Modular forms, Cohomology, and the Ramanujan Conjecture

DRAFT II

COMMENTS WELCOME

USE AT YOUR OWN RISK

Brian Conrad

Preface

The Ramanujan-Petersson conjecture gives optimal estimates on the absolute values of eigenvalues of Hecke operators acting on spaces of modular forms. This conjecture was first proven in weight 2 by Eichler [**16**] and Shimura [**55**] (with mild restrictions which were later removed by Igusa [**32**]). They used the Weil conjectures for abelian varieties, or more precisely the Jacobians of modular curves over finite fields. The method was to find representations in the Tate modules of these Jacobians which encoded information about modular forms. Appyling the Weil conjectures to these pieces of the Tate modules then gave the desired result.

Subsequent work of various mathematicians led to a conjectural method for proving the higher weight case of the Ramanujan-Petersson conjecture by using the Weil conjectures for appropriate higher-dimensional varieties (again arising from modular curves over finite fields). In 1968, this dream was brought to completion when Deligne proved a conjecture of Serre concerning the existence of Galois representations attached to certain modular forms. These representations were found in the étale cohomology of sheaves on modular curves. Deligne used a vast cohomological generalization of the weight 2 method in order to reduce the Ramanujan-Petersson conjecture to the Weil conjectures, via this existence theorem conjectured by Serre. From the point of view of Deligne's proof, the distinguishing feature of the weight 2 cases is that the sheaves of interest turn out to be constant.

The existence theorem of Deligne is a fundamental result in the arithmetic theory of modular forms, and has been generalized in various directions. Deligne's account of his work [**10**] treats the original Ramanujan conjecture and Serre's existence conjecture for Δ, but this case is somewhat anamolous because the entire space of cusp forms of weight 12 and level 1 is 1-dimensional. Deligne certainly knew how to handle the general case along similar lines, but several new difficulties arise that do not appear when studying Δ. More precisely, in order to reduce the general Ramanujan-Petersson conjecture to the Weil conjectures and to prove Serre's existence conjecture in general, one needs to get around problems caused by old forms and one must work harder in applying Poincaré duality to compute characteristic polynomials on pieces of étale cohomology (rather than on the entire cohomology).

In the fall of 1999, I taught a graduate course at Harvard on the general topic of Deligne's work on this problem. The goal of the course was two-fold: to explain the general reduction of the Ramanujan-Petersson conjecture to the Weil conjectures in weight > 1 (not just the case of Δ), and to motivate and explain in detail the geometric and cohomological reasoning in [**10**] which forms a vital tool in making the precise connection between the classical world of modular forms over $\mathbf{C}$ and the modern ideas of étale cohomology over finite fields. There was interest expressed in distributing copies of the lecture notes, and one of the students at the time (Tom Weston) graciously offered to type up the hand-written lecture notes. This book is essentially a modified version of those notes, and it assumes prior experience with the classical theory of modular forms (Hecke operators, newforms, classical modular curves) and familiarity with basic notions in complex analytic and algebraic geometry (coherent cohomology, base change, flatness, abelian varieties, spectral sequences). Previous experience with étale cohomology is not assumed, though we can only summarize and explain (rather than prove) the facts from étale cohomology which we need to use.

One can view the ideas of Deligne, Shimura, et al. as special cases of vastly more general considerations in the representation theory of automorphic forms, but for many applications within the classical theory of elliptic modular forms it seems of interest to have available an exposition that focuses on some geometric aspects of Deligne's original work. The aim of these notes is to provide such an exposition. We pass over in silence many important related topics (such as adelizations). I learned a lot by preparing these notes, and I hope that others may find them helpful in acquiring a better understanding of the beautiful ideas that went into the resolution of the Ramanujan-Petersson conjecture for classical modular forms.

Contents

Introduction

We begin with some history. Choose $z \in \mathbf{C}$ with $\operatorname{Im} z > 0$. Define $\Lambda_z = \mathbf{Z} \oplus \mathbf{Z}z \subseteq \mathbf{C}$ and let

$$E = \mathbf{C}/\Lambda_z \overset{e^{2\pi i(\cdot)}}{\simeq} \mathbf{C}^\times/q^{\mathbf{Z}}$$

with $q = e^{2\pi i z}$. Via the meromorphic function

$$\wp_z(\tau) = \wp_{\Lambda_z}(\tau) = \frac{1}{\tau^2} + \sum_{l \in \Lambda_z - \{0\}} \frac{1}{(\tau - l)^2},$$

we get a closed immersion $E \hookrightarrow \mathbf{CP}^2$ defined by

$$\tau \mapsto [\wp_{\Lambda_z}(\tau), \wp'_{\Lambda_z}(\tau), 1],$$

giving an isomorphism of E with the smooth projective curve whose 'affine' part is

$$y^2 = 4x^3 - g_2(z)x - g_3(z). \tag{0.0.0.1}$$

Here

$$g_2(z) = 60 \sum_{l \in \Lambda_z - \{0\}} \frac{1}{l^4} = \frac{(2\pi)^4}{2^2 \cdot 3}\left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n\right) \tag{0.0.0.2}$$

$$g_3(z) = 140 \sum_{l \in \Lambda_z - \{0\}} \frac{1}{l^6} = \frac{(2\pi)^6}{2^3 \cdot 3^3}\left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n\right). \tag{0.0.0.3}$$

See [**53**, VII, §4.2] for details on these identities.

Note that

$$\begin{aligned} g_2(z+1) &= g_2(z) \\ g_3(z+1) &= g_3(z) \\ g_2\left(-\frac{1}{z}\right) &= z^4 g_2(z) \\ g_3\left(-\frac{1}{z}\right) &= z^6 g_3(z). \end{aligned}$$

The cubic in (0.0.0.1) has discriminant

$$D(z) = \frac{1}{16}\big(g_2^3(z) - 27g_3^2(z)\big) = \frac{(2\pi)^{12}}{16}(q - 24q^2 + \cdots)$$

It is clear that

$$\begin{aligned} D(z+1) &= D(z) \\ D\left(-\frac{1}{z}\right) &= z^{12} D(z) \end{aligned}$$

due to the transformation properties of g_2, g_3. Thus, D lies in the 1-*dimensional* space $S_{12}(\Gamma(1), \mathbf{C})$ of weight 12 cusp forms for the group $\Gamma(1) = \mathrm{SL}_2(\mathbf{Z})$. Jacobi showed [**39**, Ch 18, §4] that

$$g_2^3 - 27g_3^2 = (2\pi)^{12} q \cdot \prod_{n\geq 1} (1-q^n)^{24}.$$

Ramanujan considered the related q-expansion

$$\Delta(q) \stackrel{\text{def}}{=} q \cdot \prod_{n\geq 1} (1-q^n)^{24} = \sum_{n\geq 1} \tau(n) q^n.$$

Based on numerical computations, Ramanujan conjectured $\tau(nm) = \tau(n)\tau(m)$ for relatively prime integers $n, m \geq 1$ and

$$\tau(p^{r+1}) = \tau(p)\tau(p^r) - p^{11}\tau(p^{r-1})$$

for any integer $r \geq 1$ and any prime p. Equivalently, he conjectured the identity of formal Dirichlet series

$$\sum_{n\geq 1} \tau(n) n^{-s} = \prod_p (1 - \tau(p)p^{-s} + p^{11-2s})^{-1}.$$

Perhaps by considering the associated quadratic polynomial $X^2 - \tau(p)X + p^{11}$ and observing that it had non-real roots for small primes p, Ramanujan was led to make the deeper conjecture:

CONJECTURE 0.0.0.1 (Ramanujan). *$|\tau(p)| \leq 2p^{11/2}$ for all primes p.*

MAIN THEOREM 1 (Deligne).

1. *The Weil conjectures imply the Ramanujan conjecture.*
2. [**11**] *The Weil conjectures are true.*

This requires a lot of Grothendieck-style mathematics. More generally there is the conjecture:

CONJECTURE 0.0.0.2 (Ramanujan-Petersson). *If*

$$f = \sum_{n\geq 1} a_n q^n \in S_k(\Gamma_1(N), \mathbf{C})$$

is a cusp form of weight $k \geq 1$ for the group $\Gamma_1(N)$, then for all $\varepsilon > 0$

$$|a_n| = O_\varepsilon\left(n^{((k-1)/2)+\varepsilon}\right)$$

for $(n, N) = 1$.

For $k = 12$, $N = 1$ this recovers the Ramanujan conjecture via the following lemma.

LEMMA 0.0.0.3. *Fix k, $N \geq 1$. The following are equivalent:*

1. *The Ramanujan-Petersson conjecture holds for all cusp forms of weight k for the group $\Gamma_1(N)$;*
2. *For all $p \nmid N$ and all characters $\chi : (\mathbf{Z}/N\mathbf{Z})^\times \to \mathbf{C}^\times$, the roots of*

$$X^2 - t_p X + \chi(p)p^{k-1} = 0$$

have absolute value $p^{(k-1)/2}$, where t_p is any eigenvalue of the Hecke operator T_p on $S_k(\Gamma_1(N), \chi)$;

3. $|a_n| \leq d(n)n^{(k-1)/2}$ *for all* $(n,N)=1$, *where* $\sum a_n q^n \in S_k(\Gamma_1(N),\chi)$ *is a normalized simultaneous eigenform of all Hecke operators* T_n *with* $(n,N)=1$*; here* $\chi : (\mathbf{Z}/N)^\times \to \mathbf{C}^\times$ *is a character and* $d(n)$ *denotes the number of positive integral divisors of* n*;*
4. $|a_p| \leq 2p^{(k-1)/2}$ *for all primes* $p \nmid N$, *with* a_p *as in* (3).

PROOF. We first prove **(1)** $\Rightarrow$ **(2)**. The Hecke operators T_n on $S_k(\Gamma_1(N),\chi)$ are commuting normal operators, so they can be simultaneously diagonalized, giving a common eigenvector

$$f = \sum_{n\geq 1} a_n q^n$$

with $T_p(f) = t_p f$. If $T_n f = t_n f$ for all $(n,N)=1$, then $a_n = a_1 t_n$ for all $(n,N)=1$ and $a_1 \neq 0$, so we may normalize f to arrange that $t_p = a_p$ and a_n is the eigenvalue of T_n on f for $(n,N)=1$. By (1), for any fixed $\varepsilon > 0$

$$|a_{p^r}| = O_\varepsilon(p^{((k-1)/2)r+\varepsilon}) \tag{0.0.0.4}$$

as r goes to infinity. If u, v are the roots of $X^2 - a_p X + \chi(p)p^{k-1}$, the recursion for T_{p^r} in terms of the Hecke operators T_p and $\langle p\rangle$ gives

$$a_{p^r} = \frac{u^{r+1} - v^{r+1}}{u - v} = u^r + u^{r-1}v + \cdots + v^r.$$

Since $|uv| = p^{k-1}$, if $|u| \neq |v|$ then this contradicts (0.0.0.4). Thus $|u| = |v| = p^{(k-1)/2}$, completing the proof of (2).

It is clear that **(2)** $\Rightarrow$ **(4)**, and the proof that **(4)** $\Rightarrow$ **(3)** uses the recursion for the T_n's to reduce to the case where n is a prime p not dividing N. Finally, to show **(3)** $\Rightarrow$ **(1)**, we note that any $f \in S_k(\Gamma_1(N),\mathbf{C})$ is a finite linear combination of simulatenous eigenvectors in the $S_k(\Gamma_1(N),\chi)$'s with respect to the T_n's with $(n,N)=1$. Since $d(n) = O_\varepsilon(n^\varepsilon)$ for all $\varepsilon > 0$, this completes the proof. $\square$

It is statement (2) in Lemma 0.0.0.3 which is our point of departure. We should remark that for $p|N$, estimates on $|a_p|$ can be established by more direct methods. Following ideas of Kuga, Sato, Shimura, and others, what Deligne really proved was the following general conjecture of Serre:

MAIN THEOREM 2 (Deligne). *Let* $f \in S_k(\Gamma_1(N),\chi)$ *be a normalized newform of weight* $k \geq 2$. *Let* $f = \sum a_n q^n$, *so the field* $K_f = \mathbf{Q}(a_n) \subseteq \mathbf{C}$ *is a number field* (*which in fact contains the values of* χ *as well*). *Choose a prime* λ *of* K_f *with residue characteristic* l.

There exists a 2*-dimensional* $K_{f,\lambda}$*-vector space* $V_{f,\lambda}$ *and a continuous representation*

$$\rho_{f,\lambda} : G_{\mathbf{Q}} \to \mathrm{Aut}_{K_{f,\lambda}}(V_{f,\lambda}) \simeq \mathrm{GL}_2(K_{f,\lambda}),$$

unramified at all $p \nmid Nl$, *with arithmetic Frobenius* Frob_p *for such* p *having characteristic polynomial*

$$X^2 - a_p X + p^{k-1}\chi(p) \in K_f[X] \subseteq K_{f,\lambda}[X].$$

In particular, $\det \rho_{f,\lambda} = \chi \cdot \varepsilon_l^{k-1}$, *where* $\varepsilon_l : G_{\mathbf{Q}} \to \mathbf{Z}_l^\times$ *is the* l*-adic cyclotomic character.*

The goal of this book is to explain the analytic and cohomological background that underlies and motivates Deligne's proof of Main Theorem 2 and to then explain Deligne's proof of this theorem, as well as his use of this result to reduce the Ramanujan-Petersson conjecture to the Weil conjectures (which he subsequently proved).

REMARK 0.0.0.4.

1. By the Cebatorev density theorem and the Brauer-Nesbitt thoerem, the semisimplification of $\rho_{f,\lambda}$ in Main Theorem 2 is uniquely determined by the characteristic polynomial conditions. By a result of Ribet [**49**, Theorem 2.3], it is known that $\rho_{f,\lambda}$ is also irreducible.
2. Deligne and Serre [**15**] proved the analogue of Main Theorem 2 for cusp forms of weight 1 by using congruences with *higher weight* modular forms and Main Theorem 2. They obtained an Artin representation $G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{C})$.
3. The *method* of the construction of $\rho_{f,\lambda}$ is what enables one to prove that the Weil conjectures imply the Ramanujan conjecture. Actually, the dual (contragredient) representation $\rho_{f,\lambda}^{\vee}$ is what is found in étale cohomology, where geometric Frobenius elements Frob_p^{-1} plays the prominent role.

Assuming familiarity with the classical theory of modular forms and the basic concepts of scheme theory, we will begin in Chapter 1 by formulating the classical analytic theory of modular forms and Hecke operators in terms of *geometry* and line bundles on universal *analytic* moduli spaces. This requires a lot of effort if one wants to prove the most natural formulation for the main geometric results (e.g., results for relative complex tori, not just relative elliptic curves). This geometric setup motivates the purely *algebraic* construction of the theory of modular forms by Katz and others, and the topological cohomology aspects lead naturally to the considerations with étale cohomology later on.

Our analytic discussion may seem overblown relative to the applications we have in mind. For example, it is not really necessary to consider families of complex tori of relative dimension > 1 or to prove that the classical Sigel upper half-spaces are really bases for universal families which represents suitable functors on the category of complex analytic spaces. In fact, most (if not all) classical treatments are content to simply establish that these upper half-spaces can be identified as *sets* with the set of isomorphism classes of certain structures on complex tori. But it seems to me much more aesthetically satisfying to follow the Grothendieck point of view and verify that this mere identification of sets can be enhanced to something much more mathematically significant: representability of a functor. Since infinitesimal methods (e.g., cohomology and base change) are as useful in analytic geometry as they are in algebraic geometry, one really ought to establish suitable universality for the Sigel upper half-spaces in the category of complex analytic spaces and not just the category of complex manifolds. Algebraic geometers know from experience that it is not adequate to establish moduli space properties on categories of reduced (or smooth) schemes, so there is no reason to relax one's standards in complex analytic geometry. It is exactly the insistence on this point which makes Chapter 1 much longer than is really necessary for our ultimate goals. However, the pleasing nature of the final analytic conclusions seems (to me) to justify this extra effort, and these conclusions also permit more elegant arguments later on when comparing the analytic and algebraic theories.

In Chapter 2 we exhaustively work out the geometric and cohomological aspects of modular forms and modular curves from the complex analytic point of view. In particular, we develop an intrinsic theory of generalized elliptic curves over complex analytic spaces and we set the stage for Deligne's construction of 2-dimensional Galois representations by working out all of the topological ingredients which will be needed in the construction. For example, we actually *construct* the underlying 2-dimensional ℓ-adic spaces in topological cohomology which Deligne's method will later re-discover in étale cohomology (together with the auxiliary data of a Galois action). In the special case of *weight* 2, one can reformulate many of these constructions using Jacobians instead of the apparatus of cohomology. More specifically, the l-adic Tate module of the Jacobian $J_1(N)_{/\mathbf{Q}} = \mathrm{Pic}^0_{X_1(N)/\mathbf{Q}}$ can be used in place of the étale cohomology group $\mathrm{H}^1_{\text{ét}}(X_1(N)_{/\overline{\mathbf{Q}}}, \mathbf{Q}_l)$. In Chapter 3 this perspective of Jacobians is developed and is used to prove Main Theorem 2 in the case $k=2$. This is technically simpler and has a concrete geometric interpretation:

THEOREM 0.0.0.5 (Shimura). *If $f = \sum a_n q^n \in S_2(\Gamma_1(N), \chi)$ is a newform, then there exists a quotient $A_{f/\mathbf{Q}}$ of $J_1(N)_{/\mathbf{Q}}$ of dimension $[K_f : \mathbf{Q}]$ equipped with an action of K_f in the "up-to-isogeny" category such that the $K_f \otimes \mathbf{Q}_l$-module*

$$V_l(A_{f/\overline{\mathbf{Q}}}) \stackrel{\text{def}}{=} \mathbf{Q}_l \otimes_{\mathbf{Z}_l} T_l(A_{f/\overline{\mathbf{Q}}})$$

is free of rank 2, with an arithmetic Frobenius element Frob_p *for $p \nmid Nl$ having characteristic polynomial*

$$X^2 - a_p X + p\chi(p) \in K_f[X] \subseteq (K_f \otimes \mathbf{Q}_l)[X].$$

Using the isomorphism $K_f \otimes \mathbf{Q}_l \simeq \prod_{\lambda | l} K_{f,\lambda}$, we obtain from Theorem 0.0.0.5 the desired representation $\rho_{f,\lambda}$ in the weight 2 case. Note that the representation space being considered in this theorem is canonically identified with $\mathrm{H}^1_{\text{ét}}(A_{f/\overline{\mathbf{Q}}}, \mathbf{Q}_l)^\vee$. The $\mathbf{Q}_l$-characteristic polynomial of Frob_p on $V_l(A_{f/\overline{\mathbf{Q}}})$ is

$$\mathrm{N}_{\mathbf{Q}_l}^{K_f \otimes \mathbf{Q}_l}(X^2 - a_p(f)X + p\chi(p)),$$

which is equal to $\mathrm{N}_{\mathbf{Q}}^{K_f}(X^2 - a_p(f)X + p\chi(p)) \in \mathbf{Z}[X]$. Thus, by the *Weil conjectures* for the fiber of the Néron model of A_f over $\mathbf{F}_p$, we get $a_p(f) = \alpha + \beta$ for p-Weil numbers α, β of weight $1/2$. Thus, $|a_p| \leq 2\sqrt{p}$; i.e., Theorem 0.0.0.5 implies Conjecture 0.0.0.2 in weight 2.

There are natural maps of $\mathbf{Q}_l[G_{\mathbf{Q}}]$-modules

$$\begin{tikzcd} V_l(A_{f/\overline{\mathbf{Q}}})^\vee \arrow[r, "\simeq"] & \mathrm{H}^1_{\text{ét}}(A_{f/\overline{\mathbf{Q}}}, \mathbf{Q}_l) \arrow[r, hook] & \mathrm{H}^1_{\text{ét}}(J_1(N)_{/\overline{\mathbf{Q}}}, \mathbf{Q}_l) \arrow[d, "\simeq"] \\ \rho_{f,\lambda}^\vee \arrow[u, hook] & & \mathrm{H}^1_{\text{ét}}(X_1(N)_{/\overline{\mathbf{Q}}}, \mathbf{Q}_l) \end{tikzcd}$$

and there is a natural $\mathbf{C}$-linear isomorphism

$$\begin{aligned} \mathrm{H}^1(X_1(N)^{\text{an}}, \mathbf{C}) &\simeq \mathrm{H}^0(X_1(N)^{\text{an}}, \Omega^1) \oplus \mathrm{H}^0(X_1(N)^{\text{an}}, \overline{\Omega}^1) \\ &= S_2(\Gamma_1(N), \mathbf{C}) \oplus \overline{S_2(\Gamma_1(N), \mathbf{C})}. \end{aligned}$$

With these identifications, Deligne's general arguments in higher weight become *exactly* Shimura's arguments in weight 2.

After we have thoroughly discussed the case of weight 2, we will give a survey in Chapter 4 of the basic facts we need from the arithmetic theory of modular curves at level N (mostly over $\mathbf{Z}[1/N]$) and we begin the preparations for the applications of étale cohomology in this context. Chapter 5 begins with a rather pedestrian overview of étale cohomology, supplying some motivation for the general theory and a meager discussion of some proofs. This is not a book on étale cohomology, so we do not aim to explain the strongest possible results, and the reader must keep in mind that the *proofs* of some specific results which we cite require much stronger versions of other results which we have stated. Hopefully the exposition we give will at least make the novice reader comfortable with the formidable cohomological apparatus that we need to use. Building on these foundations, we will then explain Deligne's proofs of Main Theorem 1(1) and Main Theorem 2. We say nothing about Deligne's proof of Main Theorem 1(2), as this is of totally different nature.

0.1. Notation and conventions

For an $\mathcal{O}_X$-module $\mathcal{F}$ on a ringed space $(X, \mathcal{O}_X)$ and an *ordered* open covering $\mathfrak{U} = \{U_i\}_{i \in I}$ of X, we denote by $\mathcal{C}^\bullet(\mathfrak{U}, \mathcal{F})$ the sheaf Čech complex resolving $\mathcal{F}$, while we denote by $\mathrm{C}^\bullet(\mathfrak{U}, \mathcal{F})$ the corresponding Čech complex of global sections and we denote by $\mathrm{H}^\bullet(\mathfrak{U}, \mathcal{F})$ the cohomology of this latter complex (which is canonically independent of the ordering on the cover). We denote the direct limit (over open covers) of these cohomology objects by $\check{\mathrm{H}}^\bullet(X, \mathcal{F})$. More specifically, for each $\underline{i} = (i_0, \dots, i_p) \in I^{p+1}$ with $i_0 < \cdots < i_p$ we define

$$U_{\underline{i}} = U_{i_0} \cap \cdots \cap U_{i_p},$$

where $j_{\underline{i}} : U_{\underline{i}} \hookrightarrow X$ denotes the canonical open immersionm, and the degree p term of $\mathcal{C}^\bullet(\mathfrak{U}, \mathcal{F})$ is defined to be

$$\prod_{i_0 < \cdots < i_p} j_{\underline{i}*}(\mathcal{F}|_{U_{\underline{i}}}).$$

A typical section of this sheaf is denoted $s = (s_{\underline{i}})$. The differential in degree p is given by the habitual formula

$$\mathrm{d}^p(s)_{(i_0, \dots, i_{p+1})} = \sum_{j=0}^{p+1} (-1)^j s_{(i_0, \dots, \widehat{i}_j, \dots, i_{p+1})}|_{U_{(i_0, \dots, i_{p+1})}}$$

and the augmentation from $\mathcal{F}$ in degree 0 is defined in the obvious way without the intervention of signs. It goes without saying (though we say it anyway) that this choice must be remembered whenever we explicit compute cohomology using Čech theory (as we often will do in the analytic considerations). If $\mathcal{F}$ is locally free of finite rank, we denote by $\mathcal{F}^\vee$ the dual sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$.

For any topological space X and any abelian group G, we denote by $\underline{G}$ the constant sheaf on X attached to G. If A is a ring, we define a *local system* of A-modules to be an $\underline{A}$-module which is locally constant. If M is a local system of finite free A-modules (i.e., locally free of finite rank as an $\underline{A}$-module) then we may form the dual local system $M^\vee$ as above (viewing $(X, \underline{A})$ as a ringed space).

If $(X, \mathcal{O}_X)$ is a ringed space, we define the isomorphism of groups

$$\mathrm{Pic}(X) \simeq \check{\mathrm{H}}^1(X, \mathcal{O}_X^\times) \simeq \mathrm{H}^1(X, \mathcal{O}_X^\times) \tag{0.1.0.5}$$

as follows. Let $\mathcal{L}$ be an invertible sheaf and let $\mathfrak{U} = \{U_i\}$ be an ordered open covering with trivializations $\varphi_i : \mathcal{O}_{U_i} \simeq \mathcal{L}|_{U_i}$. The automorphism $\varphi_i^{-1} \circ \varphi_j$ of $\mathcal{O}_{U_i \cap U_j}$

is multiplication by some $c_{ij} \in \mathrm{H}^0(U_i \cap U_j, \mathcal{O}_X^\times)$, and this defines a cohomology class $(c_{ij})_{i<j}$ in $\mathrm{H}^1(\mathfrak{U}, \mathcal{O}_X^\times)$. The image of this class in $\check{\mathrm{H}}^1(X, \mathcal{O}_X^\times)$ only depends on the isomorphism class of $\mathcal{L}$, and this defines the first map (0.1.0.5) and agrees with [**29**, 0_{I}, §3.3]. The second map in (0.1.0.5) is a special case of the general isomorphism between Čech and derived functor cohomology arising from the Čech to derived functor cohomology spectral sequence.

Our convention for double complexes (e.g., as in the construction of spectral sequences) is that they involve *anti-commutative* squares. This choice (as opposed to using commutative squares, as in EGA) influences coboundary maps up to universal signs, and in particular affects any map arising from a degenerate spectral sequence. One "application" of our choice, and in fact the reason for using it, is that the $(\partial, \overline{\partial})$-double complex from complex analysis really *is* a double complex in our sense. This manifests itself in some of our explications of general theory in the complex analytic case.

For Weil pairings on abelian schemes we follow the sign convention in [**48**], though the sign doesn't really matter for what we will do with Weil pairings (e.g., formation of an adjoint is unaffected by negating a given bilinear pairing). For abelian varieties over an algebraically closed field this agrees with the convention in [**45**] and for elliptic curves over a base scheme this agrees with [**35**]. We make extensive reference to these works in what follows. In terms of the Tate curve model $\mathbf{C}^\times/q^{\mathbf{Z}}$, this corresponds to the pairing $\langle \zeta, q^{1/N} \rangle = \zeta$ for every $\zeta \in \mu_N(\mathbf{C})$ and every Nth root of q. Beware that this is opposite the sign in [**57**]. For concreteness, when we fix $i = \sqrt{-1} \in \mathbf{C}$ and $E = \mathbf{C}/[1, z]$ is an elliptic curve with $\mathrm{Im}(z) > 0$, then $\langle 1/N, z/N \rangle_N = e^{2\pi i/N}$. This follows from [**58**, Ex 1.15] by explicit calculations with elliptic functions, or it can be verified by more conceptual (but lengthy) considerations with Jacobians of compact Riemann surfaces.

We denote by $\mathbf{C}$ an algebraic closure of $\mathbf{R}$, fixed for all time. This field is endowed with a unique absolute value extending that on $\mathbf{R}$, and in fact a unique structure of topological field over $\mathbf{R}$. This is the only structure relevant for the basic definitions of complex analysis. At this point it is convenient to slightly digress and discuss an extra structure which most people think is implicit when working with $\mathbf{C}$: the choice of $\sqrt{-1} \in \mathbf{C}$. One does *not* need to specify a choice of $\sqrt{-1} \in \mathbf{C}$ to get complex analysis off the ground, much as one does not need to specify a choice of $\sqrt{-1}$ is the completion of the algebraic closure of $\mathbf{Q}_p$ to get p-adic analysis off the ground. For example, the Cauchy integral formula is valid regardless of which such choice one makes (with the change in choice effecting a change in meaning of "counterclockwise orientation"). We may sometimes refer to making a choice of $\sqrt{-1} \in \mathbf{C}$ and denote it by i, and in fact we will often work a lot with upper half-spaces whose definition depends on such a choice. However, we will on occasion be somewhat careful about the specification of such a choice. The reason for such apparent pedantry is quite simple: if we aim to show that some construction in complex analytic geometry is compatible with a general algebraic construction in étale cohomology, then the manifestly algebraic nature of the latter demands that the analytic object of comparison cannot possibly depend on the peculiarity of a choice of $\sqrt{-1}$ (a choice which is never made on the algebro-geometric side).

The reader may feel that $\mathbf{C} = \mathbf{R}[X]/(X^2+1)$ by definition and that i denotes the residue class of X in here, but the auxiliary specification of this $\sqrt{-1}$ has absolutely no relevance to the topological nature of an algebraic closure of $\mathbf{R}$. We really just regard $\mathbf{C}$ as some fixed algebraic closure of $\mathbf{R}$ and have no need for a "specific" model. This point is perhaps more of philosophical rather than technical nature since one can probably argue from a purely logical point of view that it is impossible to actually make a "concrete choice" of $\mathbf{C}$ without implicitly having specified a preferred choice of $\sqrt{-1}$. Indeed, $\mu_4(\mathbf{C}) - \{\pm 1\}$ is a $\mathbf{Z}/2$-torsor and it seems impossible to concretely construct a $\mathbf{Z}/2$-torsor without having implicitly specified an element. However, I feel quite strongly that paying careful attention to dependencies on a choice of $\sqrt{-1} \in \mathbf{C}$ provides greater conceptual clarity when considering certain classical analytic constructions and prevents (or minimizes) errors when guessing how to correctly translate something into étale cohomology.

