6 Cohomology with compact supports, and Poincaré duality

One of the fundamental results concerning de Rham cohomology is Poincaré duality. In some examples that we have seen (the S^n and the $\mathbb{P}^n(\mathbb{C})$) one notes that $\mathrm{H}^i_{\mathrm{dR}}(X)$ and $\mathrm{H}^j_{\mathrm{dR}}(X)$ have the same dimension if $i+j = \dim_X$. On the other hand, this is not so for $\mathbb{R}^n$. We will see that this comes from the non-compactness of $\mathbb{R}^n$.

In order to set up Poincaré duality, we need a version of de Rham cohomology: the one with compact supports.

6.1 Definition. *Let $k \geq 0$, X a C^k-manifold and E a C^k-vector bundle on X. Then, for $U \subset X$ open, we let $E_c(U)$ denote the sub-$\mathbb{R}$-vector space of $E(U)$ consisisting of the sections with compact support. In particular, on a C^∞-manifold we have the complex $(\Omega_c(X), d)$.*

6.2 Definition. *Let X be a paracompact Hausdorff C^∞-manifold. Then we define $\mathrm{H}_{\mathrm{dR,c}}(X)$, the de Rham cohomology with compact supports, to be the homology of the complex $(\Omega_c(X), d)$. Explicitly, for $i \geq 0$ one has:*

$$\mathrm{H}^i_{\mathrm{dR,c}}(X) = \frac{\ker(d\colon \Omega^i_c(X) \to \Omega^{i+1}_c(X))}{\mathrm{im}(d\colon \Omega^{i-1}_c(X) \to \Omega^i_c(X))}.$$

Of course, if X is compact, then $\Omega_c(X) = \Omega(X)$, and $\mathrm{H}_{\mathrm{dR,c}}(X) = \mathrm{H}_{\mathrm{dR}}(X)$.

We note that a pullback of a differential form with compact support need not have compact support. Hence $X \mapsto (\Omega_c(X), d)$ is not a contravariant functor via pullback for arbitrary morphisms between C^∞-manifolds. Of course, if $f\colon X \to Y$ is such that the inverse image of a compact subset of Y is compact, then f does induce a morphism f^* of differential graded-commutative $\mathbb{R}$-algebras from $\Omega_c(Y)$ to $\Omega_c(X)$.

6.3 Definition. *A morphism $f\colon X \to Y$ of topological spaces is called* proper *if for every compact subset K of Y the inverse image $f^{-1}K$ is compact.*

With this definition it is obvious that:

> $\mathrm{H}_{\mathrm{dR,c}}$ is a contravariant functor for proper morphisms.

As often happens, when you lose something (such as pullback functoriality for arbitrary morphisms), you gain something else. Here it is:

> on a Hausdorff space, taking sections with compact support is covariant for open immersions.

Explicitly, let $U \subset V$ be open, with V a Hausdorff manifold, let E be a vector bundle on V and s in $E_c(U)$. Then the extension of s by zero is in $E_c(V)$ (note that $\mathrm{Supp}(s)$ is closed in V hence that s is zero in the neighborhood $V - \mathrm{Supp}(s)$ of $V - U$). We will denote the extension by zero of s again by s.

We will now establish the analogs for de Rham cohomology with compact supports of the Mayer–Vietoris sequence and the homotopy invariance. We start with the Mayer-Vietoris sequence.

Let X be a paracompact Hausdorff C^∞-manifold. Suppose that X is the union of two open subsets U and V:

$$X = U \cup V.$$

Then we have a diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_c(U\cap V) & \xrightarrow{i} & \Omega_c(U)\oplus\Omega_c(V) & \xrightarrow{s} & \Omega_c(X) & \longrightarrow & 0 \\ & & & & (\alpha,\beta) & \longmapsto & \alpha+\beta & & \\ & & \omega & \longmapsto & (\omega,-\omega) & & & & \end{array} \tag{6.4}$$

It is clear that this sequence is exact, except possibly at $\Omega_c(X)$. The next proposition takes care of that.

