consider the $\mathbb{C}$-algebra $\mathcal{O}(X)$ of global functions on it to be the $\mathbb{C}$-algebra of rational functions $f \in \mathbb{C}(t)$ that have no pole outside $\{\infty, a_1, \ldots, a_r\}$. We have:

$$\mathcal{O}(X) = \mathbb{C}[t, 1/(t - a_1), \ldots, 1/(t - a_r)].$$

A $\mathbb{C}$-basis of $\mathcal{O}(X)$ is given by the t^n with $n \geq 0$ together with the $(t - a_i)^{n_i}$ with $1 \leq i \leq r$ and $n_i < 0$. The algebraic de Rham complex $\Omega(X)$ is then:

$$0 \longrightarrow \mathcal{O}(X) \xrightarrow{d} \mathcal{O}(X) \cdot dt \longrightarrow 0,$$

with f in $\mathcal{O}(X)$ mapping to $f' \cdot dt$, with f' the usual derivative of f. Using the $\mathbb{C}$-basis given above, it is easy to compute that $\mathrm{H}^0(\Omega(X)) = \mathbb{C}$ and $\mathrm{H}^1(\Omega(X))$ is of dimension r, and represented by the forms $(t - a_i)^{-1} dt$. Surprisingly, we have $\mathrm{H}(\Omega(X)) = \mathbb{C} \otimes_{\mathbb{R}} \mathrm{H}_{\mathrm{dR}}(X)$, with $\mathrm{H}_{\mathrm{dR}}(X)$ the de Rham cohomology of X viewed as a (real) C^∞-manifold.

This example is no coincidence. It is in fact true for any affine non-singular complex algebraic variety X that $\mathrm{H}(\Omega(X)) = \mathbb{C} \otimes_{\mathbb{R}} \mathrm{H}_{\mathrm{dR}}(X)$ (a theorem due to Grothendieck). The proof of that is quite complicated, and uses resolution of singularities.