We define $\mathbf{Z}(1) \subseteq \mathbf{C}$ to be the subgroup

$$\mathbf{Z}(1) = \{2\pi i n \,|\, n \in \mathbf{Z}\},$$

where i denotes either primitive 4th root of unity in $\mathbf{C}$; this is independent of the choice of i and is a free $\mathbf{Z}$-module of rank 1. Quite generally, for any abelian group M and integer d we define

$$M(d) = M \otimes_{\mathbf{Z}} \mathbf{Z}(1)^{\otimes d}.$$

Once we have specified a choice of $i = \sqrt{-1} \in \mathbf{C}$, we can uniquely write $z = \mathrm{Re}(z) + i\mathrm{Im}(z)$ for $\mathrm{Re}(z), \mathrm{Im}(z) \in \mathbf{R}$ and we will denote by $\mathfrak{h}$ the corresponding upper half-plane

$$\mathfrak{h} \stackrel{\text{def}}{=} \{z \in \mathbf{C} \,|\, \mathrm{Im}(z) > 0\}.$$

Whenever we use $\mathfrak{h}$ it is always to be understood (if not explicitly mentioned) that a choice of $i = \sqrt{-1} \in \mathbf{C}$ has been fixed. Perhaps $\mathfrak{h}_i$ would be a better notation, but it is too clumsy and there seems to be little risk of confusion. We have resisted the option of systematically writing $\sqrt{-1}$ rather than i, but we occasionally use i as an indexing variable. We have tried to avoid doing this in cases where it may cause confusion.

In discussions of analytic models of (open) modular curves as quotients of $\mathfrak{h}$, we will usually reserve the letter z for the parameter in $\mathfrak{h}$ and the letters τ, t for the variable "in the fibers" of the universal elliptic curve over our model. That is, a typical elliptic curve will be denoted $\mathbf{C}/[1, z]$ with $[1, z] \stackrel{\text{def}}{=} \mathbf{Z} \oplus \mathbf{Z}z$, $z \in \mathfrak{h}$, and τ will denote the parameter on the universal cover $\mathbf{C}$ of this elliptic curve while t will denote the parameter on $\mathbf{C}^\times$ if we view the elliptic curve in Tate form $\mathbf{C}^\times/q^{\mathbf{Z}}$ for $q = e^{2\pi i z}$ (so $t = e^{2\pi i \tau}$). It appears to be more classical to reverse the roles of the letters z and τ as we have defined them, but I prefer to think of classical modular functions on $\mathfrak{h}$ as $f(z)$ rather than $f(\tau)$. Hopefully this will not cause confusion.

We define $\mathrm{GL}_2^+(\mathbf{R}) = \{g \in \mathrm{GL}_2(\mathbf{R}) \,|\, \det(g) > 0\}$. This acts on $\mathfrak{h}$ on the left in the standard way, via linear fractional transformations. That is, $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ acts as

$$z \mapsto \frac{az + b}{cz + d}.$$

For an integer $N \geq 1$, we define groups $\Gamma(N) \subseteq \Gamma_1(N) \subseteq \mathrm{SL}_2(\mathbf{Z}) \subseteq \mathrm{GL}_2^+(\mathbf{R})$ as follows:

$$\Gamma(N) = \{m \in \mathrm{SL}_2(\mathbf{Z}) \,|\, m \equiv 1 \bmod N\},$$
$$\Gamma_1(N) = \left\{ m \in \mathrm{SL}_2(\mathbf{Z}) \,|\, m \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod N \right\}.$$

For any integer $k \geq 1$, we denote by $S_k(\Gamma_1(N), \mathbf{C})$ the finite-dimensional $\mathbf{C}$-vector space of cusp forms of weight k for $\Gamma_1(N)$. For any character $\chi : (\mathbf{Z}/N)^\times \to \mathbf{C}^\times$, we denote by $S_k(\Gamma_1(N),\chi)$ the χ-eigenspace for the action of $(\mathbf{Z}/N)^\times$ on $S_k(\Gamma_1(N),\mathbf{C})$ via the diamond operators. It is assumed that the reader has some experience with this space and the various classical Hecke operators that act on it. We refer to [**44**] for the basic classical analytic aspects, but we will construct the modern cohomological formulation of this theory in terms of geometry on modular curves.

For $i=\sqrt{-1}\in\mathbf{C}$, the *i-orientation* for a complex manifold M means the canonical orientation on the underlying real manifold which is induced by requiring $1\wedge i>0$ for the real manifold $\mathbf{C}$. We refer the reader to [**20**] for the basic definitions and results in the theory of complex analytic spaces and coherent analytic sheaves. Analytic spaces generalize complex manifolds sort of in the same way that schemes generalize classical algebraic varieties. We refer to [**19**] for the standard definitions of the sheaves of C^∞ type (p,q)-forms and the Dolbeault double complex of such sheaves (i.e., the operators ∂, $\overline{\partial}$). For a map of analytic spaces $X\to S$, the relative de Rham complex $\Omega^\bullet_{X/S}$ may be constructed by imitating the algebraic method in [**28**, §16], and we leave it to the reader to check that base change to completions at stalks gives compatibilities between the formal de Rham complex and the algebraic de Rham complex of a locally finite type $\mathbf{C}$-scheme (via the analytification functor). If A is a finitely generated $\mathbf{C}$-algebra, we denote by $\mathrm{Sp(A)}=\mathrm{Spec(A)^{an}}$ the analytic space attached to $\mathrm{Spec}(A)$. For example, $\mathrm{Sp}(\mathbf{C}[T_1,\dots,T_n])=\mathbf{C}^n$ and if A is a finite local $\mathbf{C}$-algebra, then $\mathrm{Sp(A)}=\mathrm{Spec(A)}$ is a 1-point space with structure sheaf having global sections A. In the analytic context we will typically write $\mathrm{Sp}(\mathbf{C})$ rather than $\mathrm{Spec}(\mathbf{C})$; the two notions are literally the same, but there seems to be a modest psychological distinction.

0.2. Acknowledgements

As mentioned in the Preface, I am grateful to the audience that attended my Harvard course on this topic and pestered me with many questions about it, especially Adebisi Agboola, Matt Baker, Gebhard Boeckle, Tom Graber, Jacob Lurie, Jason Starr, and Tonghai Yang. I am most indebted to Tom Weston, who took on the thankless task of typing up these notes with remarkable speed and accuracy, and to King Fai Lai and J-P. Serre for their numerous comments on an earlier draft. Finally, I would like to thank Pierre Deligne, Nick Katz, and Andrew Wiles for the inspiring nature of their work in this subject.

CHAPTER 1

Analytic theory

This chapter sets up the framework of analytic geometry and cohomology upon which everything else rests. In the classical theory of modular curves, one attaches to each $z \in \mathfrak{h}$ the elliptic curve $\mathbf{C}/[1, z]$, This allows one to identify the points of various quotients of $\mathfrak{h}$ with *sets* of isomorphism classes of classical elliptic curves with supplementary "level structure". We want a more precise result, namely a *relative* theory of analytic elliptic curves (and even relative complex tori) and an identification of the classical quotients of $\mathfrak{h}$ with *representing objects* (i.e., moduli spaces) for certain functors on analytic spaces. Such a perspective on the classical theory will naturally motivate both the arithmetic theory of modular curves and the considerations with étale cohomology that pervade [**10**]. This will take a lot of effort to set up, but the subsequent chapters should convince the reader that this is effort well spent. It is recommended that the reader skip some of the longer analytic cohomology proofs on a first reading.

Much as it is technically useful to allow oneself the generality of (locally) finite type $\mathbf{C}$-schemes instead of restricting attention to smooth varieties over $\mathbf{C}$, we prefer to work with arbitrary complex analytic spaces instead of just complex manifolds. The advantage of this extra generality lies not only in the greater naturality of the conclusions, but also in the ease with which we will be able to work with non-smooth generalized elliptic curves and use algebro-geometric techniques (such as infinitesimal fiber arguments) in our analytic considerations. Such an approach forces us to make essential use of non-trivial results in the cohomology theory of coherent analytic sheaves. The statements of these cohomological results (which we will review) are quite similar to what one knows in the theory of schemes, so the inexperienced reader should have little difficulty accepting these basic facts on faith.

We refer the reader to [**20**] for a polished and self-contained account of the theory of complex analytic spaces, including discussion of fiber products, dimension, local and global irreducible components, quasi-finite maps, coherent sheaves, and the Nullstellensatz in the analytic context. It is probable that the reader who prefers to work only with complex manifolds and admits some of the basic cohomological results should have little difficulty checking that many the arguments in this chapter work within the more restrictive setting of complex manifolds, provided one adds the additional hypothesis (satisfied in many applications of interest later on) that various maps are submersions. However, we also make essential use of the theory of generalized elliptic curves and analytic Proj constructions, both of which require us to work with analytic spaces that might be non-smooth. This could be avoided, but it would be rather unnatural to do so. Restricting one's attention to manifolds will cause a number of technical headaches later on.

This chapter is essentially concerned with results discussed in [**10**, §2]. Although [**10**, §2] is only several pages long, this chapter is by far the longest in the book. The reason is that these analytic foundations lie at the heart of the dictionary between the classical analytic point of view and its algebraic counterpart (in terms of which algebro-geometric methods are used to deduce arithmetic results about modular forms), so we felt compelled to give a complete treatment of this material because the relevant results do not seem to be conveniently gathered in any one place in the published literature. Moreover, some results appear to be unavailable in their most natural general form in the published literature. Needless to say, everything we discuss in this chapter is well-known to experts.

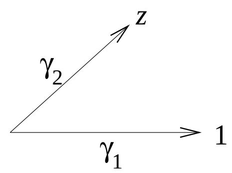
The first four sections are devoted to providing the foundations and proof for [**10**, Prop 2.2], which clarifies the universal role of classical Weierstrass theory in the analytic study of elliptic curves (and leads naturally to the analytic construction of modular curves in terms of the upper half-plane). Then in §1.5 we explain the geometric aspects of modular curves and modular forms as in [**10**, §2.3–§2.5]. The relative analytic theory of Tate curves and generalized elliptic curves is treated in §1.6. These two topics form an essential tool in connecting the algebraic and analytic theories of modular forms, in addition to providing a conceptual framework for thinking about the meaning of the compactifications of modular curves. This material includes the all-important Kodaira-Spencer isomorphism which we need for calculations near cusps, and we recover the results in [**10**, §2.6–§2.8]. Finally, in §1.7 we address the topic of the Eichler-Shimura isomorphism, in the form given in [**10**, §2.9–§2.10]. This isomorphism leads naturally to the link between étale cohomology and classical modular forms, and this is the setting in which Deligne constructs the representations whose general existence was conjectured by Serre.

1.1. Basic constructions on $\mathfrak{h}$

1.1.1. Relative elliptic curves. Let S be an arbitrary complex analytic space. We want to define the notion of an "elliptic curve over S" and then, upon choosing $i = \sqrt{-1} \in \mathbf{C}$, we want to "glue" the classical elliptic curves $\mathbf{C}/[1, z]$ for $z \in \mathfrak{h}$ into an elliptic curve $\mathcal{E}^{\mathrm{an}} \to \mathfrak{h}$ over $\mathfrak{h}$ (called $E_X \to X$ in [**10**, p. 5]) with a suitable universal property. In order to put things in perspective, we will carry this out for "relative complex tori" of any (fixed) relative dimension g, getting a suitably universal object over a Siegel upper half-space. The reader who only cares about relative dimension 1 can simply set $g = 1$ in all of our arguments, but this leads to no substantial simplification in the proofs (and even runs the risk of causing conceptual misunderstandings).

To carry out our program, we must analyze the extra data inherent in an elliptic curve of the form $\mathbf{C}/[1, z]$. In other words, we need to determine what kind of *intrinsic* structure on a classical (analytic) elliptic curve E is specified by an isomorphism $\alpha : E \simeq \mathbf{C}/[1, z]$ for $z \in \mathfrak{h}$. There are two essential such pieces of data:

1. the tangent space map at the origin $T_0(\alpha) : T_0(E) \simeq \mathbf{C}$ gives a basis of $T_0(E)$, or dually gives a non-zero differential $\omega \in \mathrm{H}^0(E, \Omega^1_E)$ corresponding to the standard differential $d\tau$ on $\mathbf{C}$;
2. $\mathrm{H}_1(\alpha) : \mathrm{H}_1(E, \mathbf{Z}) \simeq [1, z]$ with $z \in \mathfrak{h}$ gives an ordered basis $\{\gamma_1, \gamma_2\}$ of $\mathrm{H}_1(E, \mathbf{Z})$ whose ordered dual basis in $\mathrm{H}^1(E, \underline{\mathbf{Z}})$ has cup product in $\mathrm{H}^2(E, \underline{\mathbf{Z}})$ which is positive with respect to the i-orientation of E.



We will later make precise how such structures on E are related to the specification of such α's, but our first order of business is to set up a good relative notion of a "family of elliptic curves" so that we can make precise how $\mathcal{E}^{\mathrm{an}} \to \mathfrak{h}$ (once it is constructed) is going to be universal; formalizing this universality will amount to relativizing the two properties above.

One consequence of this study is an improved understanding of the classical theory of modular curves. Recall that this classical theory identifies the underlying set of $\Gamma_1(N)\backslash\mathfrak{h}$ with isomorphism classes of pairs (E, P) consisting of a classical elliptic curve E and a point $P \in E$ with exact order N, via the assignment $z \mapsto (\mathbf{C}/[1, z], 1/N)$. We will use $\mathcal{E}^{\mathrm{an}} \to \mathfrak{h}$ and descent to "glue" these objects into a global geometric structure lying over $\Gamma_1(N)\backslash\mathfrak{h}$, recovering the classical objects on fibers. In the same way that Grothendieck's reformulation of moduli spaces in terms of representable functors gave the theory of moduli more strength and flexibility, having a relative analytic theory of elliptic curves will put the analytic theory of modular curves in a more natural light.

Before we can intelligently and naturally "glue" the $\mathbf{C}/[1, z]$'s over $\mathfrak{h}$, we need to spend some time reviewing basic cohomological and geometric facts concerning complex analytic spaces. This will allow us to define and study the relative notion of elliptic curve in a purely analytic context.

Definition 1.1.1.1. A continuous map $f : X \to Y$ of topological spaces is *universally closed* if $X \times_Y Z \to Z$ is closed for "all" continuous maps $Z \to Y$.

In here and what follows, we recall that $X \times_Y Z$ in the category of topological spaces is simply the subset of $X \times Z$ which projects to the diagonal in $Y \times Y$, and this is (readily checked to be) a fiber product in the topological category. As in algebraic geometry, we refer to the operation of passing from a map $f : X \to Y$ to the induced map $X \times_Y Z \to Z$ as (topological) *base change*. The general topological notion of properness which we are about to discuss is due to Bourbaki, so the reader is referred to [**3**, I, §10] for an exhaustive (and rather abstract) discussion of properness in the topological category. An alternative reference which is adequate for our needs is [**38**, Ch 2].

Remark 1.1.1.2. Using [**38**, Exer 24–26, Ch 2], one sees that in the definition of universal closedness one does not need to check "all" Z, and in fact for a continuous map $f : X \to Y$, the map f is universally closed in the strong sense of the definition above if and only if it is closed with compact fibers. This in turn implies that the preimage of a compact set is compact, and the converse is true for locally compact Hausdorff spaces. Some of the proofs in [**38**, Exer 24, 26, Ch 2] involve working with some rather bizarre topological spaces, so it is perhaps worth noting that within the category of locally compact Hausdorff spaces one can show that a universally closed morphism is still closed with compact fibers (and hence is universally closed within the larger category of topological spaces).

Definition 1.1.1.3. A map $f : X \to Y$ of topological spaces is *separated* if the diagonal map $\Delta_f = \Delta_{X/Y} : X \to X \times_Y X$ is a closed embedding.

It follows from the definition that separatedness is preserved by topological base change, so if $f : X \to Y$ is separated, then the preimage of a Hausdorff subspace of Y is a Hausdorff subspace of X. For example, X is separated over a point if and only if X is Hausdorff. Since the underlying topological space of an analytic fiber product is the topological fiber product, it is easy to see that for a map $f : X \to Y$ of analytic spaces, Δ_f is always an immersion (here we use that analytic spaces are locally Hausdorff), so f is separated if and only if Δ_f is a closed immersion of analytic spaces.

DEFINITION 1.1.1.4. A map $f : X \to Y$ of topological spaces is *proper* if it is universally closed and separated.

It is clear from the definition that the notion of properness is preserved under base change; this is not obvious for the "usual" definition of properness in terms of compact sets (see Remark 1.1.1.2). Nevertheless, we will use the 'preimage of a compact is compact' property of proper maps when we study maps between analytic spaces (such as families of Tate curves).

Among the conditions we will impose on a morphism $E \to S$ to call it an elliptic curve over S is the condition that it should be proper. Such a morphism should also satisfy the requirements of the following fundamental definition which is inspired by the algebro-geometric relative notion of smoothness:

DEFINITION 1.1.1.5. A map $f : X \to Y$ of analytic spaces is *smooth* if f is flat (in the sense of maps of locally ringed spaces) and all analytic fibers of f are manifolds. We further say that f has *relative dimension* d if all fibers of f have pure dimension d.

This definition enjoys familiar properties as in algebraic geometry. For a map $f : X \to Y$ of analytic spaces, the following conditions are equivalent:

1. f is smooth in a neighborhood of $x \in X$;
2. $\widehat{\mathcal{O}}_{X,x} \simeq \widehat{\mathcal{O}}_{Y,f(x)}[\![t_1, \dots, t_d]\!]$ as $\mathbf{C}$-algebras, for some $d \geq 0$;
3. at x, the map f satisfies the "artinian" lifting property over Y as in algebraic geometry;
4. near x and $f(x)$, there is a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "\simeq"] \arrow[dr] & & U \times Y \arrow[dl] \\ & Y & \end{tikzcd}$$

   where $U \subseteq \mathbf{C}^d$ is an open ball, for some $d \geq 0$ (necessarily the same as in the second condition above).

When these conditions hold, f has relative dimension d near x. If Y is a manifold near $f(x)$, then these are all equivalent to X being a manifold near x and f being a submersion near x; see [**22**] for proofs of these basic facts. Associated to any smooth map f, we get a locally constant "relative dimension" function on X.

The local constancy of the relative dimension function of a smooth morphism implies that a proper smooth map with connected fibers has relative dimension function locally constant *over the base.* We may therefore reasonably speak of proper smooth families of curves, for example. The following definition is basic to all that follows.

DEFINITION 1.1.1.6. Let S be an analytic space. An *elliptic curve over* S is a proper smooth map $p : E \to S$ of relative dimension 1, with a specified section $e : S \to E$ and connected fibers of genus 1. We will often omit the section e from the notation.

Our first task is to prove that an elliptic curve $E \to S$ has a natural structure of S-group object. This will be addressed in Theorem 1.1.1.15, but first we must review some results on cohomology and base change in the analytic setting. These are analogues of results of Grothendieck in the scheme case. In the discussion that follows, we let $f : X \to Y$ denote a proper map of analytic spaces and let $\mathcal{F}$ be a coherent sheaf on X. The following result is due to Grauert and requires delicate analytic arguments.

THEOREM 1.1.1.7. *The higher direct images* $\mathrm{R}^i f_*(\mathcal{F})$ *are coherent.*

PROOF. For an elegant exposition of the proof in English, see [**20**, §10]. □

We will often find it useful to reduce general questions to the case of infinitesimal fibers. For example, let X and Y be two analytic spaces over S, and $f, g : X \to Y$ two S-morphisms. On several occasions we will have to prove such maps coincide in a general situation. Note that it always suffices to consider the case of an infinitesimal fiber (i.e., S having a single point). Indeed, suppose f and g coincide on infinitesimal fibers. These maps then agree topologically and the structure sheaf maps

$$\mathcal{O}_Y \to f_*\mathcal{O}_X = g_*\mathcal{O}_X$$

coincide because two sections of $\mathcal{O}_X$ on a common open agree if they induce the same sections on infinitesimal fibers: work locally on X and note that for each $x \in X$ over $s \in S$, the Krull intersection theorem ensures that the local noetherian ring $\mathcal{O}_x$ is separated for the topology defined by the maximal ideal $\mathfrak{m}_s$ of $\mathcal{O}_s$ (so two elements of $\mathcal{O}_x$ which have the same images in all quotients $\mathcal{O}_x/\mathfrak{m}_s^n$ are necessarily equal).

In order to reduce cohomological questions to the setting of infinitesimal fibers, it will be useful to invoke:

THEOREM 1.1.1.8 (Theorem on formal functions). *For integers* $n \geq 1$*, let* X_n *denote the* n*th infinitesimal neighborhood of the fiber* X_y *in* X *and let* $\mathcal{F}_n$ *be the pullback of* $\mathcal{F}$ *to a coherent sheaf on* X_n*. For* $y \in Y$*, the natural map of topological* $\widehat{\mathcal{O}}_y$*-modules*

$$\widehat{\mathcal{O}}_y \otimes_{\mathcal{O}_y} \mathrm{R}^i f_*(\mathcal{F})_y \to \varprojlim \ \mathrm{H}^i(X_n, \mathcal{F}_n)$$

is an isomorphism.

The proof of this theorem is very long, but there are just a handful of ideas: essentially induction on the number of generators of an ideal and fiddling with inverse limits so as to push through the induction. Unfortunately there are a number of technical details which cause the argument to become very long when written out completely. We will make extensive use of this theorem.

PROOF. One can prove this following Serre's method in the proof of [**26**, 4.1.7], provided the statement to be proven is generalized to include completion along general ideals in $\mathcal{O}_{Y,y}$ (and one then inducts on the number of generators of the ideal). This idea is carried out in the context of rigid analytic geometry by Kiehl in

[**37**], and his arguments are readily adapted to the complex analytic case. However, since the complex analytic case is not as immediately connected to rings as the algebraic or rigid analytic cases, we now give an exposition of Kiehl's method in the context of complex analytic spaces.

Since any point in a complex analytic space is closed, there exists a coherent ideal sheaf $\mathcal{I}$ on Y cutting out $\{y\}$ with $\mathcal{I}_y = \mathfrak{m}_y \subseteq \mathcal{O}_{Y,y}$. Thus, it is enough to show in general that if $\mathcal{I}$ is a coherent ideal sheaf on Y and y is in its zero locus, then the map of topological $\varprojlim \mathcal{O}_y/\mathcal{I}_y^n$-modules

$$\varprojlim \mathrm{R}^i f_*(\mathcal{F})_y/\mathcal{I}_y^n \to \varprojlim \mathrm{R}^i f_*(X_n, \mathcal{F}_n)_y$$

is an isomorphism, where $X_n = X \times_Y Y_n$ with Y_n the analytic space cut out by $\mathcal{I}^n$, $\mathcal{F}_n = \mathcal{F}|_{X_n}$, and $\mathrm{R}^i f_*(X_n, \cdot)$ denotes the ith higher direct image sheaf relative to $X_n \to Y_n$.

Working in a neighborhood of y, we may suppose $\mathcal{I}$ is generated by global sections $f_1, \ldots, f_r$ for some $r \geq 1$ and we will now induct on r. It is exactly this inductive nature of the proof which forces us to consider a rather more general situation than the specific one of interest in the statement of the theorem.

We first treat the case $r = 1$ because in this case we will prove a more precise statement which is crucial for the success of the induction. Suppose $\mathcal{I}$ is a globally principal ideal sheaf, say with generator a. Since f is proper, every neighborhood of the compact fiber $f^{-1}(y)$ contains $f^{-1}(V)$ for some open V around y. Thus, by the noetherian property of coherent analytic sheaves on X [**20**, 5.6/1], upon shrinking Y around y and replacing a with a suitable power we may assume

$$\mathcal{K} = \ker(a^n : \mathcal{F} \to \mathcal{F})$$

is the same for all $n \geq 1$. In particular, $\mathcal{F}/\mathcal{K}$ has no non-zero a-torsion. By the Artin-Rees lemma and standard coherence arguments, the system $\{a^n\mathcal{F} \cap \mathcal{K}\}$ cuts out the a-adic topology on $\mathcal{K}$. Since $a\mathcal{K} = 0$, by working locally on $f^{-1}(y)$ and shrinking Y around y we may assume (after replacing a with some power) that $a\mathcal{F} \cap \mathcal{K} = 0$. Thus,

$$0 \to \mathcal{K} \to \mathcal{F} \to \mathcal{F}/\mathcal{K} \to 0$$

is an exact sequence and the natural map $a^n\mathcal{F} \to a^n(\mathcal{F}/\mathcal{K})$ is an isomorphism for all $n \geq 1$.

Letting $S = \bigoplus_{n\geq 0} \mathcal{I}_y^n$, we therefore get an exact sequence of graded S-modules

$$\mathrm{R}^i f_*(\mathcal{K})_y \to \bigoplus_{n\geq 0} \mathrm{R}^i f_*(a^n\mathcal{F})_y \to \bigoplus_{n\geq 0} \mathrm{R}^i f_*(a^n \cdot \mathcal{F}/\mathcal{K})_y \tag{1.1.1.1}$$

in which the terms are finite $\mathcal{O}_y$-modules in each degree. Since multiplication by a^n induces an isomorphism from $\mathcal{F}/\mathcal{K}$ onto $a^n(\mathcal{F}/\mathcal{K})$ for all $n \geq 1$, it follows that the graded S-module

$$M = \bigoplus_{n\geq 0} \mathrm{R}^i f_*(a^n \cdot (\mathcal{F}/\mathcal{K}))_y$$

is a *finite* S-module, as it is generated by its 0th graded piece (since $a^n \in \mathcal{I}_y^n = S_n$ maps M_0 isomorphically onto M_n as a module over $\mathcal{O}_y = S_0$). Since S is *noetherian*, we conclude from (1.1.1.1) that $\bigoplus_{n\geq 0} \mathrm{R}^i f_*(a^n\mathcal{F})_y$ is a *finite* S-module.

Now quite generally, without assume $\mathcal{I}$ to be principal, we claim that if the graded module $\bigoplus_{n\geq 0} \mathrm{R}^i f_*(\mathcal{I}^n\mathcal{F})_y$ over the noetherian ring $S = \bigoplus_{n\geq 0} \mathcal{I}_y^n$ is a *finite* module then the natural map of topological S-modules

$$\varprojlim \mathrm{R}^i f_*(\mathcal{F})_y/\mathcal{I}_y^n \to \varprojlim \mathrm{R}^i f_*(\mathcal{F}/\mathcal{I}^n)_y \tag{1.1.1.2}$$

is an isomorphism. Applying this in the above situation with principal $\mathcal{I}$ would then take care of the case $r = 1$. In order to see that (1.1.1.2) is an isomorphism, we note that the method of proof of [**26**, 4.1.7] carries over essentially *verbatim* to show that in the exact sequence (for fixed i and variable n)

$$0 \to K_n^{(i)} \to \mathrm{R}^i f_*(\mathcal{F})_y/\mathcal{I}_y^n \to \mathrm{R}^i f_*(\mathcal{F}/\mathcal{I}^n)_y \to C_n^{(i)} \to 0, \tag{1.1.1.3}$$

the outer terms form "null systems" (i.e., there exists an n_0 such that the transition maps from stage $n+n_0$ to stage n vanish for sufficiently large n). Standard Mittag-Leffler arguments therefore imply that the induced map on inverse limits of middle terms in (1.1.1.3) is a topological isomorphism, as desired.

This argument not only verifies the case $r = 1$ but also shows something much stronger which we need for the induction. To formulate the stronger conclusion, we introduce some convenient notation. Let A denote the $\mathcal{I}_y$-adic completion of $\mathcal{O}_y$, $\mathfrak{I}$ the ideal in A generated by $\mathcal{I}_y$ (so A is an $\mathfrak{I}$-adically separated and complete noetherian ring), M the $\mathcal{I}_y$-adic completion of $\mathrm{R}^i f_*(\mathcal{F})_y$ (for a *fixed* $i \in \mathbf{Z}$), and M_n the finite $A/\mathfrak{I}^n \simeq \mathcal{O}_y/\mathcal{I}_y^n$-module $\mathrm{R}^i f_*(\mathcal{F}/\mathcal{I}^n)_y$. Viewing M with the $\mathfrak{I}$-adic topology and each M_n with the discrete topology, the preceding argument with principal $\mathcal{I}$ shows not only that that the natural map $M \to \varprojlim M_n$ is an isomorphism of *topological* A-modules, but that, at least after replacing $\mathfrak{I}$ with a suitable power (corresponding to replacing a with a suitable power) we have the following two conditions:

- the inverse system $\{M_n\}$ satisfies the Mittag-Leffler condition
- the sequence of kernels $\kappa_n = \ker(M \to M_n)$ cuts out exactly the $\mathfrak{I}$-adic topology on M.

In view of the inclusion $\mathfrak{I}^n M \subseteq \kappa_n$ and the equality $M/\mathfrak{I}^n M = \mathrm{R}^i f_*(\mathcal{F})_y/\mathcal{I}_y^n$, the fact that the kernels κ_n cut out the $\mathfrak{I}$-adic topology follows from the above-mentioned fact that the kernels $\{K_n^{(i)}\}$ in (1.1.1.3) form a "null system". The Mittag-Leffler property of the sequence $\{M_n\}$ is likewise immediate from the "null system" property of the cokernels $C_n^{(i)}$ in (1.1.1.3). Both the Mittag-Leffler condition and the $\mathfrak{I}$-adic topology condition in the $r = 1$ case will be crucial, and we do not see a pleasant way to directly establish these results for more general r since it seems tricky to verify in the complex analytic case that $\bigoplus_{n \geq 0} \mathrm{R}^i f_*(\mathcal{I}^n \mathcal{F})_y$ is a finite module over the noetherian ring $S = \bigoplus_{n \geq 0} \mathcal{I}_y^n$ (in the scheme case this sort of result is well-known essentially because one can work with the scheme $\mathrm{Proj}(S)$; perhaps the relative complex analytic theory of Proj which we discuss later could be adapted to this sort of application).