6.5 Proposition. *The map s (for sum) in (6.4) is surjective.*

Proof. Let f and g be a partition of 1 subject to the cover of X by U and V, as in the previous section. Let ω be in $\Omega_c(X)$. Then $f\omega$ has support in U, hence is in $\Omega_c(U)$. Likewise, $g\omega$ is in $\Omega_c(V)$. It follows that $(f\omega, g\omega)$ is mapped to ω under s. $\square$

As i and s above are morphisms of complexes, (6.4) is a short exact sequence of complexes, hence by the results of the previous section, we get a long exact sequence (Mayer–Vietoris for de Rham cohomology with compact supports).

6.6 Theorem. *Let X be a paracompact Hausdorff C^∞-manifold, covered by two open subsets U and V. Then the diagram in (6.4) gives, via Proposition 5.4, a long exact sequence:*

$$\to \mathrm{H}^{i-1}_{\mathrm{dR,c}}(U\cap V) \to \mathrm{H}^{i-1}_{\mathrm{dR,c}}(U)\oplus\mathrm{H}^{i-1}_{\mathrm{dR,c}}(V) \to \mathrm{H}^{i-1}_{\mathrm{dR,c}}(X) \to \mathrm{H}^{i}_{\mathrm{dR,c}}(U\cap V) \to \mathrm{H}^{i}_{\mathrm{dR,c}}(U)\oplus\mathrm{H}^{i}_{\mathrm{dR,c}}(V) \to$$

Let us now give the analog of homotopy invariance. We should say first of all that $\mathrm{H}_{\mathrm{dR,c}}$ is *not* homotopy invariant. For example, one computes directly that $\mathrm{H}^1_{\mathrm{dR,c}}(\mathbb{R}) = \mathbb{R}$ (via integration), and $\mathrm{H}^0_{\mathrm{dR,c}}(\mathbb{R}) = 0$. Let X be a paracompact Hausdorff C^∞-manifold. Let $p_X\colon \mathbb{R}\times X \to X$ be the projection. Then p_X^* does not map $\Omega_c(X)$ to $\Omega_c(\mathbb{R}\times X)$, but *integration along the fibers*, as we will explain, does send $\Omega_c(\mathbb{R}\times X)$ to $\Omega_c(X)$, and *lowers the degree by one.* We recall from section 4 that for every i we have a direct sum decomposition:

$$\Omega^i_{\mathbb{R}\times X} = p_X^*\Omega^i_X \oplus \left(p_X^*\Omega^{i-1}_X \otimes p_{\mathbb{R}}^*\Omega^1_{\mathbb{R}}\right).$$

In local coordinates, this means the following. Suppose that $x_1,\ldots,x_n$ are coordinates on $U\subset X$, and let t be the coordinate on $\mathbb{R}$. Then $\Omega^i_{\mathbb{R}\times U}$ has the basis:

$$\begin{cases} dx_{j_1}\cdots dx_{j_i}, & 1\le j_1<\cdots<j_i\le n, \\ dx_{j_1}\cdots dx_{j_{i-1}}dt, & 1\le j_1<\cdots<j_{i-1}\le n. \end{cases}$$

For ω in $\Omega^i_c(\mathbb{R}\times X)$ we can write uniquely $\omega = \omega_1 + \omega_2\,dt$, with ω_1 in $p_X^*\Omega^i_c(\mathbb{R}\times X)$ and ω_2 in $p_X^*\Omega^{i-1}_c(\mathbb{R}\times X)$. With this notation, we define:

$$p_{X,!}\colon \Omega_c(\mathbb{R}\times X) \longrightarrow \Omega_c(X), \quad \omega \longmapsto \int_{\mathbb{R}} \omega_2\,dt. \tag{6.7}$$

6.8 Theorem. *Let X be a paracompact Hausdorff C^∞-manifold. The map $p_{X,!}$ above is a morphism of complexes, and induces isomorphisms $p_{X,!}\colon \mathrm{H}^i_{\mathrm{dR,c}}(\mathbb{R}\times X) \to \mathrm{H}^{i-1}_{\mathrm{dR,c}}(X)$. An inverse is induced by the morphism of complexes $i\colon \Omega_c(X) \to \Omega_c(\mathbb{R}\times X)$, $\omega \mapsto \omega e$, with e in $\Omega^1_c(\mathbb{R})$ such that $\int_{\mathbb{R}} e = 1$.*