Now we treat the case $r > 1$. Let $\mathcal{J}$ denote the coherent ideal sheaf $(f_2, \ldots, f_r)$, so $\mathcal{I} = (f_1, \mathcal{J})$ and

$$\varprojlim \mathrm{R}^i f_*(\mathcal{F})_y/\mathcal{I}_y^n = \varprojlim \mathrm{R}^i f_*(\mathcal{F})_y/(f_1^n, \mathcal{J}_y{}^m) \tag{1.1.1.4}$$

This right side is naturally identified with the iterated inverse limit

$$\varprojlim{}_m \left(\varprojlim{}_n \mathrm{R}^i f_*(\mathcal{F})_y/(f_1^n, \mathcal{J}_y{}^m) \right).$$

But this inner inverse limit is exactly the f_1-adic completion of

$$\mathrm{R}^i f_*(\mathcal{F})_y/\mathcal{J}_y^m,$$

or equivalently the quotient by $\mathcal{J}_y^m$ of the f_1-adic completion of $\mathrm{R}^if_*(\mathcal{F})_y$. By the case $r=1$, this latter adic completion is naturally identified with $\varprojlim \mathrm{R}^if_*(X'_n,\mathcal{F}'_n)_y$, where $X'_n = X\times_Y Y'_n$ with Y'_n the analytic space cut out by $(f_1)^n$ and $\mathcal{F}'_n = \mathcal{F}|_{X'_n}$. Putting this all together, we would get a natural identification

$$\varprojlim \mathrm{R}^if_*(\mathcal{F})_y/\mathcal{I}_y^n = \varprojlim{}_m \left(\varprojlim{}_n \mathrm{R}^if_*(X'_n,\mathcal{F}'_n)_y\right)/\mathcal{J}_y^m \tag{1.1.1.5}$$

There is a natural map from the inner inverse limit to

$$\varprojlim{}_n (\mathrm{R}^if_*(X'_n,\mathcal{F}'_n)_y/\mathcal{J}_y^m),$$

but it is not clear if this map is an isomorphism, since we are trying to pass a quotient through an inverse limit which is slightly more subtle than a mere adic completion. In order to justify such a commutation of quotient and inverse limit, first note that these inverse limits over n are unaffected by replacing f_1 with a suitable power. Thus, we are in position to invoke a rather general lemma concerning quotients of inverse limits. The general lemma goes as follows. Let A be an $\mathfrak{I}$-adically separated and complete noetherian ring, M a finite A-module (endowed with the $\mathfrak{I}$-adic topology), M_n a finite $A_n = A/\mathfrak{I}^n$-module endowed with the discrete topology, and suppose we are given compatible linear maps $M_{n+1}\to M_n$ and $M\to M_n$ for all n such that

- $\{M_n\}$ satisifes the Mittag-Leffler condition,
- the kernels $\ker(M\to M_n)$ cut out the $\mathfrak{I}$-adic topology on M,
- the natural map $M\to M_n$ is an isomorphism of topological A-modules.

For example, the argument in the case $r=1$ shows that this condition holds when we take A to be the f_1-adic completion of $\mathcal{O}_y$, $\mathfrak{I}=f_1A$, M the f_1-adic completion of $\mathrm{R}^if_*(\mathcal{F})_y$, $M_n = \mathrm{R}^if_*(X'_n,\mathcal{F}'_n)$ — at least if we replace f_1 with a suitable power. We want to prove that for any ideal J in A (such as the ideal generated by $\mathcal{J}_y^m$ in our motivating example), the natural map

$$M/J \to \varprojlim (M_n/J) \tag{1.1.1.6}$$

is an isomorphism of topological A-modules. This would suffice to justify our wish to compute the right side of (1.1.1.5) by moving a quotient outside of an inverse limit.

Here is the general commutative algebra lemma we wish to prove:

LEMMA 1.1.1.9. *Let A be a noetherian ring which is separated and complete for the $\mathfrak{I}$-adic topology. Let M be a finite A-module, endowed with the $\mathfrak{I}$-adic topology, and let M_n be a finite $A/\mathfrak{I}^n$-module endowed with the discrete topology. Suppose we are given compatible linear maps $M_{n+1}\to M_n$ and $M\to M_n$ for all n, and assume moreover that the following three conditions hold:*

- *the map $M\to\varprojlim M_n$ of topological A-modules is an isomorphism,*
- *the inverse system $\{M_n\}$ satisfies the Mittag-Leffler condition,*
- *the kernels $\ker(M\to M_n)$ define the $\mathfrak{I}$-adic topology on M.*

Then on the formal scheme $\mathfrak{Y}=\mathrm{Spf}(A)$, the canonical map of topological $\mathcal{O}_{\mathfrak{Y}}$-modules

$$M^{\Delta}\to\varprojlim M_n^{\Delta} \tag{1.1.1.7}$$

is an isomorphism.

The reason this lemma suffices to prove that (1.1.1.6) is an isomorphism is that the functor $M \rightsquigarrow M^{\Delta}$ is an equivalence of categories between finite A-modules and coherent $\mathcal{O}_{\mathfrak{Y}}$-modules, behaving well with respect to formation of quotients by ideals of A (and the corresponding coherent ideal sheaves on $\mathfrak{Y}$) and satisfying $M \simeq \Gamma(\mathfrak{Y}, M^{\Delta})$; see [**29**, §10.10], especially [**29**, 10.10.8].

REMARK 1.1.1.10. When A is a local ring and $\mathfrak{I}$ is primary for the maximal ideal, then the first of the three conditions on $\{M_n\}$ in Lemma 1.1.1.9 implies the other two. Indeed, the quotients $A/\mathfrak{I}^n$ are artin local rings, so the M_n's have finite A-length and hence the Mittag-Leffler condition is automatic. Meanwhile, the kernels K_n of $M \to M_n$ are a decreasing chain of A-submodules whose intersection is 0. Since A is a complete local noetherian ring, it then follows from the module version of Chevalley's theorem [**42**, Exer. 8.7] that the K_n's define a topology at least as strong as the maximal-adic one on M (i.e., each $\mathfrak{I}^n M$ contains K_m for large m). But since K_n contains $\mathfrak{I}^n M$, the topology defined by the K_n's coincides with the maximal-adic (or equivalently, $\mathfrak{I}$-adic) one.

PROOF. (of Lemma 1.1.1.9) In order to prove that a map of *topological* sheaves (which is the category in which M^{Δ} is formed as a "sheafification"!) is an isomorphism, it suffices to check the induced maps on sections over a base of opens are *topological* isomorphisms. We will compute the topological module of sections of both sides of (1.1.1.7) over the basic opens $\mathrm{Spf}(A_{\{f\}})$ in $\mathfrak{Y}$. Since the module of sections of M^{Δ} over $\mathrm{Spf}(A_{\{f\}})$ is canonically identified with $M_{\{f\}} = M \otimes_A A_{\{f\}}$ (by [**29**, 10.8.4, 10.10.1, 10.10.8]), we just need to check that the map of topological $A_{\{f\}}$-modules

$$M_{\{f\}} \to \varprojlim (M_n)_{\{f\}} = \varprojlim (M_n)_f \tag{1.1.1.8}$$

is an isomorphism. Note that it is essential to verify this is an isomorphism in the topological category and not just in the category of $A_{\{f\}}$-modules.

Recall that for an inverse system $\{X_\alpha\}$ of discrete spaces and $\{Y_\alpha\}$ a compatible inverse system of subspaces, the map $\varprojlim Y_\alpha \to \varprojlim X_\alpha$ is a topological embedding (as both are naturally topologically embedded in $\prod X_\alpha$), so this map is a topological isomorphism if it is merely bijective. Thus, if N_n denotes the $A/\mathfrak{I}^n$-module image of M in M_n, endowed with the discrete topology, the composite

$$M \to \varprojlim N_n \to \varprojlim M_n$$

is a topological isomorphism and the right map is injective, so the right map is a topological isomorphism and hence so is the left map. Thanks to the Mittag-Leffler condition, N_n is equal to the image of $M_m \to M_n$ for sufficiently large m (perhaps depending on n), so

$$(N_n)_f = \mathrm{image}((M_m)_f \to (M_n)_f)$$

for sufficiently large m. Thus, the natural map

$$\varprojlim (N_n)_f \to \varprojlim (M_n)_f$$

is bijective and therefore a topological isomorphism. In view of the form of (1.1.1.8), we may therefore replace M_n by N_n for all n, so we are reduced to the case where the maps $M \to M_n$ is surjective for all n.

There is no loss of generality in using another inverse system of $A/\mathfrak{I}^n$-module quotients of M which is cofinal with respect to the quotients M_n and vice-versa, so by the hypothesis that the kernels of $M \twoheadrightarrow M_n$ define the $\mathfrak{I}$-adic topology on M, we may assume $M_n = M/\mathfrak{I}^n$ for all n. By [**29**, 0_{I}, 7.6.12, 7.6.13], this tranforms (1.1.1.8) into the canonical map

$$M_{\{f\}} \to \varprojlim\ (M/\mathfrak{I}^n)_{\{f\}} = \varprojlim\ M_{\{f\}}/\mathfrak{I}^n_{\{f\}}M_{\{f\}}.$$

But this map is a topological isomorphism by [**29**, 0_{I}, 7.3.6] since $M_{\{f\}}$ is a finite $A_{\{f\}}$-module, with $A_{\{f\}}$ an adic noetherian ring having ideal of definition $\mathfrak{I}_{\{f\}}$. $\square$

With the manipulation of the right side of (1.1.1.5) now justified by Lemma 1.1.1.9, we continue with the proof of Theorem 1.1.1.8. We interchange the order of inverse limits in (1.1.1.5) and then use the inductive hypothesis (applied to the ideal sheaf $\mathcal{J}$, the morphism $X'_n \to Y'_n$ and the coherent sheaf $\mathcal{F}'_n$) to identify the resulting inner inverse limit over m with

$$\varprojlim\ {}_m\mathrm{R}^if_*((X'_n)_m, (\mathcal{F}'_n)_m)_y,$$

where $(X'_n)_m \hookrightarrow X'_n$ is the analytic space cut out by $\mathcal{J}^m$ and $(\mathcal{F}'_n)_m = \mathcal{F}'_n|_{(X'_n)_m}$ is the corresponding pullback. Clearing the dust, we have obtained a canonical identification

$$\varprojlim\ \mathrm{R}^if_*(\mathcal{F})_y/\mathcal{I}^n_y = \varprojlim\ {}_n \varprojlim\ {}_m\mathrm{R}^if_*((X'_n)_m, (\mathcal{F}'_n)_m)_y.$$

But the systems of fiber spaces $\{(X'_n)_m\}$ and $\{X_\nu\}$ are cofinal with respect to each other, so this final double inverse limit is naturally (topologically) identified with the right side of the Theorem on Formal Functions, thereby completing the induction. $\square$

By the GAGA theorems [**23**, XIII], the analytification functor from proper $\mathbf{C}$-schemes to compact Hausdorff complex analytic spaces is fully faithful. We call an analytic space arising in this way *algebraic*. We will often use Theorem 1.1.1.8 to reduce general questions about analytic families of elliptic curves and principally polarized complex tori to the case of a 1 point base, where the analytic objects turn out to be algebraic (so GAGA reduces us to an algebraic situation).

Before stating the next result, we recall that an $\mathcal{O}_X$-module $\mathcal{F}$ is said to be Y*-flat* if $\mathcal{F}_x$ is a flat $\mathcal{O}_{Y,f(x)}$-module for all $x \in X$; in case this holds with $\mathcal{F} = \mathcal{O}_X$, f is a *flat* map.

THEOREM 1.1.1.11 (cohomology and base change). *Let $f : X \to Y$ be a proper morphism of analytic spaces and let $\mathcal{F}$ be a coherent Y-flat sheaf on X. For $i \in \mathbf{Z}$, if the natural map*

$$\varphi^i(y) : \mathrm{R}^if_*(\mathcal{F})_y/\mathfrak{m}_y \to \mathrm{H}^i(f^{-1}(y), \mathcal{F}|_{f^{-1}(y)})$$

is surjective for some $y \in Y$ then it is an isomorphism.

If this surjectivity condition holds for all $y \in Y$, then $\varphi^{i-1}(y)$ is surjective if and only if the coherent sheaf $\mathrm{R}^if_*(\mathcal{F})$ is locally free on some open U around y, in which case $\mathrm{R}^if_*(\mathcal{F})$ is of formation compatible with arbitrary analytic base change over U.

We remark that this statement of the theorem is not the most general form one can imagine: a stronger form to be expected (by analogy with the scheme case) is that the surjectivity of $\varphi^i(y)$ should force the surjectivity of $\varphi^i(y')$ for y' near y. However, our method of proof (which rests on using Theorem 1.1.1.8) does not yield this. The above slightly restrictive form of the theorem on cohomology and base change is adequate for nearly all of our applications because in most situations where we can prove surjectivity of some $\varphi^i(y)$, the same argument will work for all points y in the base.

PROOF. In essence the idea is to adapt the EGA proofs to this analytic situation by exploiting the analytic Theorem on Formal Functions, as given in Theorem 1.1.1.8. We begin by treating the case where Y consists of a single point and where the base changes $Y' \to Y$ under consideration have Y' also with only 1 point. That is, we assume $Y = \mathrm{Sp}(A)$ for a finite local $\mathbf{C}$-algebra A and for base change we only work with $Y' = \mathrm{Sp}(A')$ for a finite local A-algebra A'.

In this special case, X is a compact Hausdorff analytic space and $A \to \Gamma(X, \mathcal{O}_X)$ is a $\mathbf{C}$-algebra map which makes the coherent $\mathcal{O}_X$-module $\mathcal{F}$ an $\underline{A}$-flat sheaf on X. If we choose a finite ordered Stein covering $\mathfrak{U} = \{U_i\}$ of X, the cohomology of coherent sheaves on X can be computed via Čech cohomology with respect to $\mathfrak{U}$. Note that for any open Stein set $U \subseteq X$ and any finite A-module M, the natural map

$$\mathrm{H}^0(U, \mathcal{F}) \otimes_A M \to \mathrm{H}^0(U, \mathcal{F} \otimes_{\underline{A}} \underline{M})$$

is an isomorphism. Indeed, both sides are right exact functors on M and hence by using a presentation

$$A^r \to A^s \to M \to 0$$

we reduce to the trivial case $M = A$; this argument only uses that U is Stein and $\mathcal{F}$ is coherent, but not that $\mathcal{F}$ is $\underline{A}$-flat. Taking the module M to be an ideal $I \subseteq A$, the natural composite map

$$\mathrm{H}^0(U, \mathcal{F}) \otimes_A I \simeq \mathrm{H}^0(U, \mathcal{F} \otimes_{\underline{A}} \underline{I}) \to \mathrm{H}^0(U, \mathcal{F})$$

is injective, the second map being injective because $\mathcal{F}$ is $\underline{A}$-flat. Since flatness can be detected by tensoring against inclusions of an ideal into a ring [**42**, 7.7], we conclude that $\mathrm{H}^0(U, \mathcal{F})$ is a *flat* A-module (just as one would expect from the $\mathrm{Sp}(A)$-flatness of $\mathcal{F}$, due to the analogy between Stein spaces and affine schemes).

If $A \to A'$ is a finite map of local $\mathbf{C}$-algebras, then the base change map is simply

$$\mathrm{H}^i(X, \mathcal{F}) \otimes_A A' \to \mathrm{H}^i(X, \mathcal{F} \otimes_{\underline{A}} \underline{A}').$$

Since we have Čech theory available for computing coherent cohomology on X via finite Stein covers, the A-flatness of $\mathrm{H}^0(U, \mathcal{F})$ for Stein opens $U \subseteq X$ enables us to carry over the method of proof in [**30**, III, §12] essentially verbatim in the case of 1 point bases and base change to other 1 point bases (there is no need for the Theorem on Formal Functions in the proof, as the base rings are artinian).

Now consider the case of a general base Y. If $\varphi^i(y)$ is surjective, then (with the notation as in Theorem 1.1.1.8) the natural map

$$\mathrm{H}^i(X_n, \mathcal{F}_n)/\mathfrak{m}_y \to \mathrm{H}^i(X_0, \mathcal{F}_0) = \mathrm{H}^i(f^{-1}(y), \mathcal{F}|_{f^{-1}(y)})$$

is surjective for all n. By the case of a 1 point base, these maps are isomorphisms for all n. Passing to the inverse limit and using the Theorem on Formal Functions, we conclude that $\varphi^{-1}(y)$ is an isomorphism as long as we can show that the natural map

$$\left(\varprojlim \mathrm{H}^i(X_n, \mathcal{F}_n)\right)/\mathfrak{m}_y \to \varprojlim \left(\mathrm{H}^i(X_n, \mathcal{F}_n)/\mathfrak{m}_y\right)$$

is an isomorphism. By Lemma 1.1.1.9 and Remark 1.1.1.10, this natural map is indeed an isomorphism.

Now assuming that $\varphi^i(y)$ is an isomorphism (or equivalently, just surjective), let's show that $\varphi^{i-1}(y)$ is surjective if and only if the finite $\mathcal{O}_y$-module $\mathrm{R}^i f_*(\mathcal{F})_y$ is free, or equivalently is flat (so the coherent sheaf $\mathrm{R}^i f_*(\mathcal{F})$ is locally free near y). If $\varphi^{i-1}(y)$ is surjective, then

$$\mathrm{H}^j(X_n, \mathcal{F}_n)/\mathfrak{m}_y \to \mathrm{H}^j(X_0, \mathcal{F}_0)$$

is surjective for all n with $j = i, i-1$, so by the already-settled case of a 1 point base we conclude that $\mathrm{H}^i(X_n, \mathcal{F}_n)$ is a free $\mathcal{O}_y/\mathfrak{m}_y^n$-module for all n and is of formation compatible with finite local base change on $\mathcal{O}_y/\mathfrak{m}_y^n$. Combining this with the Theorem on Formal Functions, we get an expression for

$$\widehat{\mathrm{R}^i f_*(\mathcal{F})}_y = \widehat{\mathcal{O}}_y \otimes_{\mathcal{O}_y} \mathrm{R}^i f_*(\mathcal{F})_y$$

as $\varprojlim M_n$ where M_n is a finite free $\mathcal{O}_y/\mathfrak{m}_y^n$-module with $M_n/\mathfrak{m}_y^{n-1} \simeq M_{n-1}$ for all n. It follows from [**29**, 0_{I}, 7.2.9] and the local flatness criterion [**42**, 22.3] that $\widehat{\mathrm{R}^i f_*(\mathcal{F})}_y$ is a finite free $\widehat{\mathcal{O}}_y$-module with

$$\widehat{\mathrm{R}^i f_*(\mathcal{F})}_y/\mathfrak{m}_y^n \simeq \mathrm{H}^i(X_n, \mathcal{F}_n)$$

for all n. Thus, $\mathrm{R}^i f_*(\mathcal{F})_y$ is a finite flat $\mathcal{O}_y$-module, hence finite free with

$$\mathrm{R}^i f_*(\mathcal{F})_y/\mathfrak{m}_y^n \simeq \mathrm{H}^i(X_n, \mathcal{F}_n) \tag{1.1.1.9}$$

for all n.

Before we prove conversely that when $\mathrm{R}^i f_*(\mathcal{F})_y$ is free over $\mathcal{O}_y$ with $\varphi^i(y)$ surjective then $\varphi^{i-1}(y)$ is surjective, we analyze the consequences of the above considerations in the context of general base change. Since any local map $\mathcal{O}_y \to A$ to a finite local $\mathbf{C}$-algebra factors through some $\mathcal{O}_y/\mathfrak{m}_y^n$, it follows from the preceding argument (especially (1.1.1.9)) that if $\varphi^i(y)$ and $\varphi^{i-1}(y)$ are surjective then $\mathrm{R}^i f_*(\mathcal{F})$ is locally free at y and is of formation compatible with base changes $Y' \to Y$ where Y' is a 1 point space mapping to y. In particular, if $\varphi^i(y)$ and $\varphi^{i-1}(y)$ are surjective for all $y \in Y$ then $\mathrm{R}^i f_*(\mathcal{F})$ is locally free and of formation compatible with *any* base change to a 1 point space.

This implies that $\mathrm{R}^i f_*(\mathcal{F})$ is of formation compatible with arbitrary base change. To see this, consider a cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

The base change map

$$g^* \mathrm{R}^i f_*(\mathcal{F}) \to \mathrm{R}^i f'_*(\mathcal{F}')$$

of coherent $\mathcal{O}_{Y'}$-modules is an isomorphism on $\mathcal{O}_{y'}$-module stalks at $y' \in Y'$ if and only if the stalk maps are isomorphisms after applying the faithfully flat extension of scalars $\mathcal{O}_{y'} \to \widehat{\mathcal{O}}_{y'}$. This extension of scalars is identified with the natural map

$$\widehat{\mathcal{O}}_{y'} \otimes_{\widehat{\mathcal{O}}_y} (\widehat{\mathcal{O}}_y \otimes_{\mathcal{O}_y} \mathrm{R}^i f_*(\mathcal{F})_y) \to \widehat{\mathcal{O}}_{y'} \otimes_{\mathcal{O}_{y'}} \mathrm{R}^i f'_*(\mathcal{F}')$$

By the Theorem on Formal Functions, this latter map is identified with the natural map

$$\widehat{\mathcal{O}}_{y'} \otimes_{\widehat{\mathcal{O}}_y} \widehat{\mathrm{R}^i f_*(\mathcal{F})}_y \to \varprojlim \mathrm{H}^i(X'_n, \mathcal{F}'_n) \tag{1.1.1.10}$$

which we need to prove is an isomorphism (of modules). But the base change compatibility for base change to a 1 point space permits us to compute

$$\begin{aligned}\mathrm{H}^i(X'_n, \mathcal{F}'_n) &\simeq \mathcal{O}_{y'}/\mathfrak{m}_{y'}^n \otimes_{\mathcal{O}_y/\mathfrak{m}_y^n} \mathrm{H}^i(X_n, \mathcal{F}_n)\\ &\simeq \mathcal{O}_{y'}/\mathfrak{m}_{y'}^n \otimes_{\mathcal{O}_y/\mathfrak{m}_y^n} (\mathrm{R}^i f_*(\mathcal{F})_y/\mathfrak{m}_y^n)\\ &\simeq \mathcal{O}_{y'}/\mathfrak{m}_{y'}^n \otimes_{\widehat{\mathcal{O}}_y} \widehat{\mathrm{R}^i f_*(\mathcal{F})}_y,\end{aligned}$$

so (1.1.1.10) is an isomorphism.

Finally, we treat the "converse" implication that if $\mathrm{R}^i f_*(\mathcal{F})_y$ is free over $\mathcal{O}_y$ and $\varphi^i(y)$ is surjective, then $\varphi^{i-1}(y)$ is surjective. Let $\mathcal{I}$ be a coherent ideal sheaf cutting out the (closed) point $y \in Y$, with $\mathcal{I}_y = \mathfrak{m}_y \subseteq \mathcal{O}_y$. Consider the exact sequence of coherent $\mathcal{O}_X$-modules

$$0 \to \mathcal{I} \otimes_{\mathcal{O}_Y} \mathcal{F} \to \mathcal{F} \to \mathcal{F}|_{f^{-1}(y)} \to 0,$$

where exactness follows from the Y-flatness of $\mathcal{F}$. In order that $\varphi^{i-1}(y)$ be surjective, we want that the map

$$\mathrm{R}^i f_*(\mathcal{I} \otimes \mathcal{F})_y \to \mathrm{R}^i f_*(\mathcal{F})_y$$

is injective. It suffices to show that

$$\mathfrak{m}_y \otimes_{\mathcal{O}_y} \mathrm{R}^i f_*(\mathcal{F})_y \to \mathrm{R}^i f_*(\mathcal{I} \otimes \mathcal{F})_y$$

is surjective since the composite

$$\mathfrak{m}_y \otimes_{\mathcal{O}_y} \mathrm{R}^i f_*(\mathcal{F})_y \to \mathrm{R}^i f_*(\mathcal{I} \otimes \mathcal{F})_y \to \mathrm{R}^i f_*(\mathcal{F})_y \tag{1.1.1.11}$$

is injective by the $\mathcal{O}_y$-module *flatness* of the stalk $\mathrm{R}^i f_*(\mathcal{F})_y$. It is more convenient to prove such a surjectivity claim in slightly greater generality.

For any finite $\mathcal{O}_y$-module M, there is a coherent sheaf $\widetilde{M}$ on Y near y with $\widetilde{M}_y \simeq M$ as $\mathcal{O}_y$-modules, and $\widetilde{M}$ is "locally unique" near y in the sense that any two such coherent sheaves near y realizing M as their stalk at y are automatically isomorphic on a neighborhood of y. By the general yoga of coherent sheaves, this construction $M \rightsquigarrow \widetilde{M}$ is "functorial and exact" in M near y in an evident sense. We could formulate this more precisely with the language of direct limits of categories, but the basic idea seems clear enough without the need for the obsfucation of categorial formalism.

In terms of this general construction, we claim that the natural map

$$M \otimes_{\mathcal{O}_y} \mathrm{R}^i f_*(\mathcal{F})_y \to \mathrm{R}^i f_*(\widetilde{M} \otimes_{\mathcal{O}_Y} \mathcal{F})$$

is surjective for any $\mathcal{O}_y$-module M (then taking $M = \mathfrak{m}_y$ gives what we need). By the Theorem on Formal Functions, we are reduced to the case where M is of finite length over $\mathcal{O}_y$ (note $\mathfrak{m}_y$ typically does *not* have this property!). We now induct on the length of M, the case of length 1 being exactly the surjectivity hypothesis on $\varphi^i(y)$. Since tensor products are right exact (bi)functors, induction on length is trivial. Note that this final step does not use the freeness of $\mathrm{R}^i f_*(\mathcal{F})_y$, but such module freeness was crucial for verifying the injectivity of (1.1.1.11).

$\square$

A standard consequence of Theorem 1.1.1.11 is given in the next result, which we use repeatedly without comment.

COROLLARY 1.1.1.12. *Let $f : X \to Y$ be a proper flat morphism of analytic spaces with connected reduced fibers. Then the natural map $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism.*

The most basic example of an f as in this corollary is a proper smooth map (with connected fibers), or even more classically a proper submersion between complex manifolds (again, with connected fibers). One further important fact about flat maps is the fiber-by-fiber criterion for a map to be flat or an isomorphism. This will allow us to reduce relative questions to a classical situation:

LEMMA 1.1.1.13. *Let X, Y be analytic spaces over S, and $f : X \to Y$ an S-map.*

- *Assume X, Y are S-flat and $f_s : X_s \to Y_s$ is flat for all $s \in S$. Then f is flat and, moreover, f_s is an isomorphism for all $s \in S$ if and only if f is an isomorphism.*
- *If f induces a flat map (resp. an isomorphism) on* all *infinitesimal fibers over S, then f is flat (resp. an isomorphism). Also, two S-maps $f, g : X \to Y$ which coincide on all infinitesimal fibers are equal.*

PROOF. First consider the case where X and Y are S-flat. Choose $x \in X$ and let $y = f(x)$ and let s be their common image point in S. Let $\mathfrak{m}_s$ denote the maximal ideal of $\mathcal{O}_s$, so by hypothesis $\mathcal{O}_y/\mathfrak{m}_s\mathcal{O}_y \to \mathcal{O}_x/\mathfrak{m}_s\mathcal{O}_x$ is flat. Since $\mathfrak{m}_s \otimes_{\mathcal{O}_s} \mathcal{O}_y \to \mathfrak{m}_s\mathcal{O}_y$ and $\mathfrak{m}_s \otimes_{\mathcal{O}_s} \mathcal{O}_x \to \mathfrak{m}_s\mathcal{O}_x$ are isomorphisms by the S-flatness assumption, the natural map

$$\mathfrak{m}_s\mathcal{O}_y \otimes_{\mathcal{O}_y} \mathcal{O}_x \to \mathfrak{m}_s\mathcal{O}_x \tag{1.1.1.12}$$

is an isomorphism. Since $\mathcal{O}_y \to \mathcal{O}_x$ is a local map of local *noetherian* rings, this map must be flat due to the local flatness criterion [**42**, 22.3] and (1.1.1.12) being an isomorphism.

When f_s is an isomorphism, f is at least a flat map, by what we have just shown. Also, f is a quasi-finite map between analytic spaces, so locally on the source and target, it is a finite map. But a finite flat map has a "degree" which is locally constant on the target, so upon passing to fibers over S we compute this "degree" to be 1. It follows that f is a local isomorphism. But a bijective local isomorphism is an isomorphism.

Now drop the S-flatness assumption on X and Y, but assume f induces a flat map (resp. an isomorphism) on all infinitesimal fibers. By the argument of the preceding paragraph, it suffices to treat the flatness issue. By the local flatness criterion [**42**, 22.3], it suffices to prove that for all $n \geq 1$, the natural local map of local noetherian rings $\mathcal{O}_{Y,f(x)}/\mathfrak{m}_{f(x)}^n \to \mathcal{O}_{X,x}/\mathfrak{m}_{f(x)}^n$ is flat. But this is a base change of $\mathcal{O}_{Y,f(x)}/\mathfrak{m}_s^n \to \mathcal{O}_{X,x}/\mathfrak{m}_s^n$ (where x and $f(x)$ lie over $s \in S$), and this latter map is the map induced by f on nth order infinitesimal fibers over s. By hypothesis, this map is flat.

The assertion about maps coinciding on infinitesimal fibers being globally equal is an immediate consequence of the Krull intersection theorem, as we noted before the statement of the Theorem on Formal Functions.

□

Here is another fundamental and very useful theorem concerning passage from fibers to relative situations.

THEOREM 1.1.1.14. *Let $f : X \to Y$ be an S-map between proper analytic spaces over S. If X is S-flat, then the locus U of $s \in S$ for which f_s is an isomorphism is open and $f|_U : X|_U \to Y|_U$ is an isomorphism.*

PROOF. Choose $s \in U$. For all $x \in X_s$ we have $f^{-1}(f(x)) = \{x\}$, so f is quasi-finite at x. By the structure theorem for quasi-finite maps in the complex analytic category, $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is *finite*. As this map becomes an isomorphism modulo $\mathfrak{m}_s$, by Nakayama's Lemma it is a surjection. Moreover, if K is the kernel of this local ring map then

$$0 \to K \to \mathcal{O}_{f(x)} \to \mathcal{O}_x \to 0$$

is an exact sequence of $\mathcal{O}_s$-modules with $\mathcal{O}_x$ flat over $\mathcal{O}_s$, so this sequence remains exact upon passage to quotients by $\mathfrak{m}_s$. Thus, $K/\mathfrak{m}_s K = 0$ and hence $K = \mathfrak{m}_s K$, so $K = \mathfrak{m}_{f(x)} K$. But K is an ideal in the local noetherian ring $\mathcal{O}_{f(x)}$ and thus is a finite $\mathcal{O}_{f(x)}$-module. By Nakayama's Lemma again, we conclude from $K = \mathfrak{m}_{f(x)} K$ that $K = 0$. That is, f is a local isomorphism near x, so there exists an open V in X around x such that $f : V \to Y$ is étale (i.e., a local isomorphism).

Replacing S by the open set $S - f(X - V)$ around x, we may assume that f is étale. Now an étale map $X_{s'} \to Y_{s'}$ between compact Hausdorff analytic spaces must be quasi-finite (otherwise look at an accumulation point of an infinite fiber to contradict the étaleness). Thus, f is quasi-finite and proper, so it must be finite. But now f is a finite étale map, whence it has a "degree" which is a locally constant (non-negative integer-valued) function on Y. The open locus in Y where this degree is 1 contains Y_s, so upon shrinking S around s we may assume f is finite étale with constant degree 1. A finite étale map of degree 1 is an isomorphism.

□

One application of passage to infinitesimal fibers is the following fact, relativizing a well-known classical property of pointed genus 1 curves.

THEOREM 1.1.1.15. *Let $E \to S$ be an elliptic curve, defined as above. There is a unique structure of S-group object on an elliptic curve $E \to S$ with e as the identity section. This group structure is commutative, and any S-map $E_1 \to E_2$ respecting the identity sections is a homomorphism.*

PROOF. The geometric/cohomological method in [**35**, 2.1.2] for *constructing* a group structure on an elliptic curve over a scheme carries over essentially verbatim to the analytic case, thanks to the "algebraicity" of the infinitesimal fibers (1-dimensional compact Hausdorff analytic spaces are automatically algebraic) and the fact that the basic cohomological input remains true in the analytic case, as summarized in Theorem 1.1.1.7 and Theorem 1.1.1.11.

For the rest of the theorem, we are concerned with equalities of S-maps, so it suffices to consider infinitesimal fibers (i.e., S is a single point). Since 1-dimensional compact Hausdorff analytic spaces are algebraic, one could use the results in the algebraic case to complete the proof. However, we prefer to instead give a purely analytic proof which has the advantage of generalizing later to show that group structures on relative complex tori are unique.

When $S = \mathrm{Sp}\,\mathbf{C}$ is a reduced point, the commutativity and uniqueness of the group structure, as well as its compatibility with identity-preserving morphisms, are classical: the algebraic proof in [**45**, p. 43, Rigidity Lemma] applies equally well to analytic spaces, as these are locally Stein spaces and the only compact connected analytic sets in a Stein space are points [**19**, Ch IV, §3, Thm 5] (this replaces the algebraic fact that for an algebraically closed field k, the only proper connected affine variety over k is $\mathrm{Spec}(k)$).

In the general case with artinian but possibly non-reduced S, it suffices show quite generally that if G is a group-analytic space over a one point base S, and if $p : X \to S$ is proper with the natural map $\mathcal{O}_S \to p_*\mathcal{O}_X$ an isomorphism, then any S-map $f : X \to G$ with topological image supported at the identity of G factors as

$$X \xrightarrow{p} S \xrightarrow{\eta} G$$

where η is a section. This assertion is clear topologically, so it suffices to give the sheaf map $\eta^\# : \mathcal{O}_G \to \eta_*\mathcal{O}_S$. But η agrees with the identity section ε topologically, so

$$\eta_*\mathcal{O}_S = \varepsilon_*\mathcal{O}_S = \varepsilon_* p_*\mathcal{O}_X = f_*\mathcal{O}_X, \tag{1.1.1.13}$$

the last equality holding since f and $\varepsilon \circ p$ agree topologically. Taking $\eta^\#$ to agree with $f^\#$ via (1.1.1.13) now gives the desired section; this argument is the same as in the proof of [**46**, Prop 6.1]. We refer to [**46**, Cor 6.3–Cor 6.6] for similar arguments based on this infinitesimal result.

□

In relative dimension > 1, we must incorporate the group structure into the definition (whereas with elliptic curves we could "cheat"). Doing this, the same methods and conclusions as obtained above will apply to the "relative complex tori" which we discuss later. Here are the most important examples to keep in mind.

EXAMPLE 1.1.1.16.

1. Define an elliptic curve $f^{\mathrm{alg}} : E^{\mathrm{alg}} \to \mathfrak{h}$ by
$$E^{\mathrm{alg}} = \{(z, [x, y, w]) \in \mathfrak{h} \times \mathbf{CP}^2 \,|\, y^2 w = 4x^3 - g_2(z)xw^2 - g_3(z)w^3\}$$
$$f^{\mathrm{alg}}(z, [x, y, w]) = z$$
with identity section given by $e(z) = (z, [0, 1, 0])$; here g_2 and g_3 are the functions in (0.0.0.2) and (0.0.0.3) and we are giving everything the canonical analytic structure. We will often omit the projective coordinate w from the notation. For example, the complex fiber in $\mathbf{CP}^2$ of E^{alg} over $z_0 \in \mathfrak{h}$ is described as
$$\{(x, y) \,|\, y^2 = 4x^3 - g_2(z_0)x - g_3(z_0)\},$$
even though the equation should be expressed in projective homogeneous coordinates.
2. Define a "relative lattice" over $\mathfrak{h}$
$$\Lambda = \mathbf{Z}^2 \times \mathfrak{h} \hookrightarrow \mathbf{C} \times \mathfrak{h}$$
$$(m, n; z) \mapsto (mz + n; z).$$
We want to view $(\mathbf{C} \times \mathfrak{h})/\Lambda$ as an elliptic curve $\mathcal{E}^{\mathrm{an}} \to \mathfrak{h}$. To make sense of this quotient, we use the C^∞-diagram
$$\mathbf{C} \times \mathfrak{h} \simeq \mathbf{C} \times \mathfrak{h} \tag{1.1.1.14}$$
$$(x + iy; z) \mapsto (x + zy; z)$$

taking the "standard lattice"

$$\begin{aligned} \mathbf{Z}^2 \times \mathfrak{h} &\hookrightarrow \mathbf{C} \times \mathfrak{h} \\ (m,n;z) &\mapsto (mi+n;z) \end{aligned} \tag{1.1.1.15}$$

to Λ; of course, the C^∞-map (1.1.1.14) is not holomorphic. Nevertheless, it does show that the quotient topological space $(\mathbf{C}\times\mathfrak{h})/\Lambda$ by the equivalence relation $(\tau, z) \sim (\tau+\lambda, z)$ for $\lambda \in \Lambda_z$ has a unique C^∞-structure such that the projection map

$$\pi : \mathbf{C}\times\mathfrak{h} \to (\mathbf{C}\times\mathfrak{h})/\Lambda$$

is a local C^∞-isomorphism. Indeed, the projection $\mathbf{C}\times\mathfrak{h} \to (\mathbf{C}/[1,i])\times\mathfrak{h}$ is a proper covering space map since $\mathbf{C} \to \mathbf{C}/[1,i]$ is, so π is even a proper covering space map. If we give

$$\mathcal{E}^{\mathrm{an}} \overset{\mathrm{def}}{=} (\mathbf{C}\times\mathfrak{h})/\Lambda$$

the unique complex structure making π a holomorphic local isomorphism (see Lemma 1.1.2.1 below), then the projection $f^{\mathrm{an}} : \mathcal{E}^{\mathrm{an}} \to \mathfrak{h}$ is a holomorphic proper submersion of relative dimension 1. We give $\mathcal{E}^{\mathrm{an}} \to \mathfrak{h}$ the obvious holomorphic section $z \mapsto (0;z)$. To check that $\mathcal{E}^{\mathrm{an}} \to \mathfrak{h}$ is an elliptic curve, it suffices to check that the fibers are classical elliptic curves. For any $z_0 \in \mathfrak{h}$, the natural holomorphic map $\mathbf{C}/[1,z_0] \to \mathcal{E}^{\mathrm{an}}_{z_0}$ is a bijection of compact *Riemann surfaces* and thus is an isomorphism. We conclude that $f^{\mathrm{an}} : \mathcal{E}^{\mathrm{an}} \to \mathfrak{h}$ is an elliptic curve with the expected complex fibers. This relative construction will be generalized to higher dimensions in §1.2.

The classical Weierstrass theory with $\wp_z(\tau)$ and $\wp'_z(\tau)$, which are holomorphic on the open set

$$\{(\tau,z) \in \mathbf{C}\times\mathfrak{h} \,|\, \tau \notin [1,z]\} \subseteq \mathbf{C}\times\mathfrak{h},$$

gives a holomorphic map

$$\begin{aligned} \mathbf{C}\times\mathfrak{h} &\to E^{\mathrm{alg}} \\ (\tau;z) &\mapsto (z, [\wp_z(\tau), \wp'_z(\tau), 1]) \end{aligned}$$

which is invariant with respect to the Λ-action on $\mathbf{C}\times\mathfrak{h}$. We thus get a holomorphic map $\mathcal{E}^{\mathrm{an}} \to E^{\mathrm{alg}}$ over $\mathfrak{h}$ respecting the identity sections and inducing the classical map

$$(\wp_z, \wp'_z) : \mathbf{C}/[1,z_0] \to \{(x,y) \,|\, y^2 = 4x^3 - g_2(z_0)x - g_3(z_0)\}$$

on fibers. By the classical theory, we know that this fiber map is an isomorphism, so we conclude by $\mathfrak{h}$-flatness (or by using the local structure of submersions and the inverse function theorem) that the map $\mathcal{E}^{\mathrm{an}} \to E^{\mathrm{alg}}$ over $\mathfrak{h}$ is an isomorphism (of elliptic curves).

The elliptic curve $\mathcal{E}^{\mathrm{an}} \to \mathfrak{h}$ is nearly universal for being a "line bundle modulo $\underline{\mathbf{Z}}^2$" [**10**, Prop 2.2(*ii*)]. To be precise, in §1.1.2 we will describe a generalization of the above construction of $\mathcal{E}^{\mathrm{an}}$ (to be used often below) and then in §1.2.1–§1.2.3 we make some cohomological digressions before we return to the matter of formulating the universal property of $\mathcal{E}^{\mathrm{an}} \to \mathfrak{h}$ in the category of complex analytic spaces.

1.1.2. Vector bundles and lattices. Let S be an analytic space and V a rank d vector bundle on S, with $d \geq 1$. We will often identify V with its sheaf of sections. Let

$$\iota : \Lambda \hookrightarrow V$$

be an inclusion of abelian sheaves, with Λ a local system of rank $2d$ free $\mathbf{Z}$-modules (i.e., Λ is a locally constant sheaf on S, locally isomorphic to $\mathbf{Z}^{2d}$) and assume Λ induces a cocompact lattice $\Lambda_s \hookrightarrow V_s$ for all $s \in S$. Since S has a base of connected opens and connected analytic spaces are path connected, for connected S we have $\mathrm{H}^0(S, \underline{\Sigma}) = \Sigma$ for any set Σ. In particular, we can identify locally constant sheaves of sets on S with topological covering spaces of S and hence with analytic covering spaces of S. This ensures that there is a unique commutative S-group object, also denoted Λ, which is an analytic covering space of S with sheaf of holomorphic sections isomorphic to the abelian sheaf Λ.

From topological and separatedness considerations, we see that there is a unique way to define a holomorphic map $\Lambda \to V$ over S which recovers the above injective sheaf map ι on sheaves of sections, and in fact $\Lambda \to V$ is a *closed immersion* of S-group objects inducing the original $\Lambda_s \hookrightarrow V_s$ on fibers. Whenever we are given a local system Λ inside a vector bundle V as above, it is always understood that we give Λ and $\Lambda \hookrightarrow V$ these canonical analytic structures over S. Note that these constructions are compatible with arbitrary base change over S.

It is rather important that given such a $\Lambda \hookrightarrow V$, we can endow the topological quotient V/Λ with a good analytic structure (analogously to Example 1.1.1.16). Before we formulate this precisely, we introduction a convenient definition. For any smooth group object $G \to S$ with identity section e, we define the *relative tangent space* (along the identity section e) to be $T_e(G) = e^*(\Omega^1_{G/S})^\vee$, a locally free coherent sheaf (or vector bundle) on S.

LEMMA 1.1.2.1. *Let S and $\Lambda \hookrightarrow V$ be as above. Then*

1. *The projection $\pi : V \to V/\Lambda$ is a covering map of topological spaces, with $V/\Lambda \to S$ proper. Moreover, the formation of the topological space V/Λ is compatible with topological base change on S.*
2. *There is a unique complex structure on V/Λ making π a local analytic isomorphism, and $V/\Lambda \to S$ is then proper smooth, admitting a unique S-group structure making π a map of group objects. The natural map $\Lambda \to \pi^{-1}(0)$ is an analytic isomorphism, as is $V \simeq T_0(V) \to T_0(V/\Lambda)$.*
3. *Formation of the complex structures in (2) is compatible with analytic base change on S.*

PROOF. Locally on S, the map $\Lambda \hookrightarrow V$ has the topological form

$$\mathbf{Z}^{2d} \times S \hookrightarrow \mathbf{R}^{2d} \times S \tag{1.1.2.1}$$

given by continuous functions $f_{ij} : S \to \mathbf{R}$, $1 \leq i, j \leq 2d$. Since $\Lambda_s \hookrightarrow V_s$ is a cocompact lattice for all $s \in S$, $\det(f_{ij})$ must be nonvanishing on S. By Cramer's formula we can then topologically transform (1.1.2.1) into the canonical inclusion $\mathbf{Z}^{2d} \times S \hookrightarrow \mathbf{R}^{2d} \times S$, from which it is clear that (1) holds. The uniqueness in (2) implies (3), so it remains to check (2). The uniqueness of the analytic structure on V/Λ is immediate. All we have to show is that V/Λ admits an analytic S-group structure making π an analytic group map and a local analytic isomorphism (so then the covering map $\pi^{-1}(0) \to S$ is a local isomorphism and the rest of (2) follows). By uniqueness, this is a local problem over S.

To give V/Λ an analytic structure, we need to check that if S is sufficiently small, then for any $x \in V/\Lambda$, any two points $v_1, v_2 \in \pi^{-1}(x)$ have neighborhoods in V inducing the "same" analytic structure on V/Λ near x. Working locally on S, we may choose a closed immersion $S \hookrightarrow S'$ with S' a ball and we may assume that there exist Λ', V' over S' lifting Λ, V on S, as well as an S'-*map* $\Lambda' \to V'$ lifting $\Lambda \hookrightarrow V$. Indeed, shrinking S we can assume $\Lambda \simeq \mathbf{Z}^{2g} \times S$, $V \simeq \mathbf{C}^g \times S$ and we can then take $\Lambda' = \mathbf{Z}^{2g} \times S'$ and $V' = \mathbf{C}^g \times S'$ compatibly with these isomorphisms.

The map $\Lambda' \to V'$ is given by holomorphic functions f'_{ij}, $1 \leq i \leq 2g$, $1 \leq j \leq g$ with $\det(\mathrm{Re}(f'_{ij}), \mathrm{Im}(f'_{ij}))$ nonvanishing along S, and hence also nonvanishing in a neighborhood of S in S'. Thus, shrinking S' allows us to assume that $\Lambda' \to V'$ induces a cocompact lattice on fibers, so *topologically* $\Lambda' \to V'$ is a closed embedding. The analytic map $\Lambda' \to V'$ is proper and quasi-finite, hence finite, so we can check if it is an analytic closed immersion by considering fibers over S' (by Nakayama's lemma); this condition is clearly satisfied if we shrink S' around S. Thus, we have lifted our entire local situation to S'. Since we are just trying to construct *some* analytic group structure on V/Λ making π a local isomorphism of group objects, this shows that to prove the remainder of the lemma we can pass to S' and thus assume that the base S is smooth, so V is a manifold.

Since V and S are reduced (even manifolds), if V/Λ has an analytic structure making π a local isomorphism (ignoring group data), then the obvious topological S-group structure on V/Λ is compatible via π with the topological S-group structure on V, so this makes V/Λ an S-group and π a group map. Thus, we can now ignore group data and just look for an analytic structure on V/Λ over S making π a local analytic isomorphism.

Choose a small open set $U \subseteq V/\Lambda$ and let U_1, U_2 be two "copies" of U in $\pi^{-1}(U)$. Give U_1, U_2 their analytic structures as opens in V. We need to check that the composite homeomorphism

$$U_1 \simeq U \simeq U_2$$

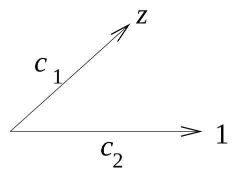
is an analytic isomorphism; we can then use these U_j's to define the desired analytic structure on U. Choose $s \in S$ and $\lambda_s \in \Lambda_s \subseteq V_s$ taking $(U_1)_s$ to $(U_2)_s$ under translation in V_s. From the local topological description of π, by shrinking S there is some $\lambda \in \Lambda(S) \subseteq V(S)$ with fiber λ_s at s; translation by λ takes U_1 to U_2 topologically and hence analytically.

□

Return now to the elliptic curve $f^{\mathrm{an}} : \mathcal{E}^{\mathrm{an}} \to \mathfrak{h}$. What are the special relative properties analogous to the the global 1-form ω and the ordered homology basis γ_1, γ_2 on an elliptic curve $\mathbf{C}/[1, z] = \mathcal{E}^{\mathrm{an}}_z$, as in the beginning of §1.1.1?

1. For the fibers $\mathcal{E}^{\mathrm{an}}_z \simeq \mathbf{C}/[1, z]$, the 1-dimensional spaces $\mathrm{H}^0(\mathcal{E}^{\mathrm{an}}_z, \Omega^1_{\mathcal{E}^{\mathrm{an}}_z})$ have a natural basis:
$$\frac{\mathrm{d}t}{t} = 2\pi i \mathrm{d}\tau = 2\pi i \frac{\mathrm{d}x}{y},$$
the different forms being given by the different incarnations of $\mathcal{E}^{\mathrm{an}}_z$:
$$\mathbf{C}^\times/q^{\mathbf{Z}} \simeq \mathbf{C}/[1, z] \simeq \{(x, y) \,|\, y^2 = 4x^3 - g_2(z)x - g_3(z)\}$$
where $t = e^{2\pi i \tau}$, $q = e^{2\pi i z}$. These all obviously vary "nicely" in $z \in \mathfrak{h}$, and thus should give a trivialization of $f^{\mathrm{an}}_*(\Omega^1_{\mathcal{E}^{\mathrm{an}}/\mathfrak{h}})$.

2. $\mathrm{H}^1(\mathcal{E}_z^{\mathrm{an}}, \underline{\mathbf{Z}})^\vee$ is isomorphic to $\mathrm{H}_1(\mathcal{E}_z^{\mathrm{an}}, \mathbf{Z})$, which has ordered basis c_1, c_2:



(the reason for the labelling will become clear later). These vary nicely in z, and thus should yield a global trivialization of $(\mathrm{R}^1 f_*^{\mathrm{an}} \underline{\mathbf{Z}})^\vee$.

To relate higher direct image sheaves for elliptic curves $f : E \to S$ with cohomology on fibers, we will need to investigate the structure of the sheaves $\mathrm{R}^1 f_* \underline{\mathbf{Z}}$ and $f_* \Omega^1_{E/S}$, especially how these sheaves behave with respect to base change (to fibers). That is, we must study higher direct images of coherent sheaves and locally constant sheaves. This will lead us to the universal property of $\mathcal{E}^{\mathrm{an}} \to \mathfrak{h}$.

1.2. Cohomological considerations

1.2.1. Higher direct images of constant sheaves. Because of the explicit global C^∞-splitting in (1.1.1.14), it is clear that the natural map

$$\mathrm{R}^1 f_*^{\mathrm{an}}(\underline{\mathbf{Z}})_z \to \mathrm{H}^1(\mathcal{E}_z^{\mathrm{an}}, \underline{\mathbf{Z}})$$

an isomorphism for all $z \in \mathfrak{h}$. More generally, how do the stalks of the higher direct images of a sheaf $\mathcal{F}$ via a map $f : X \to Y$ relate to the cohomology of the fibers? This question is simplest when Y is locally compact and locally Hausdorff, as is the case for f^{an}. However, we ideally want to imagine $\mathcal{E}^{\mathrm{an}} \to \mathfrak{h}$ extending to $\overline{\mathcal{E}}^{\mathrm{an}} \to \mathfrak{h} \cup \{\infty\}$ with a "Néron polygon with infinitely many sides" over ∞, where $\mathfrak{h} \cup \{\infty\}$ has as a base of opens around ∞ the sets

$$\{z \in \mathfrak{h} \mid \mathrm{Im}(z) > N\} \cup \{\infty\}. \tag{1.2.1.1}$$

This space $\mathfrak{h} \cup \{\infty\}$ is not locally compact near ∞, but as a "universal cover" of the punctured disk, totally ramified over the origin, it plays a critical role in the étale monodromy theory of Lefschetz pencils. By the Urysohn metrization theorem [**47**, Ch 4, §4, Thm 4.1], $\mathfrak{h} \cup \{\infty\}$ is at least metrizable. For this reason, we include the metrizable case in the following fundamental result, even though our immediate needs are just in the locally compact, locally Hausdorff case.

THEOREM 1.2.1.1 (Topological proper base change). *Consider a proper map of topological spaces $f : X \to Y$ with either:*

- *Y locally compact, locally Hausdorff;*
- *all fibers of f admitting metrizable neighborhoods.*

Then for all abelian sheaves $\mathcal{F}$ on X and all $y \in Y$, the δ-functorial map

$$\mathrm{R}^i f_*(\mathcal{F})_y \to \mathrm{H}^i(f^{-1}(y), \mathcal{F}_y)$$

is an isomorphism, where $\mathcal{F}_y = \mathcal{F}|_{f^{-1}(y)}$. The formation of $\mathrm{R}^i f_* \mathcal{F}$ is compatible with locally compact, locally Hausdorff base change on Y if Y is locally compact, locally Hausdorff.

The condition that f have metrizable neighborhoods of its compact fibers is equivalent, by the Smirnov metrization theorem [**47**, Ch 6, §5, Thm 5.1], to the condition that X be locally metrizable. Before giving the proof of Theorem 1.2.1.1, we note some consequences.

COROLLARY 1.2.1.2. *In the above situation, if G is an abelian group then the natural maps*

$$\mathrm{R}^i f_*(\underline{G})_y \to \mathrm{H}^i(f^{-1}(y), \underline{G})$$

are isomorphisms.

REMARK 1.2.1.3. Recall that if Z a paracompact Hausdorff space (for example, either metrizable or locally compact, Hausdorff, and separable) which has a base of contractible open sets, then $\mathrm{H}^i(Z, \underline{G})$ is canonically isomorphic to the singular cohomology group $\mathrm{H}^i_{\mathrm{top}}(Z, G)$ [**60**, Ch 5]. Thus, if f is a topologically trivial fibration and Y has a base of contractible opens, then Corollary 1.2.1.2 is obvious!

PROOF OF THEOREM 1.2.1.1. Without loss of generality we can assume that Y is a compact Hausdorff space (and thus that X is as well) or that X is metrizable. The $i = 0$ case of the theorem is a consequence of the following lemma.

LEMMA 1.2.1.4. *Let X be a compact Hausdorff (resp. metrizable) space and let $i : Z \hookrightarrow X$ be the inclusion of a compact (resp. arbitrary) subset. Then the natural map*

$$\varinjlim_{V \supseteq Z} \mathrm{H}^0(V, \mathcal{F}) \to \mathrm{H}^0(Z, i^{-1}\mathcal{F})$$

is an isomorphism for all abelian sheaves $\mathcal{F}$ on X, where $i^{-1}\mathcal{F}$ denotes topological pullback.

PROOF. See [**33**, Ch 3, Thm 2.2] (resp. [**18**, Ch II, §3.3, Cor 1]). □

It remains to check that the δ-functor $\mathrm{H}^*(f^{-1}(y), (\cdot)_y)$ is erasable. We show that $\mathrm{H}^i(f^{-1}(y), \mathcal{F}_y) = 0$ for all $i > 0$ and all flasque sheaves $\mathcal{F}$ on X. In the metrizable case, Lemma 1.2.1.4 implies that $i^{-1}(\mathcal{F})$ is flasque if $\mathcal{F}$ is flasque, so this case is clear. In the other case, we know that $f^{-1}(y)$ is compact Hausdorff, so ordinary cohomology agrees with compactly supported cohomology and thus [**33**, Ch 3, Cor 2.5 and Thm 2.7] now completes the proof.

□

We can use the preceding result to show that in many situations, the higher direct images of constant sheaves are themselves locally constant. Before we do this, we prove a lemma which will often allow us to reduce questions about maps of analytic spaces to the case of a smooth base. This is inspired by the algebraic analogue [**45**, Lemma, p.56]. Recall that every analytic space is locally path connected, so the connected components of an analytic space are open and locally path connected.

LEMMA 1.2.1.5. *Let S be a connected complex analytic space and let s_0 be a point of S. Then for every $s \in S$, there are finitely many points $s_0, s_1, \ldots, s_n = s \in S$ and analytic maps $C_i \to S$ for $0 \leq i < n$ with C_i a connected smooth complex analytic curve whose image in S contains s_i and s_{i+1}.*

This lemma was inspired by the rigid analytic analogue [**9**, 6.1.1].

PROOF. Since S is connected, hence path connected, it suffices to prove the result in some open (connected) neighborhood of s_0. We first wish to reduce to the case where S is globally irreducible, locally irreducible at s_0, and admits a finite

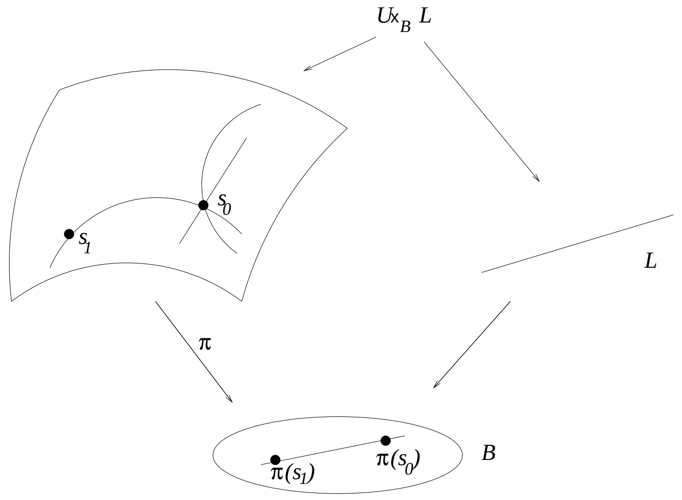


FIGURE 1. Construction of a smooth curve connecting two points

surjective map onto an open ball B in some $\mathbf{C}^d$. We will then use preimages of lines in $\mathbf{C}^d$ to construct the desired "connected analytic paths" in S (though this will require a little care).

By [**20**, Ch 9, §3.1-3.2], there are arbitrarily small neighborhoods of s_0 which are path connected and equal to a finite union of irreducible analytic spaces, each irreducible at s_0. By choosing points on intersections of these irreducible components and choosing a continuous path from s_0 to any other point of S, it suffices to consider globally irreducible S. Since any path between two points is compact, by [**20**, Ch 4,§3.1] we may assume S is irreducible at s_0. Finally, by [**20**, Ch 5, §4.2-4.3; Ch 9, §1.3] can find an irreducible neighborhood U' of s_0 admitting a finite surjective map $\pi : U' \to B'$ with B' an open set in some $\mathbf{C}^m$. In particular, all local rings on U' have dimension m. (This process is similar to Noether normalization). We may assume $m \geq 1$. The local description of finite maps between analytic spaces [**20**, Ch 2, §3.2] allows us to find a connected neighborhood U of s_0 and an open *ball* $B \subseteq B'$ around $\pi(s_0)$ such that π induces a finite surjection $U \to B$ with s_0 the *only* point in its fiber for π. Although U might not be irreducible, it is equidimensional of dimension $m \geq 1$.

Now choose $s_1 \in U$ and a line L in B through $\pi(s_0)$ and $\pi(s_1)$. Consider the preimage $\pi^{-1}(L) = U \times_B L$ of L in U. This preimage descomposes into a locally finite union of global irreducible components Z_i, each of dimension 1 or consisting of a single point. Each 1-dimensional Z_i has the property that $Z_i \to L$ is finite with image a connected analytic set, so each such $Z_i \to L$ is *surjective*. But $\pi^{-1}(\pi(s_0)) = \{s_0\}$, so each 1-dimensional Z_i contains s_0. (See Figure 1.)

We claim that there is a 1-dimensional irreducible component C of $\pi^{-1}(L)$ containing s_1. As we just noted, such C must contain s_0, so the normalization of C then gives the desired smooth connected curve joining s_0 and s_1. If no such C exists, then $\{s_1\}$ is an isolated point in $\pi^{-1}(L)$, so $\mathcal{O}_{U,s_1}/\pi^*(\mathcal{I}_L)_{s_1}$ is 0-dimensional. But $\mathcal{I}_L$ is generated by $m-1$ elements near $\pi(s_1)$ and $\dim \mathcal{O}_{U,s_1} = m$, so this contradicts Krull's Hauptidealsatz. Thus, some such C must exist, as desired.