Proof. It is clear that $p_{X,!}i$ is the identity on $\Omega_c(X)$. So it suffices now to show that $ip_{X,!}$ is homotopic to the identity on $\Omega_c(\mathbb{R}\times X)$. The homotopy operator is the following:

$$\omega = \omega_1 + \omega_2\, dt \longmapsto \left((a,x) \mapsto \int_\infty^a i_x^*\omega_2\, dt - \left(\int_{-\infty}^a e\right)\left(\int_{\mathbb{R}} i_x^*\omega_2\, dt\right)\right),$$

where $i_x\colon \mathbb{R} \to \mathbb{R}\times X$ is the map $a \mapsto (a,x)$. (I take this on faith from Bott and Tu.) The verification of this is left to the reader. □

6.9 Theorem. *Let $n \geq 0$. Then $\mathrm{H}^i_{\mathrm{dR,c}}(\mathbb{R}^n) = 0$ if $i \neq n$ and $\mathrm{H}^n_{\mathrm{dR,c}}(\mathbb{R}^n) = \mathbb{R}$ via the map $\Omega^n_c(\mathbb{R}^n) \to \mathbb{R}$, $\omega \mapsto \int_{\mathbb{R}^n}\omega$.*

Now we have the right prerequisites to formulate and prove Poincaré duality.

6.10 Theorem. (Poincaré duality) *Let X be a paracompact Hausdorff C^∞-manifold with a given orientation, of dimension n. Then the pairings:*

$$\Omega^i(X) \times \Omega^{n-i}_c(X) \longrightarrow \mathbb{R}, \quad (\omega,\eta) \longmapsto \int_X \omega\eta$$

induce isomorphisms:

$$\mathrm{H}^i_{\mathrm{dR}}(X) \longrightarrow \mathrm{H}^{n-i}_{\mathrm{dR,c}}(X)^\vee.$$

Proof. First we note that, as X is oriented, we have $\int_X\colon \Omega^n_c(X) \to \mathbb{R}$. Secondly, for $\omega \in \Omega^i(X)$ and $\eta \in \Omega^{n-i}_c(X)$ we have $\omega\eta$ in $\Omega^n_c(X)$. Hence the pairings exist. As $d(\omega\eta) = (d\omega)\eta + (-1)^i\omega d\eta$, $\omega\eta$ is closed if ω and η are closed. If $\omega = d\omega'$ and η is closed, then $d(\omega'\eta) = \omega\eta$ and $\int_X \omega\eta = 0$. If ω is closed and $\eta = d\eta'$, then $d(\omega\eta') = (-1)^i\omega\eta$ and $\int_X \omega\eta = 0$. Hence we get induced pairings:

$$\mathrm{H}^i_{\mathrm{dR}}(X) \times \mathrm{H}^{n-i}_{\mathrm{dR,c}}(X) \longrightarrow \mathbb{R}.$$

These pairings induce morphisms:

$$\mathrm{H}^i_{\mathrm{dR}}(X) \longrightarrow \mathrm{H}^{n-i}_{\mathrm{dR,c}}(X)^\vee.$$

It remains to show that these morphisms are isomorphisms. We will only prove this for X that have a finite *good cover*. For the general case, the reader is referred to the reference given in Bott and Tu: Greub, Halperin and Vanstone, "Connections, curvature, and cohomology", vol. 1, Academic Press, New York, 1972. We also note that a Poincaré duality statement for singular homology and singular cohomology with compact support that does not need the hypotheses Hausdorff and paracompact can be found in Massey's book on singular homology.

A good cover of X is a family U_i, $i \in I$ (with I an arbitrary set), such that for all non-empty finite subsets J of I, the intersection $U_J := \cap_{i\in J}U_i$ is diffeomorphic to $\mathbb{R}^n$ or empty. Each paracompact Hausdorff manifold admits a good cover. The idea to prove this is to choose a Riemannian metric on X (via a partition of 1 subject to a cover by charts), and to use geodesically convex neighborhoods of points. For some details, see Bott and Tu. Compact X have finite good covers. Good covers are useful for computing cohomology using Mayer-Vietoris sequences.