$\square$

Theorem 1.2.1.1 plays a crucial role in what follows. For example, it is used in the proof of:

THEOREM 1.2.1.6. *Let $f : X \to S$ be a proper smooth map with connected fibers of dimension d. For any torsion-free abelian group G (e.g, $\mathbf{Z}$, $\mathbf{C}$, $\mathbf{Z}(d)$), $\mathrm{R}^i f_* \underline{G}$ is locally constant and compatible with base change for all $i \leq 2d$, and vanishes for $i > 2d$. Furthermore, the natural maps*

$$\underline{G} \longrightarrow f_* \underline{G},$$
$$\underline{G} \otimes \mathrm{R}^{2d} f_* \underline{\mathbf{Z}} \longrightarrow \mathrm{R}^{2d} f_* \underline{G}$$

are isomorphisms and $\mathrm{R}^{2d} f_* \underline{\mathbf{Z}}$ is a local system of rank 1 free $\mathbf{Z}$-modules.

If S is smooth, then the map f is topologically trivial locally on S and Theorem 1.2.1.6 is easy. In fact, this is the essential case. Also, one can remove the torsion-free assumption on G if one grants the rather non-trivial fact that S has a base of contractible open sets. We also note that a special case of Theorem 1.2.1.6 is that if $f : E \to S$ is an elliptic curve over an analytic space, then $\underline{\mathbf{Z}} \simeq f_* \underline{\mathbf{Z}}$ and $\mathrm{R}^1 f_* \underline{\mathbf{Z}}$ is a local system of rank 2 free $\mathbf{Z}$-modules, of formation compatible with base change. Also, $\mathrm{R}^2 f_* \underline{\mathbf{Z}}$ and $\mathrm{R}^2 f_* \underline{\mathbf{Z}}(1)$ are invertible $\underline{\mathbf{Z}}$-modules of formation compatible with base change. We will see later that $\mathrm{R}^2 f_* \underline{\mathbf{Z}}(1)$ is *canonically* isomorphic to $\underline{\mathbf{Z}}$ (whereas defining a global trivialization of $\mathrm{R}^2 f_* \underline{\mathbf{Z}}$ requires a non-canonical choice of isomorphism $\mathbf{Z} \simeq \mathbf{Z}(1)$, which is to say a choice of $\sqrt{-1} \in \mathbf{C}$).

PROOF. Since the analytic fiber product gives the topological fiber product on underlying topological spaces and S is locally compact, locally Hausdorff, we can use Theorem 1.2.1.1. Thus

$$\mathrm{R}^i f_* (\underline{G})_s \simeq \mathrm{H}^i(X_s, \underline{G}) \simeq \mathrm{H}^i_{\mathrm{top}}(X_s, G) \tag{1.2.1.2}$$

for all $s \in S$. Furthermore, the natural map $\underline{G} \to f_* \underline{G}$ is just the map $G \to \mathrm{H}^0(X_s, \underline{G})$ on stalks. Each connected manifold fiber X_s is path connected, so the map $G \to \mathrm{H}^0(X_s, \underline{G})$ is an isomorphism for all $s \in S$. Thus, by (1.2.1.2), $\underline{G} \to f_* \underline{G}$ is an isomorphism as well.

For the rest of the theorem we first want to reduce to the case where G is finitely generated. Let $\{G_\alpha\}$ be the set of finitely generated subgroups of G, so we have an isomorphism $\varinjlim \underline{G}_\alpha \simeq \underline{G}$. The induced map

$$\varinjlim \mathrm{R}^i f_* \underline{G}_\alpha \to \mathrm{R}^i f_* \underline{G} \tag{1.2.1.3}$$

becomes

$$\varinjlim \mathrm{H}^i(X_s, \underline{G}_\alpha) \to \mathrm{H}^i(X_s, \underline{G})$$

on stalks, and this is an isomorphism since sheaf cohomology commutes with direct limits on compact Hausdorff spaces [**18**, Ch II, 4.12.1]. We conclude that (1.2.1.3) is an isomorphism, and thus we can assume that G is finitely generated. Therefore we may finally assume that G is just $\mathbf{Z}$ (this is where we use the torsion-freeness assumption; without this condition, we would run into unpleasant problems of torsion-orders varying over the space).

In this case we have isomorphisms

$$\mathrm{H}^i(X_s, \underline{\mathbf{Z}}) \simeq \mathrm{H}^i_{\mathrm{top}}(X_s, \mathbf{Z}), \tag{1.2.1.4}$$

and we know that the second of these groups is finitely generated and vanishes for $i > 2d$ by Poincaré duality. We are thus reduced to showing that $\mathrm{R}^i f_* \underline{\mathbf{Z}}$ is locally constant for all $i \leq 2d$. Indeed, (1.2.1.2) and (1.2.1.4) show that $\mathrm{R}^i f_* \underline{\mathbf{Z}}$ vanishes for $i > 2d$; in the case $i = 2d$, by (1.2.1.4) and Poincaré duality we also see that all stalks are free of rank 1 over $\mathbf{Z}$, so if $\mathrm{R}^{2d} f_* \underline{\mathbf{Z}}$ is locally constant then it must be locally free of rank 1 as a $\underline{\mathbf{Z}}$-module.

For the remainder of the argument we reduce to the case of a smooth base S as follows. Choose a point $s_0 \in S$ and shrink S around s_0 so that S is connected. We have an isomorphism $\mathrm{R}^i f_*(\underline{\mathbf{Z}})_{s_0} \simeq \mathrm{H}^i(X_{s_0}, \underline{\mathbf{Z}})$, and these are both finitely presented $\mathbf{Z}$-modules. Thus, the inverse isomorphism lifts (after possibly shrinking S around s_0 some more) to a map

$$\mathrm{H}^i(X_{s_0}, \underline{\mathbf{Z}}) \to \mathrm{H}^0(S, \mathrm{R}^i f_* \underline{\mathbf{Z}})$$

for all $i \leq 2d$.

It is now necessary and sufficient to show that the induced map

$$\mathrm{H}^i(X_{s_0}, \underline{\mathbf{Z}}) \to \mathrm{R}^i f_*(\underline{\mathbf{Z}})_s \simeq \mathrm{H}^i(X_s, \underline{\mathbf{Z}}) \tag{1.2.1.5}$$

is an isomorphism for all $s \in S$ *near* s_0, as this will then imply that the map of sheaves

$$\xi : \underline{\mathrm{H}^i(X_{s_0}, \underline{\mathbf{Z}})} \to \mathrm{R}^i f_* \underline{\mathbf{Z}}$$

is an isomorphism near S, and thus that $\mathrm{R}^i f_* \underline{\mathbf{Z}}$ is locally constant.

By Lemma 1.2.1.5 we can assume that there is a smooth connected curve $\widetilde{S}$ mapping to S whose image contains s_0 and s_1. By proper base change it suffices to check (1.2.1.5) on $\widetilde{S}$, so we may assume that S is smooth (and even a curve, although we will not need this). Recall that $\mathrm{R}^i f_* \underline{\mathbf{Z}}$ is the sheafification of the presheaf

$$U \mapsto \mathrm{H}^i(f^{-1}(U), \underline{\mathbf{Z}}).$$

Since S is smooth and f is proper smooth, locally on S we have a C^∞-splitting $X \simeq S \times X_{s_0}$. The sheaf $\mathrm{R}^i f_* \underline{\mathbf{Z}}$ is now given by

$$U \mapsto \mathrm{H}^i(f^{-1}(U), \underline{\mathbf{Z}}) \simeq \mathrm{H}^i(U \times X_{s_0}, \underline{\mathbf{Z}}) \simeq \mathrm{H}^i_{\mathrm{top}}(U \times X_{s_0}, \mathbf{Z}).$$

If U is small enough so as to be contractible as well (such opens are certainly a base of opens on the smooth S), then the natural pullback map

$$\mathrm{H}^i_{\mathrm{top}}(U \times X_{s_0}, \mathbf{Z}) \to \mathrm{H}^i_{\mathrm{top}}(X_{s_0}, \mathbf{Z})$$

is an isomorphism. All of these isomorphisms are functorial in U, so we see that $\mathrm{R}^i f_* \underline{\mathbf{Z}}$ is just the constant sheaf $\underline{\mathrm{H}^i_{\mathrm{top}}(X_{s_0}, \mathbf{Z})}$ and the map

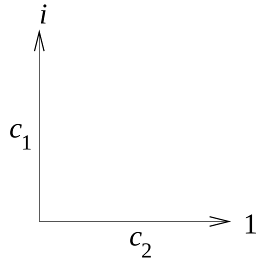
$$\underline{\mathrm{H}^i(X_{s_0}, \underline{\mathbf{Z}})} \xrightarrow{\xi} \mathrm{R}^i f_* \underline{\mathbf{Z}} \simeq \underline{\mathrm{H}^i_{\mathrm{top}}(X_{s_0}, \mathbf{Z})}$$

is just the canonical isomorphism *at* s_0. Thus, locally on the path connected S this composite map of constant sheaves is an isomorphism, so ξ is an isomorphism. It follows that (1.2.1.5) is an isomorphism near s_0, completing the proof. ☐

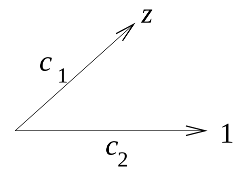
If S is contractible, all locally constant sheaves are constant (via the stalk at a fixed base point). For $f^{\mathrm{an}} : \mathcal{E}^{\mathrm{an}} \to \mathfrak{h}$ with *base point* i, we can explicitly describe the unique isomorphism

$$(\alpha^{\mathrm{an}})^\vee : \mathbf{Z}^2 \simeq \underline{\mathrm{H}_1(\mathbf{C}/[1,i], \mathbf{Z})} \simeq \mathrm{R}^1 f^{\mathrm{an}}_*(\underline{\mathbf{Z}})^\vee \tag{1.2.1.6}$$

corresponding to a specific isomorphism on stalks at $i \in \mathfrak{h}$ (the dual α^{an} of this map is what we will use later). Let c_1, c_2 be the ordered basis



of $\mathrm{H}_1(\mathcal{E}_i^{\mathrm{an}}, \mathbf{Z}) = \mathrm{H}_1(\mathbf{C}/[1,i], \mathbf{Z})$. We will see in §1.2.4 why it is convenient to take $\{c_1, c_2\}$ as the ordered basis in our homology groups. As in Example 1.1.1.16(2), the induced basis on $\mathrm{R}^1 f_*^{\mathrm{an}}(\underline{\mathbf{Z}})_z^\vee \simeq \mathrm{H}_1(\mathbf{C}/[1,z], \mathbf{Z})$ is given by



since the C^∞ commutative diagram

$$\begin{tikzcd} \mathbf{C} \times \mathfrak{h} \arrow[rr, "\simeq"] \arrow[dr] & & \mathbf{C} \times \mathfrak{h} \arrow[dl] \\ & \mathfrak{h} & \end{tikzcd}$$

sending $(x + iy; z)$ to $(x + zy; z)$ induces a C^∞-isomorphism over $\mathfrak{h}$ from $(\mathbf{C}/\mathbf{Z}^2) \times \mathfrak{h}$ to $\mathcal{E}^{\mathrm{an}}$. This describes the map $(\alpha^{\mathrm{an}})^\vee$.

There is a unique isomorphism

$$\mathrm{R}^2 f_*^{\mathrm{an}} \underline{\mathbf{Z}} \simeq \underline{\mathbf{Z}}$$

fitting into a commutative diagram

$$\begin{tikzcd} \wedge^2 \mathrm{R}^1 f_*^{\mathrm{an}} \underline{\mathbf{Z}} \arrow[r, "\wedge^2(\alpha^{\mathrm{an}})", "\simeq"'] \arrow[d, "\cup"'] & \wedge^2(\underline{\mathbf{Z}}^2) \arrow[d, "\simeq"] & (1,0) \wedge (0,1) \arrow[d, mapsto] \\ \mathrm{R}^2 f_*^{\mathrm{an}} \underline{\mathbf{Z}} \arrow[r, "\simeq"'] & \underline{\mathbf{Z}} & 1 \end{tikzcd} \tag{1.2.1.7}$$

Here the left column is an isomorphism on stalks (and thus is an isomorphism) by topological proper base change. Since the dual basis to $\{c_2, c_1\}$ is represented by $\{\mathrm{d}x - (a/b)\mathrm{d}y, \mathrm{d}y/b\}$ with $z = a + bi$, $b > 0$, we see that the *negative* of the induced map $\mathrm{H}^2(\mathcal{E}_z^{\mathrm{an}}, \underline{\mathbf{Z}}) \simeq \mathbf{Z}$ in the bottom row of (1.2.1.7) corresponds *exactly* to the i-orientation of the complex manifold $\mathcal{E}_z^{\mathrm{an}}$ via the choice of $i = \sqrt{-1} \in \mathbf{C}$ used to define $\mathfrak{h}$. This completes our general study of $\mathrm{R}^1 f_* \underline{\mathbf{Z}}$ for elliptic curves $E \to S$ over an analytic space, except for the matter of canonically trivializing $\mathrm{R}^2 f_* \underline{\mathbf{Z}}(1)$, which we will handle via considerations with coherent sheaves of relative differential forms (this is essentially a special case of Verdier's general topological duality theorem for locally compact Hausdorff spaces).

1.2.2. Higher direct images of sheaves of differentials. We turn now to the analytic base change of the coherent sheaf $f_*^{\mathrm{an}} \Omega^1_{\mathcal{E}^{\mathrm{an}}/\mathfrak{h}}$. The global relative differential form $2\pi i d\tau$ on $\mathcal{E}^{\mathrm{an}} \to \mathfrak{h}$ shows that the map

$$f_*^{\mathrm{an}}(\Omega^1_{\mathcal{E}^{\mathrm{an}}/\mathfrak{h}})_z \to \mathrm{H}^0(\mathcal{E}_z^{\mathrm{an}}, \Omega^1_{\mathcal{E}_z^{\mathrm{an}}})$$

is surjective, since $\mathrm{H}^0(\mathcal{E}_z^{\mathrm{an}}, \Omega^1_{\mathcal{E}_z^{\mathrm{an}}})$ is one-dimensional. It now follows from cohomology and base change (Theorem 1.1.1.11) that $f_*^{\mathrm{an}}(\Omega^1_{\mathcal{E}^{\mathrm{an}}/\mathfrak{h}})$ *is* invertible and commutes with base change. The *global* triviality of this invertible pushfoward sheaf (via $2\pi i d\tau$) is not an accident. Quite generally a holomorphic line bundle on a contractible Stein space (e.g., $\mathfrak{h}$) is *always* (non-canonically) globally trivial. Indeed, the Picard group of such a space X sits in an exact sequence

$$\mathrm{H}^1(X, \mathcal{O}_X) \to \mathrm{H}^1(X, \mathcal{O}_X^\times) \stackrel{e^{2\pi i(\cdot)}}{\longrightarrow} \mathrm{H}^2(X, \underline{\mathbf{Z}}) = \mathrm{H}^2_{\mathrm{top}}(X, \mathbf{Z})$$

and the first and last groups are trivial by Steinness and contractibility respectively.

The local freeness and compatibility with base change for $f_*^{\mathrm{an}}\Omega^1_{\mathcal{E}^{\mathrm{an}}/\mathfrak{h}}$ are seen more conceptually via the next lemma.

Lemma 1.2.2.1.

1. *Let $p : G \to S$ be an analytic group object with identity section e. Then there exists a canonical isomorphism $p^*e^*\Omega^1_{G/S} \simeq \Omega^1_{G/S}$ of coherent $\mathcal{O}_G$-modules, compatible with base change on S. In particular, if p is smooth of relative dimension g, then $\Omega^1_{G/S} \simeq \mathcal{O}_G^{\oplus g}$ locally over S.*
2. *If $f : X \to S$ is proper flat with connected reduced fibers, then $\mathcal{O}_S \simeq f_*\mathcal{O}_X$ and this isomorphism commutes with base change.*

Thus, if X is a proper smooth group object with pure relative dimension g and connected fibers, then $f_*\Omega^1_{X/S}$ is a rank g vector bundle commuting with any base change.

Proof. (**2**) is a restatement of Theorem 1.1.1.12. As for (**1**), this follows from [**2**, §4.2, esp. Prop 2], where the proof carries over verbatim to the analytic case since the basic algebraic properties of sheaves of relative differentials carry over to relative analytic differentials (as one checks via *faithfully flat* base change to completions of stalks). □

Let $f : E \to S$ be an elliptic curve. We define the invertible sheaf

$$\omega_{E/S} \stackrel{\mathrm{def}}{=} f_*\Omega^1_{E/S} \simeq e^*\Omega^1_{E/S}, \tag{1.2.2.1}$$

the last isomorphism coming from Lemma 1.2.2.1. The preceding work now allows us to relativize the structures on the $\mathcal{E}_z^{\mathrm{an}}$'s as near the beginning of §1.1.1. The relativization of choosing a basis of $\mathrm{H}^0(\mathcal{E}_z^{\mathrm{an}}, \Omega^1_{\mathcal{E}_z^{\mathrm{an}}})$ is choosing an isomorphism $\omega_{E/S} \simeq \mathcal{O}_S$, which can be done locally on S.

The relativization of choosing an (oriented) basis of $\mathrm{H}_1(\mathcal{E}_z^{\mathrm{an}}, \mathbf{Z})$ is choosing an isomorphism $\alpha^\vee : \underline{\mathbf{Z}}^2 \simeq (\mathrm{R}^1 f_*\underline{\mathbf{Z}})^\vee$; given such an $\alpha^\vee$ we can uniquely fill in "?" to make an anti-commutative diagram

$$\begin{tikzcd} \wedge^2\mathrm{R}^1 f_*\underline{\mathbf{Z}} \arrow[r, "\wedge^2\alpha", "\simeq"'] \arrow[d, "\simeq"', "\cup"] & \wedge^2\underline{\mathbf{Z}}^2 \arrow[d, "\simeq"] \\ \mathrm{R}^2 f_*\underline{\mathbf{Z}} \arrow[r, dashed, "?"'] & \underline{\mathbf{Z}} \end{tikzcd} \tag{1.2.2.2}$$

where one checks via topological proper base change that the cup product map $\wedge^2\mathrm{R}^1 f_*\underline{\mathbf{Z}} \to \mathrm{R}^2 f_*\underline{\mathbf{Z}}$ on the left side of (1.2.2.2) is an isomorphism. Of course, we will show that the map $\mathrm{R}^2 f_* \underline{\mathbf{Z}} \to \underline{\mathbf{Z}}$ in the bottom row of (1.2.2.2) induces the map $\mathrm{H}^2(E_s, \underline{\mathbf{Z}}) \simeq \mathbf{Z}$ on fibers corresponding to the i-orientation of each analytic fiber manifold E_s. To check this, we will use the second part of the following theorem. Before stating the result, we make a remark about intrinsic integration on a complex manifold. Let M be a complex manifold with pure dimension d and let ω be a compactly supported, $\mathbf{C}$-valued, C^∞ differential form on M of degree $2d$. Endowing M with the i-orientation for a choice of $i = \sqrt{-1} \in \mathbf{C}$, the integral

$$\frac{1}{(2\pi i)^d} \int_M \omega$$

is independent of the choice of i. Whenever such an integral expression is written down, it will be implicitly understood that the integral is computed with respect to the i-orientation on M.

THEOREM 1.2.2.2. *Let $f : X \to S$ be a proper smooth map with pure relative dimension d and connected fibers. Then there is a canonical isomorphism of $\mathcal{O}_S$-modules*

$$\mathrm{R}^d f_* \Omega^d_{X/S} \simeq \mathrm{R}^{2d} f_* \underline{\mathbf{C}} \otimes_{\underline{\mathbf{C}}} \mathcal{O}_S \tag{1.2.2.3}$$

compatible with base change.

Also, there exists a unique isomorphism $\mathrm{R}^{2d} f_* \underline{\mathbf{C}} \simeq \underline{\mathbf{C}}$ inducing the map

$$\frac{1}{(2\pi i)^d} \int_{X_s} : \mathrm{H}^{2d}(X_s, \underline{\mathbf{C}}) = \mathrm{H}^{2d}_{\mathrm{dR}}(X_s, \mathbf{C}) \simeq \underline{\mathbf{C}}$$

on fibers. This induces a canonical isomorphism $\mathrm{R}^{2d} f_* \underline{\mathbf{Z}}(d) \simeq \underline{\mathbf{Z}}$ via the injection $\underline{\mathbf{Z}}(d) \hookrightarrow \underline{\mathbf{C}}$. Consequently, upon making a choice of i which we use to identify $\mathbf{Z}(1) = \mathbf{Z}$ via $2\pi i \mapsto 1$ (and hence $\mathbf{Z}(r) = \mathbf{Z}$ for all integers r), there is a canonical isomorphism $\mathrm{R}^{2d} f_* \underline{\mathbf{Z}} \simeq \underline{\mathbf{Z}}$ inducing the isomorphism $\mathrm{H}^{2d}(X_s, \underline{\mathbf{Z}}) \simeq \underline{\mathbf{Z}}$ corresponding to the i-orientation on X_s for all $s \in S$ (or equivalently, inducing the map

$$\int_{X_s} : \mathrm{H}^{2d}_{\mathrm{top}}(X_s, \mathbf{Z}) \to \mathbf{Z}$$

of integration with respect to the i-orientation on the manifold X_s).

PROOF. The comparison between de Rham and topological cohomology via integration shows that for a choice of orientation on the connected manifold X_s, the diagram

$$\begin{tikzcd}
\mathrm{H}^{2d}(X_s, \underline{\mathbf{Z}}) \arrow[rr] \arrow[d, equal] & & \mathrm{H}^{2d}(X_s, \underline{\mathbf{C}}) \arrow[d, "\simeq"', "\int_{X_s}"] \\
\mathrm{H}^{2d}_{\mathrm{top}}(X_s, \mathbf{Z}) \arrow[r, "\simeq"] & \mathbf{Z} \arrow[r, hook] & \mathbf{C}
\end{tikzcd}$$

commutes, where $\mathrm{H}^{2d}_{\mathrm{top}}(X_s, \mathbf{Z}) \simeq \mathbf{Z}$ and $\int_{X_s} : \mathrm{H}^{2d}(X_s, \underline{\mathbf{C}}) \simeq \mathbf{C}$ are computed via the same orientation. Thus (using Theorem 1.2.1.6) we just have to show that a trivialization $\mathrm{R}^{2d} f_* \underline{\mathbf{Z}} \simeq \underline{\mathbf{Z}}$ induces isomorphisms $\mathrm{H}^{2d}(X_s, \underline{\mathbf{Z}}) \simeq \mathbf{Z}$ which correspond to the *same* i-orientation on nearby fibers (i.e., the orientation obtained via a common choice of $i \in \mathbf{C}$) and that there is a canonical isomorphism (1.2.2.3) compatible with base change. For $\mathcal{E}^{\mathrm{an}} \to \mathfrak{h}$ we saw this orientation compatibility by hand; see the calculation below (1.2.1.7).

First we construct the isomorphism (1.2.2.3). For any $\underline{\mathbf{C}}$-module $\mathcal{F}$ on S, we have maps

$$\mathcal{F} \otimes_{\underline{\mathbf{C}}} \mathrm{R}^i f_* \underline{\mathbf{C}} \to f_* f^{-1}\mathcal{F} \otimes_{\underline{\mathbf{C}}} \mathrm{R}^i f_* \underline{\mathbf{C}} \overset{\cup}{\to} \mathrm{R}^i f_*(f^{-1}\mathcal{F}) \tag{1.2.2.4}$$

which induce the canonical complex linear map

$$\mathcal{F}_s \otimes_{\mathbf{C}} \mathrm{H}^i(X_s, \underline{\mathbf{C}}) \to \mathrm{H}^i(X_s, (f^{-1}\mathcal{F})|_{X_s}) = \mathrm{H}^i(X_s, \underline{\mathcal{F}_s}) \tag{1.2.2.5}$$

on fibers. In general, if V is any complex vector space (for example, $\mathcal{F}_s$), then the map

$$V \otimes_{\mathbf{C}} \mathrm{H}^i(X_s, \underline{\mathbf{C}}) \to \mathrm{H}^i(X_s, \underline{V})$$

is an isomorphism. Indeed, we can choose a *basis* for V over $\mathbf{C}$ and use that $\mathrm{H}^i(X_s, \cdot)$ commutes with direct limits, as X_s is a compact Hausdorff space. Thus, (1.2.2.5) and then (1.2.2.4) are isomorphisms. Applying (1.2.2.4) with $\mathcal{F} = \mathcal{O}_S$ (which has stalks of infinite dimension over $\mathbf{C}$ in general), we get a canonical $\mathcal{O}_S$-linear isomorphism

$$\mathrm{R}^{2d} f_* \underline{\mathbf{C}} \otimes_{\underline{\mathbf{C}}} \mathcal{O}_S \simeq \mathrm{R}^{2d} f_*(f^{-1}\mathcal{O}_S)$$

compatible with base change over S. Thus, to construct (1.2.2.3) it suffices to construct a natural isomorphism of $\mathcal{O}_S$-modules

$$\mathrm{R}^{2d} f_*(f^{-1}\mathcal{O}_S) \simeq \mathrm{R}^d f_* \Omega^d_{X/S}$$

compatible with base change on S.

Consider the relative de Rham complex on X (constructed just like the scheme analogue in [**28**, §16])

$$0 \to f^{-1}\mathcal{O}_S \to \mathcal{O}_X \overset{\mathrm{d}}{\to} \Omega^1_{X/S} \overset{\mathrm{d}}{\to} \cdots \overset{\mathrm{d}}{\to} \Omega^d_{X/S} \to 0 \tag{1.2.2.6}$$

given as one expects in local holomorphic coordinates. This is a complex of *flat* $f^{-1}\mathcal{O}_S$-modules (but not a complex of $\mathcal{O}_X$-modules) and is compatible with base change via closed immersions $Z \hookrightarrow S$. It is also local on X. We claim that (1.2.2.6) is exact for any smooth (not necessarily proper) $f : X \to S$. Working locally on X and S we have $X \simeq S \times T$ and a closed immersion $S \hookrightarrow B$ for open balls T and B. Thus, it is enough to study maps of the form $f : X = B \times T \to B$. In this case the *usual* proof of the Poincaré lemma on star-shaped regions can be used to prove the exactness of (1.2.2.6), as the standard homotopy operator does preserve holomorphicity (by the Cauchy-Riemann equations).

The exact complex (1.2.2.6) yields an $\mathcal{O}_S$-linear isomorphism

$$\mathrm{R}^{2d} f_*(f^{-1}\mathcal{O}_S) = \mathrm{H}^{2d}(\mathbf{R}f_*(f^{-1}\mathcal{O}_S)) \simeq \mathrm{H}^{2d}(\mathbf{R}f_*(\Omega^\bullet_{X/S})) \tag{1.2.2.7}$$

(where we regard $f^{-1}\mathcal{O}_S$ as a one-term complex of sheaves concentrated in degree 0 and where we view f as a flat map of ringed spaces $(X, f^{-1}\mathcal{O}_S) \to (S, \mathcal{O}_S)$; this has no effect on the higher direct images in which we are interested). The hypercohomology spectral sequence

$$E_2^{pq} = \mathrm{R}^p f_* \Omega^q_{X/S} \Rightarrow \mathrm{H}^{p+q}(\mathbf{R}f_* \Omega^\bullet_{X/S}) \tag{1.2.2.8}$$

has $E_2^{pq} = 0$ for $q > d$ (which is clear) and for $p > d$ (by using cohomology and base change to reduce to the case of fibers which are d-dimensional manifolds, in which case we can use the standard *soft* resolution of $\Omega^q_{X_s}$ by C^∞ type-$(q, \cdot)$ forms as in [**19**, Ch 2, §4.1]). Thus, (1.2.2.7) and the spectral sequence (1.2.2.8) yield a canonical $\mathcal{O}_S$-linear isomorphism

$$\mathrm{R}^d f_* \Omega^d_{X/S} = E_2^{d,d} \simeq \mathrm{H}^{2d}(\mathbf{R}f_*(\Omega^\bullet_{X/S})) \simeq \mathrm{R}^{2d} f_*(f^{-1}\mathcal{O}_S) \tag{1.2.2.9}$$

as desired.