Now assume that X has a finite good cover $U_1, \ldots, U_r$. Then we will prove by induction on r that the Poincaré duality morphisms are isomorphisms. For $r = 0$ or 1 this follows from our knowledge of the cohomology of $\emptyset$ and of $\mathbb{R}^n$. Now let $U := U_1 \cup \cdots \cup U_{r-1}$, and let $V := U_r$. Then we know that the Poincaré duality morphisms for U and V are isomorphisms. We have a morphism between long exact sequences (checking the commutativity is left to the reader):

$$\begin{array}{ccccccc}
\mathrm{H}^{i-1}_{\mathrm{dR}}(U\cap V) & \to & \mathrm{H}^{i}_{\mathrm{dR}}(X) & \to & \mathrm{H}^{i}_{\mathrm{dR}}(U) \oplus \mathrm{H}^{i}_{\mathrm{dR}}(V) & \to & \mathrm{H}^{i}_{\mathrm{dR}}(U\cap V) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{H}^{n-i+1}_{\mathrm{dR,c}}(U\cap V)^\vee & \to & \mathrm{H}^{n-i}_{\mathrm{dR,c}}(X)^\vee & \to & \mathrm{H}^{n-i}_{\mathrm{dR,c}}(U)^\vee \oplus \mathrm{H}^{n-i}_{\mathrm{dR,c}}(V)^\vee & \to & \mathrm{H}^{n-i}_{\mathrm{dR,c}}(U\cap V)^\vee
\end{array}$$

The four vertical arrows surrounding the Poincaré duality morphisms for X are isomorphisms. The *five lemma* below finishes the proof. $\square$

6.11 Lemma. *Suppose we have a commutative diagram (of modules over some ring, say) with exact rows:*

$$\begin{array}{ccccccccc}
A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
F & \longrightarrow & G & \longrightarrow & H & \longrightarrow & I & \longrightarrow & J
\end{array}$$

such that the four outer vertical arrows are isomorphisms, then $C \to H$ is an isomorphism.

Proof. An ordinary diagram chase. $\square$

The proof of Theorem 6.10 also proves the following result.

6.12 Theorem. *Let X be a Hausdorff manifold with a finite good cover. Then the vector spaces $\mathrm{H}_{\mathrm{dR}}(X)$ and $\mathrm{H}_{\mathrm{dR,c}}(X)$ are finite dimensional. In particular, this holds for compact X.*

6.13 Example. Let $X := \mathbb{R}^2 - \mathbb{Z}$. Then $\mathrm{H}^1_{\mathrm{dR,c}}(X)$ has a basis indexed by $\mathbb{Z}$ (think of small loops around the integers). But $\mathrm{H}^1_{\mathrm{dR}}(X)$ is the dual of this space, and hence of dimension $|\mathbb{R}|$.

6.14 Remark. We note that Poincaré duality can be formulated on a non-orientable manifold if one uses the *orientation sheaf* $\Omega_X^{\dim X} \otimes \mathrm{Vol}_X^\vee$. See the book "Cohomology of sheaves" by Iversen.

There is some space left on this page. We use it to state one last theorem about de Rham cohomology: the Künneth formula for the cohomology of a product.

6.15 Theorem. (Künneth) *Let X and Y be paracompact C^∞-manifolds. Then the maps induced by the projections p_X and p_Y:*

$$\mathrm{H}_{\mathrm{dR}}(X) \otimes \mathrm{H}_{\mathrm{dR}}(Y) \longrightarrow \mathrm{H}_{\mathrm{dR}}(X \times Y), \quad \mathrm{H}_{\mathrm{dR,c}}(X) \otimes \mathrm{H}_{\mathrm{dR,c}}(Y) \longrightarrow \mathrm{H}_{\mathrm{dR,c}}(X \times Y)$$

are isomorphisms of graded-commutative algebras (possibly without unit in the second case).

For a proof we refer to Bott and Tu.