To see the base change compatibility of (1.2.2.9), let

$$\begin{tikzcd} & X_T \arrow[dl, "g'"'] \arrow[d, "f'"] \\ X \arrow[d, "f"'] & T \arrow[dl, "g"] \\ S & \end{tikzcd}$$

be a cartesian diagram. The map g is initially a map of ringed spaces $(T, \mathcal{O}_T) \to (S, \mathcal{O}_S)$, but we will instead view it as a flat map $(T, g^{-1}\mathcal{O}_S) \to (S, \mathcal{O}_S)$, and likewise for g'. We have a commutative diagram of $f'^{-1}g^{-1}\mathcal{O}_S$-modules

$$\begin{tikzcd} g'^{-1}f^{-1}\mathcal{O}_S = f'^{-1}g^{-1}\mathcal{O}_S \arrow[r] \arrow[d] & f'^{-1}\mathcal{O}_T \arrow[d] \\ g'^{-1}\Omega^\bullet_{X/S} \arrow[r] & \Omega^\bullet_{X_T/T} \end{tikzcd}$$

By the flat base change compatibility of the hypercohomology spectral sequence, we get a commutative diagram over $g^{-1}\mathcal{O}_S \to \mathcal{O}_T$:

$$\begin{tikzcd} g^{-1}\mathrm{R}^d f_*\Omega^d_{X/S} \arrow[r, "\simeq"] \arrow[d] & \mathrm{H}^{2d}(g^{-1}\mathbf{R}f_*\Omega^\bullet_{X/S}) \arrow[r, "\simeq"] \arrow[d] & \mathrm{H}^{2d}(g^{-1}\mathbf{R}f_*(f^{-1}\mathcal{O}_S)) \arrow[d] \\ \mathrm{R}^d f'_*(g'^{-1}\Omega^d_{X/S}) \arrow[d] & \mathrm{H}^{2d}(\mathbf{R}\overline{f}'_*(g'^{-1}\Omega^\bullet_{X/S})) \arrow[r] \arrow[d] & \mathrm{H}^{2d}(\mathbf{R}\overline{f}'_*(g'^{-1}f^{-1}\mathcal{O}_S)) \arrow[d] \\ \mathrm{R}^d f'_*\Omega^d_{X_T/T} \arrow[r, "\simeq"'] & \mathrm{H}^{2d}(\mathbf{R}f'_*\Omega^\bullet_{X_T/T}) \arrow[r, "\simeq"'] & \mathrm{H}^{2d}(\mathbf{R}f'_*(f'^{-1}\mathcal{O}_T)) \end{tikzcd}$$

where $\overline{f}' : (X_T, g'^{-1}f^{-1}\mathcal{O}_S = f'^{-1}g^{-1}\mathcal{O}_S) \to (T, g^{-1}\mathcal{O}_S)$ is flat. We thus get a commutative diagram

$$\begin{tikzcd} g^{-1}\mathrm{R}^d f_*\Omega^d_{X/S} \arrow[r, "\simeq"] \arrow[d] & g^{-1}\mathrm{R}^{2d} f_*(f^{-1}\mathcal{O}_S) \arrow[d] \\ \mathrm{R}^d f'_*\Omega^d_{X_T/T} \arrow[r, "\simeq"'] & \mathrm{R}^{2d} f'_*(f'^{-1}\mathcal{O}_T) \end{tikzcd}$$

over $g^{-1}\mathcal{O}_S \to \mathcal{O}_T$, so finally we get an $\mathcal{O}_T$-*linear* commutative diagram

$$\begin{tikzcd} g^*\mathrm{R}^d f_*\Omega^d_{X/S} \arrow[r, "\simeq"] \arrow[d] & g^*\mathrm{R}^{2d} f_*(f^{-1}\mathcal{O}_S) \arrow[r, "\simeq"] \arrow[d] & g^*(\mathrm{R}^{2d} f_*(\underline{\mathbf{C}}) \otimes \mathcal{O}_S) \arrow[d] \\ \mathrm{R}^d f'_*\Omega^d_{X_T/T} \arrow[r, "\simeq"'] & \mathrm{R}^{2d} f'_*(f'^{-1}\mathcal{O}_T) \arrow[r, "\simeq"'] & \mathrm{R}^{2d} f'_*(\underline{\mathbf{C}}) \otimes \mathcal{O}_T \end{tikzcd}$$

where the columns *are* the canonical maps. This completes the construction of (1.2.2.3).

Now we turn to the second part of the theorem. We *fix* a choice of $i \in \mathbf{C}$ to define the i-orientation on complex manifolds. After shrinking S, we choose an isomorphism $\mathrm{R}^{2d} f_* \underline{\mathbf{Z}} \simeq \underline{\mathbf{Z}}$ inducing the i-orientation on some fiber X_{s_0}. We must check that the induced orientation on all fibers in the (open) connected component of s_0 agrees with the i-orientation. By the same trick as in the proof of Theorem 1.2.1.6 (using Lemma 1.2.1.5) we reduce to the case of a smooth S, so we can take S to be a small ball such that:

1. there exists a diagram
$$\begin{tikzcd} S \times U \arrow[rr, hook, "j", "\text{open}"'] \arrow[dr] & & X \arrow[dl] \\ & S & \end{tikzcd}$$
of complex manifolds (the fact that S is smooth is irrelevant here); and
2. there is a C^∞-splitting
$$\begin{tikzcd} X \arrow[rr, "\varphi", "\simeq"'] \arrow[dr] & & S \times X_{s_0} \arrow[dl] \\ & S & \end{tikzcd}$$
with φ *orientation-preserving* with respect to the i-orientations on X, S and X_{s_0}.

   Note that the orientation compatibility of φ can be ensured by simply looking at the *construction* of φ using vector flow, together with the fact that orientation compatibility can be checked on tangent spaces and the map
$$T_x(X)_{\mathbf{R}} \simeq T_x(X_{\mathbf{R}})$$
$$\frac{\partial}{\partial z} \mapsto \frac{\partial}{\partial x} - \frac{\partial}{\partial y}$$
$$i\frac{\partial}{\partial z} \mapsto \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$$
is orientation preserving. Here, $T_x(X)$ receives its orientation from its structure as complex vector space (and $(\cdot)_{\mathbf{R}}$ denotes "underlying object over $\mathbf{R}$").

Via φ (resp. j) the i-orientation on X corresponds to the product orientation of the i-orientations on S and X_{s_0} (resp. S and U). Passing to the fiber at a point $s \in S$, the *composite*
$$U \overset{j_s}{\hookrightarrow} X_s \xrightarrow{\varphi_s} X_{s_0}$$
(where j_s is holomorphic and φ_s is only C^∞) respects the orientations corresonding to the complex structures on U and X_{s_0} while the holomorphic j_s is orientation preserving with respect to the complex structure on U and X_s. We conclude that $\varphi_s : X_s \to X_{s_0}$ is orientating-preserving with respect to the complex structures. The topological isomorphism φ_s is exactly what propogates $\mathrm{H}^{2d}(X_{s_0}, \underline{\mathbf{Z}}) \simeq \mathbf{Z}$ to $\mathrm{H}^{2d}(X_s, \underline{\mathbf{Z}}) \simeq \mathbf{Z}$. Since the first isomorphism corresponds to the i-orientation on X_{s_0} (by the way in which $\mathrm{R}^{2d} f_* \underline{\mathbf{Z}} \simeq \underline{\mathbf{Z}}$ was defined), we conclude that the second isomorphism does as well. This completes the proof. $\square$

1.2.3. Residues and duality. When X is a compact Riemann surface, there is another way to define an isomorphism $\mathrm{H}^1(X, \Omega^1_X) \xrightarrow{\simeq} \mathbf{C}$ as in (1.2.2.3) using residues. Since this will be used later on to compute a Kodaira-Spencer map (with X an elliptic curve), we briefly digress to recall the construction of this (analytic) residue isomorphism and to check that it agrees with the isomorphism

$$\frac{1}{2\pi i}\int_X : \mathrm{H}^1(X, \Omega^1_X) \xrightarrow{\simeq} \mathbf{C}$$

of Theorem 1.2.2.2.

Let $\mathcal{M}$ be the sheaf of meromorphic functions on X. Since any section of $\mathcal{M}(U)$ is holomorphic outside of a discrete subset of U, there is a canonical isomorphism

$$\mathcal{M}/\mathcal{O}_X \simeq \bigoplus_{x\in X} \mathcal{M}_x/\mathcal{O}_{X,x}$$

with $\mathcal{M}_x/\mathcal{O}_{X,x}$ viewed as a sheaf supported at x. Since every invertible sheaf $\mathcal{L}$ on X has the form $\mathcal{O}_X(D)$ for some divisor D on X, the proof that $\mathrm{H}^1(X, \mathcal{M}) = 0$ using Čech cohomology (as in [**19**, Ch 7, §5.3]) can be adapted to show that $\mathrm{H}^1(X, \mathcal{L} \otimes \mathcal{M}) = 0$ for any invertible $\mathcal{L}$. The short exact sequence of sheaves

$$0 \to \Omega^1_X \to \Omega^1_X \otimes \mathcal{M} \to \bigoplus_{x\in X} (\Omega^1_{X,x} \otimes \mathcal{M}_x)/\Omega^1_{X,x} \to 0 \tag{1.2.3.1}$$

(with the middle term the sheaf of meromorphic differentials on X) is an acyclic resolution of Ω^1_X (relative to the global sections functor), so, as with any acyclic resolution, we can use this sequence to *compute* the cohomology of Ω^1_X. This yields a canonical presentation of $\mathbf{C}$-vector spaces

$$\mathrm{H}^0(X, \Omega^1_X \otimes \mathcal{M}) \to \bigoplus_{x\in X} (\Omega^1_{X,x} \otimes \mathcal{M}_x)/\Omega^1_{X,x} \to \mathrm{H}^1(X, \Omega^1_X) \to 0 \tag{1.2.3.2}$$

since $\mathrm{H}^0(X, \cdot)$ commutes with direct limits (in particular, with direct sums), as X is compact Hausdorff. We warn the careful reader that the quotient map in (1.2.3.2), which Gabber proposes calling the "definition homomorphism", is the *negative* of the connecting homomorphism in which arises from applying the δ-functor of cohomology to the short exact sequence (1.2.3.1). This is essentially due to an analogue of [**31**, V, Prop 7.1] for injective resolutions and covariant functors, and can be a source of extreme confusion if one wishes to avoid sign errors in subsequent calculations.

By the (analytic) residue theorem,

$$\sum_{x\in X} \mathrm{res}_x : \bigoplus_{x\in X} (\Omega^1_{X,x} \otimes \mathcal{M}_x)/\Omega^1_{X,x} \to \mathbf{C}$$

kills $\mathrm{H}^0(X, \Omega^1_X \otimes \mathcal{M})$, so via (1.2.3.2) we get a non-zero residue map

$$\mathrm{res}_X : \mathrm{H}^1(X, \Omega^1_X) \to \mathbf{C}$$

which must be an isomorphism for dimension reasons.

If we view X as the analytification of a proper smooth connected curve over $\operatorname{Spec} \mathbf{C}$, the analytification compatibilities among differentials, rational and meromorphic functions, and algebraic/analytic residues imply that the analytically constructed res_X above is compatible (via the GAGA isomorphisms) with the similarly constructed algebraic residue map (which is the classical residue map as considered by Serre and others).

The "compatibility" of Theorem 1.2.2.2 with residues is given by the next result.

COROLLARY 1.2.3.1. *For X as above, the diagram*

$$\begin{tikzcd} \mathrm{H}^1(X,\Omega^1_X) \arrow[d, "\simeq"'] \arrow[dr, "\mathrm{res}_X"] & \\ \mathrm{H}^2(X,\underline{\mathbf{C}}) \arrow[r, "\frac{1}{2\pi i}\int_X"'] & \mathbf{C} \end{tikzcd} \tag{1.2.3.3}$$

commutes, where the vertical isomorphism is that of (1.2.2.3).

A vastly more general statement is true in higher dimensions relating integration maps with Grothendieck duality, but this 1-dimensional version with the residue map is adequate for our purposes.

PROOF. Let $x_0, x_1 \in X$ be two distinct points and set $U_j = X - \{x_j\}$. By the Riemann-Roch theorem, there exists a meromorphic function on X with its only pole at x_j. This function yields a finite map $U_j \to \mathbf{C}$. Each open U_j is therefore a Stein space, so $\mathfrak{U} = \{U_0, U_1\}$ is an ordered Stein covering of X and therefore computes the cohomology of coherent sheaves on X via Čech cohomology.

Let φ_0 be a C^∞-function on X with $\varphi_0 = 1$ near x_0 but φ_0 compactly supported in a small disk near x_0; in particular, we want $\varphi_0 = 0$ near x_1. Define $\varphi_1 = 1 - \varphi_0$, so $\varphi_1 = 1$ near x_1 and is compactly supported. Beware that φ_0 has its support inside of U_1! The soft resolution

$$0 \to \Omega^1_X \to \mathcal{A}^{1,0} \xrightarrow{\overline{\partial}} \mathcal{A}^{1,1} \to 0$$

(with $\mathcal{A}^{p,q}$ the sheaf of C^∞ type-(p,q) forms on X) computes the cohomology of Ω^1_X, as does the Čech resolution

$$0 \to \Omega^1_X \to \mathcal{C}^0(\mathfrak{U},\Omega^1_X) \to \mathcal{C}^1(\mathfrak{U},\Omega^1_X) \to 0.$$

We will construct an explicit map of complexes $\mathcal{C}^\bullet(\mathfrak{U},\Omega^1_X) \to \mathcal{A}^{1,\bullet}$ compatible with the augmentations from Ω^1_X, so the induced map on cohomology must be the canonical identification of $\mathrm{H}^p(\mathrm{C}^\bullet(\mathfrak{U},\Omega^1_X))$ with $\mathrm{H}^p(X,\Omega^1_X)$. Composing with $\mathcal{A}^{1,\bullet} \to \mathrm{Tot}^\oplus(\mathcal{A}^{\bullet,\bullet})$ then gives a map

$$\mathrm{H}^1(\mathrm{C}^\bullet(\mathfrak{U},\Omega^1_X)) \simeq \mathrm{H}^1(\Gamma(X,\mathcal{A}^{1,\bullet})) \to \mathrm{H}^2(\mathrm{Tot}^\oplus(\Gamma(X,\mathcal{A}^{\bullet,\bullet})))$$

which must be *exactly* the map $\mathrm{H}^1(X,\Omega^1_X) \simeq \mathrm{H}^2(X,\underline{\mathbf{C}})$ in (1.2.2.3).

The map $\mathcal{C}^\bullet(\mathfrak{U},\Omega^1_X) \to \mathcal{A}^{1,\bullet}$ is defined to be

$$(\omega_0,\omega_1) \mapsto \varphi_1\omega_0 + (1-\varphi_1)\omega_1$$

in degree 0 and

$$\eta \mapsto -\overline{\partial}\varphi_1 \wedge \eta$$

in degree 1 (note that $\overline{\partial}\varphi_1$ vanishes near x_0 and x_1). This is a map of complexes because

$$-\overline{\partial}\varphi_1 \wedge (\omega_1 - \omega_0) = \overline{\partial}(\varphi_1\omega_0 + (1-\varphi_1)\omega_1)$$

for holomorphic ω_0, ω_1.

Consider now an element $\eta \in \mathrm{C}^1(\mathfrak{U},\Omega^1_X) = \mathrm{H}^0(U_0 \cap U_1, \Omega^1_X)$ which is meromorphic on X and has $\mathrm{res}_{x_0}(\eta) \neq 0$ (so $\mathrm{res}_{x_1}(\eta) = -\mathrm{res}_{x_0}(\eta) \neq 0$). Such an η exists by the Riemann-Roch theorem. The map $\mathrm{H}^1(X,\Omega^1_X) \simeq \mathrm{H}^2(X,\underline{\mathbf{C}})$ sends $[\eta]$ to $-\overline{\partial}\varphi_1 \wedge \eta = -\,\mathrm{d}(\varphi_1\eta)$ (since η is holomorphic on $U_0 \cap U_1$, so $\mathrm{d}\eta = 0$ and

$\partial\varphi_1 \wedge \eta = 0$). Let $\Delta \subseteq X$ be a small closed disc around x_1 on the interior of which φ_1 is supported, so

$$\frac{1}{2\pi i}\int_X -\overline{\partial}\varphi_1 \wedge \eta = -\frac{1}{2\pi i}\int_\Delta \overline{\partial}\varphi_1 \wedge \eta = -\frac{1}{2\pi i}\int_{\Delta-\Delta_1} \mathrm{d}(\varphi_1\eta)$$

with $\Delta_1 \subseteq \Delta$ an open disc around x_1 where $\varphi_1 = 1$. Since $\varphi_1|_{\partial\Delta} = 0$ and $\varphi_1|_{\partial\Delta_1} = 1$, by Stokes' theorem this is equal to

$$-\frac{1}{2\pi i}\int_{\partial\Delta}\varphi_1\eta + \frac{1}{2\pi i}\int_{\partial\Delta_1}\varphi_1\eta = \frac{1}{2\pi i}\int_{\partial\Delta_1}\eta = \mathrm{res}_{x_1}(\eta) \neq 0.$$

Thus, $[\eta] \in \mathrm{H}^1(X, \Omega^1_X)$ is non-zero and consequently is a basis of this 1-dimensional space, so the composite map

$$\mathrm{H}^1(X, \Omega^1_X) \to \mathrm{H}^2(X, \underline{\mathbf{C}}) \to \mathbf{C}$$

is determined by $[\eta] \mapsto \mathrm{res}_{x_1}(\eta)$.

It remains to check that $\mathrm{res}_X[\eta] = \mathrm{res}_{x_1}(\eta)$. We are therefore reduced to showing that if $\mathfrak{V} = \{V_0, V_1\}$ is an ordered open covering of X with $V_i = X - \{x_i\}$ for distinct points $x_0, x_1 \in X$ and $\eta \in \Gamma(V_0 \cap V_1, \Omega^1_X)$ is a meromorphic differential on X with pole set inside of $\{x_0, x_1\}$, then the composite map

$$\check{\mathrm{H}}^1(\mathfrak{V}, \Omega^1_X) \simeq \mathrm{H}^1(X, \Omega^1_X) \xrightarrow{\mathrm{res}_X} \mathbf{C} \tag{1.2.3.4}$$

takes the class of η to $\mathrm{res}_{x_1}(\eta)$. Note that since $\mathfrak{V}$ is ordered, there is an a priori distinction between x_0 and x_1. The first map in (1.2.3.4) is the edge map in the Čech to derived functor cohomology spectral sequence.

In our situation, Ω^1_X has a rather concrete acyclic resolution given by

$$\Omega^1_X \to \underline{\Omega}^1_X \otimes \mathcal{M} \to \bigoplus_{x \in X^0} (\Omega^1_{X,x} \otimes \mathcal{M}_x)/\Omega^1_{X,x}) \to 0, \tag{1.2.3.5}$$

defined without an intervention of signs. If $j_i : V_i \hookrightarrow X$, $j : V_0 \cap V_1 \hookrightarrow X$ are the canonical open immersions, then the Čech resolution $\mathcal{C}^\bullet(\mathfrak{V}, \Omega^1_X)$ relative to $\mathfrak{V}$ is

$$\Omega^1_X \to j_{0*}\Omega^1_{V_0} \oplus j_{1*}\Omega^1_{V_1} \to j_*\Omega^1_{V_0 \cap V_1} \to 0, \tag{1.2.3.6}$$

with the second map given by "$(\eta_0, \eta_1) \mapsto \eta_1 - \eta_0$". We can define a map of complexes

$$\begin{tikzcd} \Omega^1_X \arrow[r] \arrow[d, equal] & j_{0*}\Omega^1_{V_0} \oplus j_{1*}\Omega^1_{V_1} \arrow[r] \arrow[d] & j_*\Omega^1_{V_0 \cap V_1} \arrow[d] \\ \Omega^1_X \arrow[r] & \Omega^1_X \otimes \mathcal{M} \arrow[r] & \bigoplus_{x \in X^0} (\Omega^1_{X,x} \otimes \mathcal{M}_x)/\Omega^1_{X,x} \end{tikzcd} \tag{1.2.3.7}$$

extending the identity on Ω^1_X, where the map in the middle column sends (η_0, η_1) to η_1 and the map in the right column sends η to the element whose x_1th coordinate is represented by η and whose xth coordinate is 0 for all $x \neq x_1$. The diagram (1.2.3.7) computes the edge map in (1.2.3.4). By the *definition* of res_X in terms of (1.2.3.5), the map (1.2.3.4) must therefore send the class of η to $\mathrm{res}_{x_1}(\eta) \in \mathbf{C}$, as desired.

$\square$

1.2.4. Relative complex tori and i-permissible trivializations. We are now in position to formulate the universal property of $\mathcal{E}^{\mathrm{an}} \to \mathfrak{h}$. Let us fix a choice of $i = \sqrt{-1} \in \mathbf{C}$. Following Deligne (and using Theorem 1.2.2.2) if $f : E \to S$ is an elliptic curve and $\alpha : \mathrm{R}^1 f_* \underline{\mathbf{Z}} \simeq \underline{\mathbf{Z}}^2$ is an isomorphism, we call α an *i-permissible* isomorphism (or just *permissible* if the choice of i is understood) if the diagram

$$\begin{tikzcd}
\wedge^2 \mathrm{R}^1 f_* \underline{\mathbf{Z}} \arrow[r, "\wedge^2\alpha", "\simeq"'] \arrow[d, "\simeq"', "\cup"] & \wedge^2(\underline{\mathbf{Z}}^2) \arrow[d, "\simeq"] & (1,0) \wedge (0,1) \arrow[d, mapsto] \\
\mathrm{R}^2 f_* \underline{\mathbf{Z}} \arrow[r, tail, dash, "\simeq"'] & \underline{\mathbf{Z}} & 1
\end{tikzcd}$$

anti-commutes, where the bottom isomorphism is induced by the i-orientation on fibers, as in Theorem 1.2.2.2.

The basic reason for the anti-commutativity requirement above is due to the compatibility with the higher dimensional theory of moduli spaces of principally polarized abelian varieties. More precisely, recall that any classical elliptic curve E admits a *unique* principal polarization $E \simeq \widehat{E} \stackrel{\mathrm{def}}{=} \underline{\mathrm{Pic}}^0_{E/\mathrm{Sp}\,\mathbf{C}}$ given on points by $x \mapsto t_x^* \mathcal{L} \otimes \mathcal{L}^{-1}$ for any fixed line bundle $\mathcal{L}$ of degree 1. (Here t_x is the "translation by x" map $y \mapsto y + x$ on E and $\widehat{E}$ is the dual elliptic curve, to be discussed in a relative and functorial manner for complex tori of arbitrary relative dimension later on). Taking $\mathcal{L} = \mathcal{O}(e)$ with e the identity point of E, we get the map $x \mapsto \mathcal{O}(x) \otimes \mathcal{O}(e)^{-1}$, which is the "usual" autoduality of an elliptic curve (at least if one follows the general conventions of Mumford as in his book [**45**] on abelian varieties; this choice is not universally accepted, but is the one I am used to using; clearly this choice influences explications of the theory).

From a more analytic point of view, if we choose an isomorphism $E \simeq \mathbf{C}/[1, z]$ then this principal polarization is defined by the imaginary part of the positive definite Riemann form $H : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ given by

$$H(w_1, w_2) = \frac{1}{\mathrm{Im}\, z} w_1 \overline{w}_2$$

(cf. [**45**, pp. 35-36] and the discussion in §1.4 below); the matrix of the imaginary part of H is

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

with respect to the ordered (i-oriented) basis $\{1, z\}$ of $\mathrm{H}_1(E, \mathbf{Z})$, which is *dual* to the basis

$$\{\mathrm{d}x - (a/b)\mathrm{d}y, (1/b)\mathrm{d}y\}$$

of $\mathrm{H}^1(E, \underline{\mathbf{Z}})$, where $z = a + bi$, $b > 0$ (the homology basis $\{1, z\}$ is *not* i-permissible, since $b > 0$; note that $\{z, 1\}$ is i-permissible).

But the higher dimensional theory of moduli spaces for principally polarized abelian varieties with fixed homology basis (via Siegel upper half-spaces) usually standardizes the imaginary part of the Riemann form of the principally polarization to have matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

with respect to the chosen basis of $\mathrm{H}_1(\mathbf{Z})$. For this reason, it is actually preferable to take $\{z, 1\}$ as a "standard basis" of $\mathrm{H}_1(\mathbf{C}/[1, z], \mathbf{Z})$ rather than $\{1, z\}$ (with $\mathrm{Im}\, z > 0$). This seems to be what motivates Deligne's notion of i-permissibility.

More generally one can check that if X is a compact Riemann surface, then the canonical principal polarization on $\mathrm{Pic}^0_X \stackrel{\mathrm{def}}{=} \mathrm{H}^1(X, \mathcal{O}_X)/\mathrm{H}^1(X, \underline{\mathbf{Z}})$ defined via the exponential sequence

$$0 \to \mathbf{Z} \to \mathcal{O}_X \stackrel{e^{2\pi i(\cdot)}}{\longrightarrow} \mathcal{O}_X^\times \to 0 \tag{1.2.4.1}$$

on X is induced by the Hermitian form on $\mathrm{H}^1(X, \mathcal{O}_X)$ whose restriction to the lattice $\mathrm{H}^1(X, \underline{\mathbf{Z}})$ is the skew-symmetric duality $\mathrm{H}^1(X, \underline{\mathbf{Z}}) \times \mathrm{H}^1(X, \underline{\mathbf{Z}}) \to \mathbf{Z}$ which is the *negative* of Poincaré duality (with respect to the i-orientation, where i is as used in the map $e^{2\pi i(\cdot)}$ in (1.2.4.1).)

For conceptual clarity in the next section, we introduce the following definition.

DEFINITION 1.2.4.1. A *relative complex torus* (of relative dimension g) is a proper smooth group object $f : A \to S$ with connected fibers (of dimension g). This is also called a *complex torus over* S. (Recall that proper smooth maps with connected fibers always have relative dimension function locally constant over the base.)

As we noted immediately after Theorem 1.1.1.15, relative complex tori have a unique group structure given a specified identity section, and maps of relative complex tori respecting identity sections respect the group structures. In particular, these group structures are commutative. When S is a reduced point, a relative complex torus is the same thing as the classical notion of a complex torus.

The reason we introduce the relative notion of complex torus is that the universal property of $\mathcal{E}^{\mathrm{an}} \to \mathfrak{h}$ is a special case of a more general representability result in the setting of relative complex tori. Arguing in this greater generality below will clarify what is going on.

The final piece of cohomological information we will need in our study of relative complex tori is the following fact.

COROLLARY 1.2.4.2. *Let $f : A \to S$ be a relative complex torus. The coherent sheaves $\mathrm{R}^i f_* \mathcal{O}_A$ and $\mathrm{R}^i f_* \Omega^1_{A/S}$ are locally free and of formation compatible with arbitrary base change.*

PROOF. By Lemma 1.2.2.1 and working locally on S, it suffices to study the sheaves $\mathrm{R}^i f_* \mathcal{O}_A$. By cohomology and base change, it suffices to show that the map

$$(\mathrm{R}^i f_* \mathcal{O}_A)_s \to \mathrm{H}^i(A_s, \mathcal{O}_{A_s})$$

is surjective for all $s \in S$ and all i. Consider the map $f^{-1}\mathcal{O}_S \to \mathcal{O}_A$. Applying $\mathrm{R}^i f_*$, by functoriality of base change we get a commutative diagram

$$\begin{tikzcd}
\mathrm{R}^i f_*(\underline{\mathbf{C}})_s \otimes_{\mathbf{C}} \mathcal{O}_{S,s} \arrow[r] \arrow[d, "\simeq"] & \mathrm{R}^i f_*(f^{-1}\mathcal{O}_S)_s \arrow[r] \arrow[d] & (\mathrm{R}^i f_* \mathcal{O}_A)_s \arrow[d] \\
\mathrm{H}^i(A, \underline{\mathbf{C}}) \otimes_{\mathbf{C}} \mathcal{O}_{S,s} \arrow[r] & \mathrm{H}^i(A_s, \underline{\mathcal{O}_{S,s}}) \arrow[r] & \mathrm{H}^i(A_s, \mathcal{O}_{A_s})
\end{tikzcd}$$

where the left column is an isomorphism by Theorem 1.2.1.1. Using this diagram, it suffices to check that $\mathrm{H}^i(A, \underline{\mathbf{C}}) \to \mathrm{H}^i(A, \mathcal{O}_A)$ is surjective for a classical complex torus A. By general nonsense, this map is *exactly* the edge map in the Hodge to de Rham cohomology spectral sequence for A. Since

$$\dim \mathrm{H}^n(A, \underline{\mathbf{C}}) = \sum_{p+q=n} \dim \mathrm{H}^p(A, \Omega^q_A)$$

for all n [**45**, p. 9], this spectral sequence degenerates. Thus, the above edge map is surjective as desired.

$\square$

1.3. Generalities on relative complex tori

1.3.1. The dual relative complex torus. In the classical theory of complex tori, the exponential map

$$\mathrm{H}^0(A,\Omega^1_A)^\vee \xrightarrow{\simeq} T_0(A) \twoheadrightarrow A$$

sets up an isomorphism of complex Lie groups

$$\mathrm{H}^0(A,\Omega^1_A)^\vee/\mathrm{H}_1(A,\mathbf{Z}) \simeq A. \tag{1.3.1.1}$$

We want to generalize this, showing that any relative complex torus $f : A \to S$ of relative dimension g can be *functorially* realized as $(e^*\Omega^1_{A/S})^\vee/\mathrm{R}^1f_*(\underline{\mathbf{Z}})^\vee$ for some suitable inclusion $\mathrm{R}^1f_*(\underline{\mathbf{Z}})^\vee \hookrightarrow (e^*\Omega^1_{A/S})^\vee$ recovering the classical picture (1.3.1.1) on fibers. In other words, we want to show that the construction of complex tori in Lemma 1.1.2.1 is rather universal.

A technical problem is that we do not see how to *directly* set up a relative theory of the exponential map for families, especially if we allow the base to be non-reduced. However, we will set up an exponential sequence of *sheaves* in order to construct the "dual" $\mathrm{Pic}^0_{A/S} \to S$ of $A \to S$ and then essentially by "double duality" we will recover our desired relativization of (1.3.1.1). For relative dimension 1 this will lead to the universal property of $\mathcal{E}^{\mathrm{an}} \to \mathfrak{h}$. In general, we will be led to a universal family over a Siegel upper half-space.

To get things started, we want to define a short exact sequence of sheaves

$$0 \to \underline{\mathbf{Z}}(1) \to \mathcal{O}_X \xrightarrow{\exp_X} \mathcal{O}_X^\times \to 1 \tag{1.3.1.2}$$

for any analytic space X. It will turn out later that by using $\underline{\mathbf{Z}}(1)$ and "exp" (rather than $\underline{\mathbf{Z}}$ and "$\exp(2\pi i(\cdot))$") we will have a canonical compatibility with (1.3.1.1) without the annoyance of a choice of $i \in \mathbf{C}$. For reduced X, there already is a standard exponential sequence

$$0 \to \underline{\mathbf{Z}}(1) \to \mathcal{C}_X \xrightarrow{e^{(\cdot)}} \mathcal{C}_X^\times \to 0 \tag{1.3.1.3}$$

where $\mathcal{C}_X$ is the the sheaf of $\mathbf{C}$-valued C^∞-functions on X. Note that the natural map $\mathcal{O}_X \to \mathcal{C}_X$ is *injective* by the analytic Nullstellensatz. We would obviously like an analytic exponential sequence to agree with the restriction of (1.3.1.3) to $\mathcal{O}_X \subseteq \mathcal{C}_X$, $\mathcal{O}_X^\times \subseteq \mathcal{C}_X^\times$ for reduced X.

Lemma 1.3.1.1. *There is a unique way to define a map of sheaves of groups* $\exp_X : \mathcal{O}_X \to \mathcal{O}_X^\times$ *which is functorial in* X *and recovers the usual exponential map for reduced* X. *The diagram*

$$0 \to \underline{\mathbf{Z}}(1) \to \mathcal{O}_X \xrightarrow{\exp_X} \mathcal{O}_X^\times \to 1 \tag{1.3.1.4}$$

is short exact.

The sequence (1.3.1.4) is called the *exponential sequence* on X.

Proof. We may work locally on X, so we can assume that there is a closed immersion $X \hookrightarrow Z$ with Z a smooth space. If $\mathcal{I}$ is the coherent ideal sheaf on Z cutting out X, for the existence of $\exp_X$ we just need to check that the usual exponential e^f lies in $1+\mathcal{I}$ for $f \in \mathcal{I}$. (The independence of the choice of $X \hookrightarrow Z$ is then clear by comparison with a product of such immersions, as is functoriality and uniqueness.) This condition $e^f \in 1+\mathcal{I}$ follows immediately from the convergent power series for $e^{(\cdot)}$ on the smooth space Z.

To check that (1.3.1.4) is exact, we note that this is clear on X_{red} from (1.3.1.3) and it follows from the construction above that $\exp_X$ is surjective and (1.3.1.4) is a complex with injectivity on the left. It remains to check that $\underline{\mathbf{Z}}(1) \subseteq \ker(\exp_X)$ is an equality. Since we know this in the reduced case, we just need to show that if s is nilpotent and if $\exp_X(s) = 1$, then $s = 0$. By construction and the *nilpotence* of s it is easy to see that

$$\exp_X(s) = \sum_{n \geq 1} s^n/n! = 1 + s(1 + \sum_{n \geq 2} s^{n-1}/n!),$$

where this is just a finite sum. The relation $\exp_X(s) = 1$ now forces the nilpotent s to vanish, since the sum of 1 and a nilpotent is a unit. $\square$

Let $f : A \to S$ be a complex torus with identity section $\varepsilon : S \to A$. Let $\mathrm{Pic}^0(A/S)$ denote the group of isomorphism classes of pairs $(\mathcal{L}, \iota)$ where $\mathcal{L}$ is a line bundle on A which is topologically trivial on fibers (that is, $\mathcal{L}|_{A_s} \simeq A_s \times \mathbf{C}$ as topological line bundles on A_s for all $s \in S$) and $\iota : \varepsilon^*\mathcal{L} \simeq \mathcal{O}_S$ is an isomorphism, also called a "rigidification" along the identity section. One checks immediately that for a classical complex torus A, viewed as a relative complex torus over $\mathrm{Sp}\,\mathbf{C}$, the natural map of groups $\mathrm{Pic}^0(A/\mathrm{Sp}\,\mathbf{C}) \to \mathrm{Pic}^0(A)$ is an isomorphism since $\mathrm{H}^0(A, \mathcal{O}_A^\times) = \mathbf{C}^\times$.

We claim that the above fiber condition on $\mathcal{L}$ says exactly that $\mathcal{L}$ is killed by the coboundary map $\mathrm{R}^1 f_* \mathcal{O}_A^\times \to \mathrm{R}^2 f_* \underline{\mathbf{Z}}(1)$ induced by the exponential sequence on A. To see this, we begin with the classical case. Let A be classical complex torus. Since the sequence of global sections

$$0 \to \mathbf{Z}(1) \to \mathbf{C} \to \mathbf{C}^\times \to 0$$

of the exponential sequence on A is exact, we obtain an exact sequence

$$0 \to \mathrm{H}^1(A, \underline{\mathbf{Z}}(1)) \to \mathrm{H}^1(A, \mathcal{O}_A) \to \ker\big(\mathrm{Pic}(A) \to \mathrm{H}^2(A, \underline{\mathbf{Z}}(1))\big) \to 0.$$

We claim that the kernel of $\mathrm{Pic}(A) \to \mathrm{H}^2(A, \underline{\mathbf{Z}}(1))$ consists precisely of those line bundles $\mathcal{L}$ which are topologically trivial. To see this, let $\mathcal{C}_A^{\mathrm{top}}$ denote the sheaf of continuous $\mathbf{C}$-valued functions on A as before, so we have a commutative diagram of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \underline{\mathbf{Z}}(1) \arrow[r] \arrow[d, equal] & \mathcal{O}_A \arrow[r] \arrow[d] & \mathcal{O}_A^\times \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \underline{\mathbf{Z}}(1) \arrow[r] & \mathcal{C}_A^{\mathrm{top}} \arrow[r] & \mathcal{C}_A^{\mathrm{top}\,\times} \arrow[r] & 0
\end{tikzcd}$$

This yields a commutative diagram

$$\begin{tikzcd}
\mathrm{H}^1(A, \mathcal{O}_A) \arrow[r] \arrow[d] & \mathrm{H}^1(A, \mathcal{O}_A^\times) \arrow[r] \arrow[d] & \mathrm{H}^2(A, \underline{\mathbf{Z}}(1)) \arrow[d, equal] \\
\mathrm{H}^1(A, \mathcal{C}_A^{\mathrm{top}}) \arrow[r] & \mathrm{H}^1(A, \mathcal{C}_A^{\mathrm{top}\,\times}) \arrow[r] & \mathrm{H}^2(A, \underline{\mathbf{Z}}(1))
\end{tikzcd}$$

Since $\mathcal{C}_A^{\mathrm{top}}$ is a soft sheaf (via partitions of unity), we have $\mathrm{H}^1(A, \mathcal{C}_A^{\mathrm{top}}) = 0$. Thus

$$\ker\bigl(\mathrm{Pic}(A) \to \mathrm{H}^2(A, \underline{\mathbf{Z}}(1))\bigr) = \ker\bigl(\mathrm{Pic}(A) \to \mathrm{H}^1(A, \mathcal{C}_A^{\mathrm{top}\,\times})\bigr)$$

as claimed. The case of a general base S now follows immediately by topological proper base change and the fact that $\mathrm{R}^2 f_* \underline{\mathbf{Z}}(1)$ is locally constant.

Since the isomorphism $\mathcal{O}_S = f_*\mathcal{O}_A$ is compatible with base change, we see that pairs $(\mathcal{L}, \iota)$ admit *no* nontrivial automorphisms. Indeed, any global automorphism of $\mathcal{L}$ is given by multiplication by an element of

$$\mathrm{H}^0(A, \mathcal{O}_A^\times) = \mathrm{H}^0(S, f_*\mathcal{O}_A^\times) = \mathrm{H}^0(S, \mathcal{O}_S^\times),$$

so any such non-trivial automorphism would change ι. Let $\underline{\mathrm{Pic}}^0_{A/S}$ denote the functor on S-spaces $T \to S$ given by $T \mapsto \mathrm{Pic}^0(A \times_S T/T)$ (with the functorial structure defined in the obvious way). Since pairs $(\mathcal{L}, \iota)$ have no non-trivial automorphisms, it is clear that $\underline{\mathrm{Pic}}^0_{A/S}$ (restricted to the category of open subspaces of S) defines an abelian *sheaf* on S. We also clearly have $\underline{\mathrm{Pic}}^0_{A/S} \times_S T \simeq \underline{\mathrm{Pic}}^0_{A \times_S T/T}$ as functors on T-spaces.

1.3.2. The Picard functor. When $S = \mathrm{Sp}\,\mathbf{C}$ is a reduced point, the classical analytic duality theorem for complex tori (as in, for example, [**45**, Ch 1]) gives an isomorphism of groups

$$\mathrm{Pic}^0(A/S) \simeq \mathrm{H}^1(A, \mathcal{O}_A)/\mathrm{H}^1(A, \underline{\mathbf{Z}}(1))$$

via the exponential sequence (1.3.1.4). This relativizes as follows. Suppose $f : A \to S$ has relative dimension g and consider the exponential sequence on A. This gives rise to an exact sequence

$$0 \to f_*\underline{\mathbf{Z}}(1) \to f_*\mathcal{O}_A \to f_*\mathcal{O}_A^\times \to \mathrm{R}^1 f_*\underline{\mathbf{Z}}(1) \to \mathrm{R}^1 f_*\mathcal{O}_A$$

and the first three (non-zero) terms constitute exactly the exponential sequence on S (by Theorem 1.2.1.6 and Lemma 1.2.2.1(2)). Thus, we get an inclusion

$$\mathrm{R}^1 f_*\underline{\mathbf{Z}}(1) \hookrightarrow \mathrm{R}^1 f_*\mathcal{O}_A$$

from a local system of rank $2g$ free $\mathbf{Z}$-modules into a rank g vector bundle, inducing a cocompact lattice

$$\mathrm{H}^1(A_s, \underline{\mathbf{Z}}(1)) \hookrightarrow \mathrm{H}^1(A_s, \mathcal{O}_{A_s})$$

on fibers (cf. Corollary 1.2.4.2). The co-compactness follows from the uniformization of A_s in the classical theory. Thus, Lemma 1.1.2.1 ensures that

$$\widehat{A} \stackrel{\mathrm{def}}{=} \mathrm{R}^1 f_*\mathcal{O}_A/\mathrm{R}^1 f_*\underline{\mathbf{Z}}(1)$$

is a relative complex torus over S (of relative dimension g) compatible with *base change* on S. The key fact is the following.

THEOREM 1.3.2.1. *The functor $\underline{\mathrm{Pic}}^0_{A/S}$ is represented by $\widehat{A}$, compatibly with respect to base change on S. If $(\mathcal{P}, i_A)$ is the universal rigidified line bundle on $A \times_S \widehat{A}$ with $i_A : \mathcal{P}|_{A \times \{0\}} \simeq \mathcal{O}_A$ the canonical rigidification* (*this data is provided by the universality of $\widehat{A}$ and the definition of the* 0*-section of $\widehat{A}$*)*, then $(\mathcal{P}, i_{\widehat{A}})$ induces a map $A \to \hat{\hat{A}}$ and this is an isomorphism of relative complex tori.*

PROOF. The exponential sequence gives an exact sequence of sheaves on S

$$0 \to \mathrm{R}^1 f_* \underline{\mathbf{Z}}(1) \to \mathrm{R}^1 f_* \mathcal{O}_A \to \ker\left(\mathrm{R}^1 f_* \mathcal{O}_A^\times \to \mathrm{R}^2 f_* \underline{\mathbf{Z}}(1)\right) \to 0.$$

We also have a natural map of sheaves

$$\underline{\mathrm{Pic}}^0_{A/S} \to \mathrm{R}^1 f_* \mathcal{O}_A^\times$$

which is compatible with the usual injection $\mathrm{Pic}^0(A_s) \hookrightarrow \mathrm{H}^1(A_s, \mathcal{O}_{A_s}^\times)$ on fibers. This map has image in the kernel of the map $\mathrm{R}^1 f_* \mathcal{O}_A^\times \to \mathrm{R}^2 f_* \underline{\mathbf{Z}}(1)$, by the definition of $\underline{\mathrm{Pic}}^0_{A/S}$. We claim that this sets up an isomorphism of sheaves on S

$$\underline{\mathrm{Pic}}^0_{A/S} \xrightarrow{\simeq} \ker\left(\mathrm{R}^1 f_* \mathcal{O}_A^\times \to \mathrm{R}^2 f_* \underline{\mathbf{Z}}(1)\right) \tag{1.3.2.1}$$

and thus an exact sequence of abelian sheaves on S

$$0 \to \mathrm{R}^1 f_* \underline{\mathbf{Z}}(1) \to \mathrm{R}^1 f_* \mathcal{O}_A \to \underline{\mathrm{Pic}}^0_{A/S} \to 0. \tag{1.3.2.2}$$

Injectivity of (1.3.2.1) follows from the obvious fact that any element $(\mathcal{O}_A, \iota)$ in $\mathrm{Pic}^0(A/S)$ is isomorphic to $(\mathcal{O}_A, 1)$, where $1 : \varepsilon^* \mathcal{O}_A \simeq \mathcal{O}_S$ is the canonical isomorphism.

For surjectivity of (1.3.2.1), we must show locally on S that an arbitrary line bundle $\mathcal{L}$ in the kernel of $\mathrm{R}^1 f_* \mathcal{O}_A^\times \to \mathrm{R}^2 f_* \underline{\mathbf{Z}}(1)$ can be endowed with a trivialization along the identity section. Since $\varepsilon^* \mathcal{L}$ is locally trivial on S, this is clear (globally replace $\mathcal{L}$ with $\mathcal{L} \otimes f^* \varepsilon^* \mathcal{L}^{-1}$). By Lemma 1.1.2.1(3), we have an isomorphism $\mathrm{R}^1 f_* \mathcal{O}_A / \mathrm{R}^1 f_* \underline{\mathbf{Z}}(1) \xrightarrow{\simeq} \widehat{A}$ as *sheaves on S* and this is *compatible with base change on S*. Thus by (1.3.2.2) we get an isomorphism of sheaves on S

$$\widehat{A} \simeq \underline{\mathrm{Pic}}^0_{A/S}$$

compatible with base change on S. We therefore arrive at an isomorphism $\widehat{A} \simeq \underline{\mathrm{Pic}}^0_{A/S}$ as functors on analytic spaces over S.

In order to define the map $A \to \hat{\hat{A}}$ using $(\mathcal{P}, i_{\widehat{A}})$, we have to check that $\mathcal{P}$ has topologically trivial restriction to all fibers of $p_1 : A \times_S \widehat{A} \to A$. That is, we need to check that the image P_a of each $\mathcal{P}|_{\{a\} \times \widehat{A}}$ in $\mathrm{R}^2 p_{1*} \underline{\mathbf{Z}}(1)_a \simeq \mathrm{H}^2(\widehat{A}_a, \underline{\mathbf{Z}}(1))$ is trivial. By the existence of i_A we know that the image of P_a in the stalk $\mathrm{R}^2 p_{1*}(\underline{\mathbf{Z}}(1))_0$ is trivial. Since $\mathrm{R}^2 p_{1*} \underline{\mathbf{Z}}(1)$ is locally constant and all fibers of $A \to S$ are connected manifolds which meet the identity section, this implies that the image P_a in $\mathrm{R}^2 p_{1*} \underline{\mathbf{Z}}(1)_a$ is trivial, as claimed. Thus, the map $A \to \hat{\hat{A}}$ makes sense.

To show that $A \to \hat{\hat{A}}$ is an isomorphism, Theorem 1.1.1.14 reduces us to checking on fibers. When $S = \mathrm{Sp}\,\mathbf{C}$, both sides are compact complex manifolds, so it suffices to check bijectivity on points. Using group structures and equality of dimensions, it suffices to check injectivity. That is, if $\mathcal{P}_{\{a\} \times \widehat{A}}$ is trivial on $\widehat{A}$, we must show that $a = 0$. This follows from the argument of [**45**, Prop 2, p. 81] (which is the algebraic analogue in characteristic 0 of our desired result) as this algebraic argument easily adapts to our analytic setting (for example, [**45**, Cor 2, p. 74] carries over). $\square$

1.3.3. Compatibility with classical constructions. When $S = \mathrm{Sp}\,\mathbf{C}$ there now arises a question of compatibility with the classical setting. From the exponential sequence

$$0 \to \underline{\mathbf{Z}}(1) \to \mathcal{O}_A \xrightarrow{\exp_A} \mathcal{O}_A^\times \to 1$$

we obtained an analytic isomorphism

$$\mathrm{H}^1(A,\mathcal{O}_A)/\mathrm{H}^1(A,\underline{\mathbf{Z}}(1)) \simeq \underline{\mathrm{Pic}}^0_A \stackrel{\mathrm{def}}{=} \underline{\mathrm{Pic}}^0_{A/\operatorname{Sp}\mathbf{C}},$$

so we get an isomorphism of complex vector spaces

$$\begin{aligned}\mathrm{H}^1(A,\mathcal{O}_A) &\simeq T_0(\underline{\mathrm{Pic}}^0_A)\\ &= \ker\bigl(\underline{\mathrm{Pic}}^0_A(\operatorname{Sp}\mathbf{C}[\varepsilon]) \to \underline{\mathrm{Pic}}^0_A(\operatorname{Sp}\mathbf{C})\bigr)\\ &= \ker\bigl(\mathrm{Pic}^0(A[\varepsilon]) \to \mathrm{Pic}^0(A)\bigr)\\ &= \ker\bigl(\mathrm{Pic}(A[\varepsilon]) \to \mathrm{Pic}(A)\bigr).\end{aligned} \tag{1.3.3.1}$$

Here $\mathbf{C}[\varepsilon] = \mathbf{C}[\varepsilon]/\varepsilon^2$ and $A[\varepsilon] = A \times_{\operatorname{Sp}\mathbf{C}} \operatorname{Sp}\mathbf{C}[\varepsilon]$. But the "algebraic" exponential sequence

$$0 \longrightarrow \mathcal{O}_A \longrightarrow \mathcal{O}^\times_{A[\varepsilon]} \longrightarrow \mathcal{O}^\times_A \longrightarrow 1$$

determined by

$$f \longmapsto 1 + f\varepsilon$$

$$a + b\varepsilon \longmapsto a$$

induces another map

$$\mathrm{H}^1(A,\mathcal{O}_A) \xrightarrow{\simeq} \ker\bigl(\mathrm{Pic}(A[\varepsilon]) \to \mathrm{Pic}(A)\bigr). \tag{1.3.3.2}$$

LEMMA 1.3.3.1. *The maps* (1.3.3.1) *and* (1.3.3.2) *are equal.*

This lemma is very important.

PROOF. Fix a choice of $i = \sqrt{-1} \in \mathbf{C}$, so we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \underline{\mathbf{Z}} \arrow[r] \arrow[d, "2\pi i"] & \mathcal{O}_A \arrow[r, "e^{2\pi i(\cdot)}"] \arrow[d, "2\pi i"] & \mathcal{O}^\times_A \arrow[r] \arrow[d, equal] & 1 \\ 0 \arrow[r] & \underline{\mathbf{Z}}(1) \arrow[r] & \mathcal{O}_A \arrow[r, "e^{(\cdot)}"] & \mathcal{O}^\times_A \arrow[r] & 1 \end{tikzcd} \tag{1.3.3.3}$$

In [**45**, Ch 1, pp. 85–87] a *direct* analytic construction is given for a line bundle $\mathcal{P}$ on $A \times \widehat{A} = A \times (\mathrm{H}^1(A,\mathcal{O}_A)/\mathrm{H}^1(A,\underline{\mathbf{Z}}))$ which is trivial along both zero sections $A \times \{0\}$, $\{0\} \times \widehat{A}$ and whose fibers over the points of $\mathrm{H}^1(A,\mathcal{O}_A)/\mathrm{H}^1(A,\underline{\mathbf{Z}})$ are in bijection with all elements of $\mathrm{Pic}^0(A)$. It follows that the resulting map of compact complex manifolds

$$\mathrm{H}^1(A,\mathcal{O}_A)/\mathrm{H}^1(A,\underline{\mathbf{Z}}) \to \underline{\mathrm{Pic}}^0_A$$

is bijective and thus is an isomorphism. The details of this construction will be reviewed shortly because we will need them to complete the proof.

Mumford compares this construction with the map $\mathrm{H}^1(A,\mathcal{O}_A) \to \mathrm{Pic}(A)$ arising from the exponential sequence

$$0 \to \underline{\mathbf{Z}} \to \mathcal{O}_A \xrightarrow{e^{2\pi i(\cdot)}} \mathcal{O}^\times_A \to 1. \tag{1.3.3.4}$$

By calculating the Čech cocycles for the analytic fiber of Mumford's explicit $\mathcal{P}$ over $A[\varepsilon] \hookrightarrow A \times \underline{\mathrm{Pic}}_A^0$ (via some $v \in T_0(\underline{\mathrm{Pic}}_A^0)$) we claim (using the "experimental error" of $2i$ in [**45**, p. 87]) that the diagram

$$\begin{tikzcd} \mathrm{H}^1(A, \mathcal{O}_A) \arrow[r] \arrow[dr] & T_0(\underline{\mathrm{Pic}}_A^0) \\ & T_0(\underline{\mathrm{Pic}}_A^0) \arrow[u, "2\pi i"'] \end{tikzcd} \tag{1.3.3.5}$$

commutes; here the horizontal map is defined via Mumford's exponential sequence (1.3.3.4) and the diagonal map is defined via the algebraic exponential sequence. Grant this for a moment. From (1.3.3.3), we deduce the commutativity of the diagram

$$\begin{tikzcd} \mathrm{H}^1(A, \mathcal{O}_A) \arrow[r, "\mathrm{Mumford}"] \arrow[d, "2\pi i"] & T_0(\underline{\mathrm{Pic}}_A^0) \arrow[d, equal] \\ \mathrm{H}^1(A, \mathcal{O}_A) \arrow[r, "\mathrm{us}"'] & T_0(\underline{\mathrm{Pic}}_A^0) \end{tikzcd} \tag{1.3.3.6}$$

comparing Mumford's exponential sequence and our sequence exponential. This completes the proof of Lemma 1.3.3.1, conditional on the commutativity of (1.3.3.5).

The verification of the commutativity of (1.3.3.5) is a rather involved exercise with the analytic construction of the universal bundle over the product $A \times \widehat{A}$. To get started, we may (canonically) write $A = V/U$ for the complex vector space $V = T_0(A) = \mathrm{H}^0(A, \Omega^1)^\vee$ and the lattice of full rank $U = \mathrm{H}_1(A, \mathbf{Z})$ embedded in V via the integration map. The dual-conjugate space $\overline{V}^\vee$ is canonically identified with the space of conjugate-linear maps $V \to \mathbf{C}$, and we define $U' \subseteq \overline{V}^\vee$ to be the full rank lattice of elements $\ell \in \overline{V}^\vee$ for which $\mathrm{Im}(\ell(u)) \in \mathbf{Z}$ for all $u \in U$; note this is independent of the choice of $i \in \mathbf{C}$ for defining "Im". Mumford gives an analytic description of a line bundle on $(V/U) \times (\overline{V}^\vee/U')$ which explicitly identifies $\overline{V}^\vee/U'$ as the abstract "dual torus". This analytic construction in [**45**, p. 86] rests on the map

$$\mathrm{H}^1(A, \mathcal{O}_A) \simeq \mathrm{H}^0(A, \Omega_A^{0,1}) \simeq \overline{V}^\vee \simeq T_0(\overline{V}^\vee/U') \tag{1.3.3.7}$$

where the first isomorphism is constructed as in [**45**, p. 11], as we now recall. Any

$$\omega \in \mathrm{H}^0(A, \Omega^{0,1})$$

can locally be written as $\omega|_{U_i} = \overline{\partial} f_i$ for a smooth $\mathbf{C}$-valued function f_i on U_i (with $\{U_i\}$ an ordered open covering of A), and then $\overline{\partial}(f_j - f_i) = 0$ on $U_{ij} = U_i \cap U_j$. Thus, $f_j - f_i$ is a *holomorphic* function on U_{ij} and so these differences for $i < j$ define an element $(f_j - f_i)_{i<j} \in \mathrm{H}^1(\mathfrak{U}, \mathcal{O}_A)$. Using the canonical map

$$\mathrm{H}^1(\mathfrak{U}, \mathcal{O}_A) \to \check{\mathrm{H}}^1(A, \mathcal{O}_A) \simeq \mathrm{H}^1(A, \mathcal{O}_A),$$

this construction defines an element in $\mathrm{H}^1(A, \mathcal{O}_A)$ denoted $\delta(\omega)$, and the inverse of

$$\delta : \mathrm{H}^0(A, \Omega^{0,1}) \to \mathrm{H}^1(A, \mathcal{O}_A)$$

is the first step in (1.3.3.7).

The commutativity of the diagram (1.3.3.5) will be proven by realizing it as the outside edge of a larger diagram whose inner pieces all commute. More specifically, consider the diagram:

(1.3.3.8)
$$\begin{tikzcd}
\mathrm{H}^0(A, \overline{\Omega}^1) \arrow[r, "\delta", "\simeq"'] \arrow[d, equal] & \check{\mathrm{H}}^1(A, \mathcal{O}_A) \arrow[dr, "\text{alg. map}"] & \\
\overline{V}^\vee \arrow[r, "\simeq", "\text{an. map}"'] \arrow[d, "\text{Hodge}"'] & T_0(\widehat{A}) \arrow[dr, "2i"] & T_0(\widehat{A}) \arrow[l, "\pi"'] \arrow[d, "2\pi i"] \\
\mathrm{H}^1(A, \mathcal{O}_A) \arrow[rr, "\simeq", "\text{exp. seq.}"'] & & T_0(\widehat{A})
\end{tikzcd}$$

The outside edge is exactly (1.3.3.5). The lower right triangle trivially commutes. The lower left rectangle compares the analytic model $\overline{V}^\vee/U'$ of $\widehat{A}$ with the analytic exponential sequence on A, and is commutative because of the magical "experimental error" of $2i$ at the bottom of [**45**, p. 87]! Thus, we are reduced to the assertion that the following diagram commutes:

(1.3.3.9)
$$\begin{tikzcd}
\mathrm{H}^0(A, \overline{\Omega}^1_A) \arrow[r, "\delta"] \arrow[d, equal] & \check{\mathrm{H}}^1(A, \mathcal{O}_A) \arrow[r, "\simeq"'] & \check{\mathrm{H}}^1(A, \mathcal{O}^\times_{A[\varepsilon]}) \\
\overline{V}^\vee \arrow[r, equal] & T_0(\overline{V}^\vee/U') \arrow[r, "\simeq", "\text{an.map}/\pi"'] & T_0(\widehat{A}) \arrow[u]
\end{tikzcd}$$

We begin the diagram chase by computing $\delta(\mathrm{d}\overline{z})$, where z is a "linear coordinate" on V (i.e., an element in $V^\vee$) and $\mathrm{d}\overline{z} \in \mathrm{H}^0(V/U, \overline{\Omega}^1_A) = \overline{V}^\vee$ corresponds to the conjugate-linear functional $V \to \mathbf{C}$ defined by

$$v \mapsto \overline{z}(v).$$

Choose a small open A around 0 in V, and cover V/U by translates

$$U_i \stackrel{\text{def}}{=} A + a_i \bmod U$$

with $a_i \in V$. Here, "small" means that A is disjoint from all of its translates by non-zero elements of U. This gives an ordered open covering $\mathfrak{U} = \{U_i\}$ of V/U, so $\mathrm{d}\overline{z}|_{U_i}$ is the "restriction" of $\overline{\partial}(\overline{z}|_{A+a_i})$ to U_i. The Čech cohomology class of $\delta(\mathrm{d}\overline{z})$ is represented by the $\mathcal{O}_A$-valued 1-cocycle (c_{ij}) relative to $\mathfrak{U}$ whose value on U_{ij} (for $i < j$) is $\overline{z}(u_{ij})$ if $A + a_j$ meets $A + a_i + u_{ij}$ for some (necessarily unique!) $u_{ij} \in U$ (i.e., if U_{ij} is non-empty), and whose value on U_{ij} is 0 otherwise. The asymmetry here in i and j is exactly due to the asymmetry in the signs for the Čech differential relative to an ordered open covering. The image of the $\mathcal{O}_A$-valued Čech class of (c_{ij}) in $\mathrm{H}^1(\mathfrak{U}, \mathcal{O}^\times_{A[\varepsilon]})$ is represented by the Čech 1-cocycle whose value on U_{ij} is $1 + \varepsilon c_{ij}$. This computes the top row of (1.3.3.9) on $\mathrm{d}\overline{z} \in \mathrm{H}^0(\mathrm{A}, \overline{\Omega}^1_{\mathrm{A}})$.

Going around (1.3.3.9) the other way will require computing with $\mathrm{Sp}(\mathbf{C}[\varepsilon])$-points of the dual complex torus and hence requires direct analysis of the universal Poincaré bundle on $A \times \widehat{A}$. More specifically, we must compute the fiber of this bundle over the tangent vector $\mathrm{d}\overline{z}$ to the analytically constructed $\widehat{A}$ at the origin. Such a fiber is an element in the kernel

$$\ker(\mathrm{Pic}(A[\varepsilon]) \to \mathrm{Pic}(A)) = \check{\mathrm{H}}^1(A, \mathcal{O}_A[\varepsilon])$$

and we wish to prove that this is represented by the Čech 1-cocycle $(1+\varepsilon\pi c_{ij})$ (so then the extra division by π in the tangent space as in the bottom row of (1.3.3.9) will eliminate this extra factor, leaving us with $1+\varepsilon c_{ij}$, just as we found from the computation of $\delta(\mathrm{d}\overline{z})$).

From the analysis in [**45**, p. 85], the universal line bundle on

$$A \times \widehat{A} \simeq V/U \times \overline{V}^{\vee}/U'$$

over a small open neighborhood $\widetilde{A} \bmod U'$ of 0 in $\overline{V}^{\vee}/U'$ may be computed over $(V/U) \times \widetilde{A}$ to be the quotient

$$(\mathbf{C} \times V \times \widetilde{A})/U$$

where $u \in U$ acts as

$$\phi'_u : (\lambda, v, \overline{\varphi}) \mapsto (\lambda e^{\pi\overline{\varphi}(u)}, u+v, \overline{\varphi}), \tag{1.3.3.10}$$

with $\overline{\varphi} \in \widetilde{A} \subseteq \overline{V}^{\vee}$ denoting the composite of a $\mathbf{C}$-linear $\varphi : V \to \mathbf{C}$ followed by complex conjugation.

Recall our sign convention for the standard isomorphism between the Picard group and the first Čech cohomology of $\mathcal{O}^{\times}$ for any ringed space. Applying this to the above restriction of the Poincaré bundle over $(V/U) \times \widetilde{A}$ yields an $\mathcal{O}^{\times}$-valued Čech class on $(V/U) \times \widetilde{A}$ which is represented by a 1-cocycle relative to the covering $\{U_{ij} \times \widetilde{A}\}$ with value $e^{\pi\varphi(u_{ij})}$ on $U_{ij} \times \widetilde{A}$. This rests on how u_{ij} was *defined* (with $u_{ij} = 0$ if U_{ij} is empty). Recalling the dictionary between classical tangent vectors at the origin and $\mathbf{C}[\varepsilon]$-points, we can compute the Čech data describing the pullback of the Poincaré bundle along the map $A[\varepsilon] \to A \times \widehat{A}$ defined by the tangent vector $\mathrm{d}\overline{z} \in \overline{V}^{\vee} = T_0(\widehat{A})$. This amounts to "plugging in" $\varepsilon\overline{z}(u_{ij})$ in the role of $\overline{\varphi}(u_{ij})$ in the general formula (1.3.3.10). Consequently, we arrive at the transition data

$$e^{\pi(\varepsilon\overline{z})(u_{ij})} = 1 + \pi\overline{z}(u_{ij})\varepsilon = 1 + \pi c_{ij}\varepsilon,$$

exactly as we desired.

□

REMARK 1.3.3.2. The compatibility in this lemma matches exactly with the fact that for the Jacobian $J = J(X)$ of a proper smooth connected curve X over $\mathrm{Spec}(\mathbf{C})$ (or compact Riemann surface) the pairing $T_0(J) \times \mathrm{Cot}_0(J) \to \mathbf{C}$ agrees with the algebraic (or analytic) residue pairing which corresponds to

$$\mathrm{H}^0(X, \Omega^1_X) \times \mathrm{H}^1(X, \mathcal{O}_X) \xrightarrow{\cup} \mathrm{H}^1(X, \Omega^1_X) \overset{\mathrm{res}_X}{\simeq} \mathbf{C}$$

which we explicated in Corollary 1.2.3.1. We refer the reader to [**7**, Appendix B] for an exhaustive discussion of this aspect of duality theory on Jacobians. Theorems B.2.2 and B.4.1 in [**7**] give the above-mentioned compatibility on Jacobians, except Theorem B.4.1 in [**7**] is accidentally missing a minus sign (the computation on the bottom of p. 288 in [**7**] mistakenly mixes up the order of the opens in the covering; correcting this leads to an extra sign, as was noted by Gabber).

A rather important consequence of Theorem 1.3.2.1 is the following analytic analogue of the unobstructedness of deformation theory for abelian schemes.

COROLLARY 1.3.3.3. *Let $A \to S$ be a relative complex torus and let $S \hookrightarrow S'$ be a closed immersion. Locally on S, the complex torus $A \to S$ lifts to a complex torus over S'.*

REMARK 1.3.3.4. If the fibers of $A \to S$ are abelian varieties (i.e., are algebraic), we do *not* claim that the "deformation" over S' can be designed to have this property. In general this is not possible.

PROOF. The construction of $\underline{\mathrm{Pic}}^0_{A/S}$ in terms of Lemma 1.1.2.1 and the isomorphism $A \simeq \hat{\hat{A}} \simeq \underline{\mathrm{Pic}}^0_{\widehat{A}/S}$ shows that locally on S we have $A \simeq V/\Lambda$ for a "cocompact lattice" $\Lambda \hookrightarrow V$ as in Lemma 1.1.2.1. Working locally on S' around a point $s \in S$ we can trivially lift $\Lambda \hookrightarrow V$ to a map $\Lambda' \to V'$ from a rank $2d$ local system of free $\mathbf{Z}$-modules to a rank d vector bundle, since as usual we may assume that $\Lambda \hookrightarrow V$ is $\mathbf{Z}^{2d} \times S \hookrightarrow \mathbf{C}^d \times S$ given by some functions $f_{ij} \in \mathrm{H}^0(S, \mathcal{O}_S)$, $1 \le i \le 2d$, $1 \le j \le d$, and we can then lift these to $f'_{ij} \in \mathrm{H}^0(S', \mathcal{O}_{S'})$.

The fact that the (f'_{ij}) define a cocompact sublattice at a point $s' \in S'$ is equivalent to the condition that the $2d \times 2d$ real matrix $(\mathrm{Re}(f_{ij}), \mathrm{Im}(f_{ij}))$ has non-vanishing determinant. This is true along S, hence also near S in S'. Since this is precisely the condition that $\Lambda' \to V'$ yields a cocompact lattice on fibers, $V'/\Lambda' \to S'$ is a relative complex torus, as desired. We saw this same argument in the proof of Lemma 1.1.2.1. $\square$

The importance of Corollary 1.3.3.3 is that it allows many base-change compatibility questions about relative complex tori to be reduced to the setting of a smooth base. For example, we can relativize Lemma 1.3.3.1: the analytic isomorphism of abelian sheaves

$$\begin{aligned} \mathrm{R}^1 f_* \mathcal{O}_A &\simeq \ker(\underline{\mathrm{Pic}}^0_{A[\varepsilon]/S[\varepsilon]} \to \underline{\mathrm{Pic}}^0_{A/S}) \\ &= \ker(\mathrm{R}^1 f_{\varepsilon *}(\mathcal{O}^\times_{A[\varepsilon]}) \to \mathrm{R}^1 f_*(\mathcal{O}^\times_A)) \end{aligned}$$

is equal to the algebraic isomorphism

$$\mathrm{R}^1 f_* \mathcal{O}_A \simeq \ker(\mathrm{R}^1 f_{\varepsilon *}(\mathcal{O}^\times_{A[\varepsilon]}) \to \mathrm{R}^1 f_*(\mathcal{O}^\times_A)).$$

Indeed, by Corollary 1.3.3.3 we may reduce to the case of a smooth (or even just reduced) base, where we can check on fibers, using Lemma 1.3.3.1.

1.3.4. The relative exponential map. Another application of Corollary 1.3.3.3 is the following theorem, which generalizes the exponential sequence in [**10**, §2.1] to the case of arbitrary relative dinmension.

THEOREM 1.3.4.1. *There is a unique way to define a map of group objects*

$$\exp_{A/S} : (e^*\Omega^1_{A/S})^\vee \to A$$

for relative complex tori $A \to S$ in a manner compatible with base change such that we recover the exponential map (1.3.1.1) *when $S = \mathrm{Sp}\,\mathbf{C}$.*

PROOF. The fiber condition forces uniqueness when S is reduced (for example, smooth) and base change compatibility then force uniqueness in general via Corollary 1.3.3.3. Thus, we are left with a question of existence. If $\widehat{f} : \widehat{A} \to S$ is the structure map of the dual of A, then the isomorphism $A \simeq \hat{\hat{A}}$ defines an isomorphism of complex tori

$$A \simeq \underline{\mathrm{Pic}}^0_{\widehat{A}/S} \simeq \mathrm{R}^1 \widehat{f}_* \mathcal{O}_{\widehat{A}} / \mathrm{R}^1 \widehat{f}_* \underline{\mathbf{Z}}(1)$$

compatible with base change on S. Thus, we get an isomorphism by pullback to the tangent space along the zero section

$$(e^*\Omega^1_{A/S})^\vee \simeq \mathrm{R}^1\widehat{f}_*\mathcal{O}_{\widehat{A}}$$

compatible with *base change on* S. We therefore arrive at a map of group objects

$$(e^*\Omega^1_{A/S})^\vee \to A$$

inducing the identity on tangent spaces along the zero section, and hence inducing the classical exponential map on the fibers. ☐

Consider the $\mathcal{O}_S$-linear edge map

$$f_*\Omega^1_{A/S} \to \mathrm{R}^1 f_* f^{-1}\mathcal{O}_S \simeq \mathrm{R}^1 f_*\underline{\mathbf{C}} \otimes_{\underline{\mathbf{C}}} \mathcal{O}_S$$

coming from the (relative) Hodge to de Rham spectral sequence and (1.2.2.4). The $\mathcal{O}_S$-dual map defines a map of abelian sheaves

$$(\mathrm{R}^1 f_*\underline{\mathbf{Z}})^\vee \to (\mathrm{R}^1 f_*\underline{\mathbf{C}})^\vee \to (\mathrm{R}^1 f_*\underline{\mathbf{C}})^\vee \otimes_{\underline{\mathbf{C}}} \mathcal{O}_S \to (f_*\Omega^1_{A/S})^\vee \tag{1.3.4.1}$$

compatible with base change on S, where the first map is extension of scalars by $\underline{\mathbf{Z}} \hookrightarrow \underline{\mathbf{C}}$. On fibers this induces a map

$$\mathrm{H}_1(A_s, \mathbf{Z}) \xrightarrow{\simeq} \mathrm{H}^1(A_s, \underline{\mathbf{Z}})^\vee \to \mathrm{H}^0(A_s, \Omega^1_{A_s})^\vee \tag{1.3.4.2}$$

which is given by integration, since that is how the isomorphism between de Rham and topological cohomology is defined. But the classical exponential map

$$\mathrm{H}^0(A_s, \Omega^1_{A_s})^\vee \simeq T_0(A_s) \twoheadrightarrow A_s$$

has kernel given by exactly this map (1.3.4.2), so (1.3.4.2) is an identification with a cocompact lattice. Thus, (1.3.4.1) induces a cocompact lattice on fibers. In fact, we have the following result.

THEOREM 1.3.4.2. *The kernel of*

$$(f_*\Omega^1_{A/S})^\vee \simeq (e^*\Omega^1_{A/S})^\vee \overset{\exp_{A/S}}{\twoheadrightarrow} A$$

is exactly $(\mathrm{R}^1 f_*\underline{\mathbf{Z}})^\vee$ *via the map* (1.3.4.1)*. This is all compatible with base change on* S*.*

PROOF. The base change compatibility follows from the fact that we've checked it all along. By Lemma 1.1.2.1(3) and Corollary 1.3.3.3 we may assume that S is reduced (or even smooth), so the composite

$$(\mathrm{R}^1 f_*\underline{\mathbf{Z}})^\vee \to A$$

vanishes (as it vanishes on fibers over S).

We therefore get a map $(\mathrm{R}^1 f_*\underline{\mathbf{Z}})^\vee \to \ker(\exp_{A/S})$ between local systems of finite free $\mathbf{Z}$-modules, both of which are analytic covering spaces over S. The discussion preceding the theorem shows that this is an isomorphism on fibers, so it is an isomorphism. ☐

1.3.5. The category of tori. Fix $g \geq 1$. Consider the additive categories C_1, C_2 consisting respectively of complex tori $f : A \to S$ of relative dimension g over S and of triples (Λ, V, ι) where $\Lambda \to S$ is a rank $2g$ local system of finite free $\mathbf{Z}$-modules, $V \to S$ is a rank g vector bundle, and $\iota : \Lambda \hookrightarrow V$ induces a cocompact lattice on fibers. If we're given (Λ, V, ι) and let $f : V/\Lambda \to S$ denote the associated analytic space, we get a functor $T' : C_2 \to C_1$. Likewise, if we're given $f : A \to S$ then $(\mathrm{R}^1 f_* \underline{\mathbf{Z}})^\vee \hookrightarrow (f_* \Omega^1_{A/S})^\vee$ gives a functor $T : C_1 \to C_2$. Using $\exp_{A/S} : (f_*\Omega^1_{A/S})^\vee / (\mathrm{R}^1 f_* \underline{\mathbf{Z}})^\vee \simeq A$, we have functorially $T' \circ T \simeq \mathrm{id}$.

Meanwhile, for $f : V/\Lambda \to S$ we claim that there is a unique functorial isomorphism $\Lambda \simeq (\mathrm{R}^1 f_* \underline{\mathbf{Z}})^\vee$ inducing the canonical isomorphism $\Lambda_s \simeq \mathrm{H}_1(V_s/\Lambda_s, \mathbf{Z})$ on fibers. As both Λ and $(\mathrm{R}^1 f_* \underline{\mathbf{Z}})^\vee$ are analytic covering spaces over S, this is a purely topological question and uniqueness is forced, so we can work locally. Thus we may (via a continuous isomorphism) assume $\Lambda \hookrightarrow V$ is $\mathbf{Z}^{2d} \times S \hookrightarrow \mathbf{R}^{2d} \times S$ via the canonical inclusion $\mathbf{Z} \hookrightarrow \mathbf{R}$, for which the assertion is clear. Functoriality in $f : V/\Lambda \to S$ of the isomorphism $\Lambda \simeq (\mathrm{R}^1 f_* \underline{\mathbf{Z}})^\vee$ can be checked on fibers, where it is clear. We have now proven the following key result.

THEOREM 1.3.5.1. *The functors $T : C_1 \to C_2$ and $T' : C_2 \to C_1$ are naturally quasi-inverses in a manner compatible with base change on S.*

This makes precise the sense in which the constructions in §1.1.2 is of universal nature. Our aim is to construct a universal "principally polarized" relative dimension g relative complex torus with an "i-permissible trivialization" (generalizing §1.2.4) over a Siegel upper half-space. This will recover $\mathcal{E}^{\mathrm{an}} \to \mathfrak{h}$ when $g = 1$, finally realizing our goal of finding the universal property of Example 1.1.1.16. Before carrying this out, it will be convenient to consider a related, but more concrete, moduli problem.

For any complex analytic space S, let $\underline{\mathrm{Incl}}(\mathbf{Z}^{2g}, \mathbf{C}^g)(S)$ denote the set

$$\{\mathbf{Z}^{2g} \times S \overset{\iota}{\hookrightarrow} \mathbf{C}^g \times S \text{ a closed immersion inducing a cocompact lattice on fibers}\}.$$

The functor $\underline{\mathrm{Incl}}(\mathbf{Z}^{2g}, \mathbf{C}^g)$ is represented by an open subset of the $2g$-fold product of $\mathbf{C}^g$. Indeed, any S-group map $\mathbf{Z}^{2g} \times S \to \mathbf{C}^g \times S$ is given by a collection of $2g$ g-tuples $(f_{i1}, \dots, f_{ig})$ of elements in $\mathrm{H}^0(S, \mathcal{O}_S)$, so we get a map of functors $\underline{\mathrm{Incl}}(\mathbf{Z}^{2g}, \mathbf{C}^g) \to (\mathbf{C}^g)^{2g}$ which factors through the open subset $U \subseteq (\mathbf{C}^g)^{2g}$ of elements (f_{ij}) for which $\det(\mathrm{Re}(f_{ij}), \mathrm{Im}(f_{ij})) \neq 0$. The map $\underline{\mathrm{Incl}}(\mathbf{Z}^{2g}, \mathbf{C}^g) \to U$ is an isomorphism because *any* S-group map $\mathbf{Z}^{2g} \times S \to \mathbf{C}^g \times S$ inducing a cocompact lattice on fibers is *automatically* a closed immersion, for reasons noted in the proof of Lemma 1.1.2.1(2). From now on, we will always consider $\underline{\mathrm{Incl}}(\mathbf{Z}^{2g}, \mathbf{C}^g)$ as a complex analytic space via this isomorphism with U. When $g = 1$, the map $\iota \mapsto -\iota(0,1)/\iota(1,0)$ sets up an isomorphism of $\underline{\mathrm{Incl}}(\mathbf{Z}^2, \mathbf{C})$ with $\mathbf{C} - \mathbf{R}$, where the component $\mathfrak{h}$ corresponds to maps $\mathbf{Z}^2 \hookrightarrow \mathbf{C}$ which induce an $\mathbf{R}$-linear isomorphism $\mathbf{R}^2 \simeq \mathbf{C}$ that is orientation reversing with respect to the i-orientation on $\mathbf{C}$ (it will become clear later why we include the sign in the formula $-\iota(0,1)/\iota(1,0)$). The component of such orientation-reversing maps is denoted $\mathrm{Hom}^+(\mathbf{R}^2, \mathbf{C})$ in [**10**, §2.1] and is Deligne's formulation of the classical upper half-space in the theory. We will generalize this component to a Siegel upper half-space after first studying $\underline{\mathrm{Incl}}(\mathbf{Z}^{2g}, \mathbf{C}^g)$ more carefully.

Let F denote the functor associating to any S the set of commutative diagrams of S-group objects

$$\begin{tikzcd} \mathbf{Z}^{2g} \times S \arrow[r, hook] \arrow[d, "\simeq"] & \mathbf{C}^g \times S \arrow[d, "\simeq"] \\ \mathbf{Z}^{2g} \times S \arrow[r, hook] & \mathbf{C}^g \times S \end{tikzcd} \tag{1.3.5.1}$$

with the two horizontal maps in $\underline{\mathrm{Incl}}(\mathbf{Z}^{2g}, \mathbf{C}^g)(S)$. By ignoring the right column we get a map of functors

$$F \to \underline{\mathrm{Incl}}(\mathbf{Z}^{2g}, \mathbf{C}^g) \times_S \underline{\mathrm{Incl}}(\mathbf{Z}^{2g}, \mathbf{C}^g) \times \mathrm{GL}_{2g}(\mathbf{Z}), \tag{1.3.5.2}$$

since $\underline{\mathrm{Aut}}_S(\mathbf{Z}^{2g} \times S)$ is represented by $S \times \mathrm{GL}_{2g}(\mathbf{Z})$, where $\mathrm{GL}_{2g}(\mathbf{Z})$ is *discrete*. The next lemma gives an essential fact about this map.

LEMMA 1.3.5.2. *The map* (1.3.5.2) *is a closed immersion; that is, F is represented by a closed analytic subspace of the right side of* (1.3.5.2).

PROOF. Fixing a matrix $M \in \mathrm{GL}_{2g}(\mathbf{Z})$, it is clear that we may restrict our attention to

$$F_M \to \underline{\mathrm{Incl}}(\mathbf{Z}^{2g}, \mathbf{C}^g) \times \underline{\mathrm{Incl}}(\mathbf{Z}^{2g}, \mathbf{C}^g) \tag{1.3.5.3}$$

where F_M is the open/closed subfunctor of F_M for which $\mathbf{Z}^{2g} \times S \to \mathbf{Z}^{2g} \times S$ is given by M.

First we claim that (1.3.5.3) is monic. If we let $\mathrm{GL}_g(\mathbf{C})$ act on the functor $\underline{\mathrm{Incl}}(\mathbf{Z}^{2g}, \mathbf{C}^g)$ in the obvious way, monicity is equivalent to checking that this $\mathrm{GL}_{2g}(\mathbf{C})$ action has no "fixed points" on $\underline{\mathrm{Incl}}(\mathbf{Z}^{2g}, \mathbf{C}^g)$, since the map (1.3.5.3) is defined by "ignoring" the right side of (1.3.5.1). In other words, we want the commutative diagram

$$\begin{tikzcd} \mathrm{GL}_g(\mathbf{C}) \times \underline{\mathrm{Incl}}(\mathbf{Z}^{2g}, \mathbf{C}^g) \arrow[r, "\pi"] & \underline{\mathrm{Incl}}(\mathbf{Z}^{2g}, \mathbf{C}^g) \times \underline{\mathrm{Incl}}(\mathbf{Z}^{2g}, \mathbf{C}^g) \\ \underline{\mathrm{Incl}}(\mathbf{Z}^{2g}, \mathbf{C}^g) \arrow[u, "1\times\mathrm{id}"] \arrow[r, equal] & \underline{\mathrm{Incl}}(\mathbf{Z}^{2g}, \mathbf{C}^g) \arrow[u, "\Delta"'] \end{tikzcd}$$

to be cartesian, where $\pi(g, \iota) = (g \circ \iota, \iota)$ and Δ is the diagonal map. On geometric points this is clear, and since the columns are closed immersions (being sections to separated maps) we get a bijective closed immersion

$$\underline{\mathrm{Incl}}(\mathbf{Z}^{2g}, \mathbf{C}^g) \to Z \tag{1.3.5.4}$$

where Z is the cartesian product of π and Δ. By the Krull intersection theorem, to check that (1.3.5.4) is an isomorphism, it suffices to check that $\underline{\mathrm{Incl}}(\mathbf{Z}^{2g}, \mathbf{C}^g) \to Z$ is bijective on points with values in finite local $\mathbf{C}$-algebras $(A, \mathfrak{m})$. That is, for any such A and any $\iota \in \underline{\mathrm{Incl}}(\mathbf{Z}^{2g}, \mathbf{C}^g)(\mathrm{Sp}\, A)$ fixed by some $T \in \mathrm{GL}_d(A)$, we must show that $T = 1$. Clearly $T \equiv 1 \pmod{\mathfrak{m}}$ since we already know the injectivity of (1.3.5.3) on geometric points. Viewing ι as an inclusion $\Lambda \hookrightarrow V$, we see that T induces an automorphism of the relative complex torus V/Λ over $\mathrm{Sp}\, A$, with this automorphism reducing to the identity modulo $\mathfrak{m}$. For reasons observed in the proof of Theorem 1.1.1.15, T must be the identity on V/Λ. Passing to the tangent space along the identity section we conclude that $T = 1$. This completes the proof that (1.3.5.2) is monic.

It remains to show that for any map $T \to S$ of analytic spaces, the pullback to T of an element of

$$\underline{\mathrm{Incl}}(\mathbf{Z}^{2g}, \mathbf{C}^g)(S) \times \underline{\mathrm{Incl}}(\mathbf{Z}^{2g}, \mathbf{C}^g)(S)$$

lies in the subset $F_M(T)$ if and only if $T \to S$ factors through some fixed closed analytic subspace of S. In other words, we must show that for any $\iota_1, \iota_2 \in \underline{\mathrm{Incl}}(\mathbf{Z}^{2g}, \mathbf{C}^g)(S)$ and $M \in \mathrm{GL}_{2g}(\mathbf{Z})$, the condition that ι_1 and $\iota_2 \circ M$ become $\mathrm{GL}_g(T)$ conjugate over some $T \to S$ is represented by a closed analytic subspace of S. One sees immediately that for $T = \mathrm{Sp}\,\mathbf{C} \to S$, $(\iota_1)_T$ is $\mathrm{GL}_g(\mathbf{C})$-conjugate to $(\iota_2)_T$ if and only if the $\mathbf{R}$-linear automorphism of $\mathbf{R}^{2g}$ induced by

$$\mathbf{Z}^{2g} \overset{(\iota_1)_T}{\hookrightarrow} \mathbf{C}^g \overset{(\iota_2)_T}{\hookleftarrow} \mathbf{Z}^{2g}$$

is $\mathbf{C}$-linear. This amounts to some linear conditions on $\mathrm{Re}(\iota_j)$'s and $\mathrm{Im}(\iota_j)$'s, and so says exactly that the map on topological spaces $T \to S$ factors through some universal *closed* subset of S. Thus, we need only work in neighborhoods of points $s_0 \in S$ where $(\iota_1)_{s_0}$ and $(\iota_2)_{s_0}$ are $\mathrm{GL}_g(\mathbf{C})$ conjugate. Given such an s_0, we may globally conjugate so that in fact $(\iota_1)_{s_0} = (\iota_2)_{s_0}$ and we can rename the basis vectors of $\mathbf{Z}^{2g}$ so that the images under each $(\iota_j)_{s_0}$ of the first g basis vectors $e_1, \dots, e_g$ of $\mathbf{Z}^{2g}$ yields a $\mathbf{C}$-basis of $\mathbf{C}^g$.

We can shrink S around s_0 so as to assume that the images under $(\iota_1)_s$ and $(\iota_2)_s$ of $e_1, \dots, e_g$ form a basis of $\mathbf{C}^g$ for all $s \in S$. Thus, ι_1 and ι_2 yield *isomorphisms* of holomorphic vector bundles

$$\mathbf{C}^g \times S \xrightarrow{\alpha_1} \mathbf{C}^g \times S$$

$$\mathbf{C}^g \times S \xrightarrow{\alpha_2} \mathbf{C}^g \times S$$

using $e_1, \dots, e_g$ as bases on the left-hand sides, so there are explicit linear formulas (with coefficients in $\mathrm{H}^0(S, \mathcal{O}_S)$) expressing each of $\iota_j(e_{g+1}), \dots, \iota_j(e_{2g})$ in terms of $\{\iota_j(e_1), \dots, \iota_j(e_g)\}$ for each j.

With the above simplifications, we now check that the existence of $h \in \mathrm{GL}_g(T)$ taking ι_1 over to ι_2 is a closed analytic condition on S. Any such h *must* be $h = (\alpha_2 \circ \alpha_1^{-1})_T$, and in order to actually take ι_1 over to ι_2 it must take $\iota_1(e_j)$ to $\iota_2(e_j)$ for all $g+1 \le j \le 2g$. This is clearly the closed analytic condition on S cut out by universal formulas in terms of the "matrix coefficients" of α_1, α_2 and the coefficients of the $\iota_1(e_j)$, $\iota_2(e_j)$ with respect to the bases $\iota_1(e_1), \dots, \iota_1(e_g)$ and $\iota_2(e_1), \dots, \iota_2(e_g)$ respectively. $\square$

The significance of Lemma 1.3.5.2 is that it ensures that we can study diagrams of the form

$$\begin{tikzcd}
\mathbf{Z}^{2g} \times S \arrow[r, hook] \arrow[d, "\simeq"] & \mathbf{C}^g \times S \arrow[d, "\simeq"] \\
\mathbf{Z}^{2g} \times S \arrow[r, hook] & \mathbf{C}^g \times S
\end{tikzcd}$$

by "ignoring" the right column, subject to imposing some universal closed analytic condition on S.

We conclude this section with the following immediate consequence [**10**, Prop 2.2(i)] of Theorem 1.3.5.1.

Corollary 1.3.5.3. *Fix $g \ge 1$. The space $\underline{\mathrm{Incl}}(\mathbf{Z}^{2g}, \mathbf{C}^g)$ represents the functor on analytic spaces which associates to S the set of isomorphism classes of triples*

(f, φ, α) *consisting of a relative complex torus* $f : A \to S$ *of relative dimension* g*, an* $\mathcal{O}_S$*-linear isomorphism* $\varphi : f_*\Omega^1_{A/S} \simeq \mathcal{O}_S^{\oplus g}$*, and an isomorphism* $\alpha : \mathrm{R}^1 f_* \underline{\mathbf{Z}} \simeq \underline{\mathbf{Z}}^{2g}$*.*

As we mentioned earlier, when $g = 1$ Deligne works with a connected component of $\underline{\mathrm{Incl}}(\mathbf{Z}^2, \mathbf{C})$ which he denotes $\underline{\mathrm{Hom}}^+(\mathbf{R}^2, \mathbf{C})$. We don't understand the geometric category in which to work with $\mathbf{R}^2 \times S$ for non-reduced S, so we avoid Deligne's $\mathrm{Hom}^+(\mathbf{R}^2, \mathbf{C})$ notation.

1.4. Moduli spaces of principally polarized relative complex tori

1.4.1. Principal polarizations. Fix an integer $g \geq 1$. We want to prove [**10**, Prop 2.2] in arbitrary relative dimension, realizing the Siegel upper half-space $\mathfrak{h}_g \subseteq \mathrm{GL}_g(\mathbf{C})$ as the base of a suitable universal analytic family of relative complex tori of relative dimension g. For $g = 1$ this will yield the sought-after universal property of $\mathcal{E}^{\mathrm{an}} \to \mathfrak{h}$, from which we will recover the analytic theory of modular curves.

For $g = 1$, the classical theory tells us that classical elliptic curves admit a *unique* principal polarization, namely the usual canonical isomorphism $E \simeq \mathrm{Pic}^0_E$ defined by $x \mapsto \mathcal{O}(x) \otimes \mathcal{O}(e)^{-1}$. To handle the case of $g > 1$, we need to first discuss principal polarizations in a relative analytic setting.

Let $f : A \to S$ be a relative complex torus of relative dimension $g \geq 1$, with $\widehat{A} = \mathrm{R}^1 f_* \mathcal{O}_A / \mathrm{R}^1 f_* \underline{\mathbf{Z}}(1)$ its dual. We want to study those A which are self-dual in a rather special way.

We begin by first considering an arbitrary S-isomorphism of tori $\varphi : A \xrightarrow{\simeq} \widehat{A}$. By Theorem 1.3.5.1, to give φ is equivalent to giving a commutative diagram over S

$$\begin{tikzcd} (f_*\Omega^1_{A/S})^\vee \arrow[r, "\simeq"] & \mathrm{R}^1 f_* \mathcal{O}_A \\ (\mathrm{R}^1 f_* \underline{\mathbf{Z}})^\vee \arrow[r, "\simeq"] \arrow[u, hook] & \mathrm{R}^1 f_* \underline{\mathbf{Z}}(1) \arrow[u, hook] \end{tikzcd} \tag{1.4.1.1}$$

where the columns are the canonical maps. We *do not* try to relativize the classical $\mathbf{C}$-linear isomorphism

$$\overline{\mathrm{H}^0(A_s, \Omega^1_{A_s})} \simeq \mathrm{H}^1(A_s, \mathcal{O}_{A_s}) \tag{1.4.1.2}$$

because complex conjugation does *not* act on the $\underline{\mathbf{R}}$-module

$$\mathrm{R}^1 f_*(f^{-1}\mathcal{O}_S) \simeq \mathrm{R}^1 f_*(\underline{\mathbf{C}}) \otimes_{\underline{\mathbf{C}}} \mathcal{O}_S$$

when S is not $\mathrm{Sp}\,\mathbf{C}$; this is analogous to the fact that there is no splitting of the relative Hodge filtration in general. We will nevertheless frequently use the isomorphism (1.4.1.2) on fibers below.

Since the columns in (1.4.1.1) are cocompact lattices, by Lemma 1.3.5.2 the data of φ is completely encoded in the bottom row of (1.4.1.1). This row can be viewed as a perfect pairing

$$(\mathrm{R}^1 f_* \underline{\mathbf{Z}})^\vee \otimes (\mathrm{R}^1 f_* \underline{\mathbf{Z}})^\vee \xrightarrow{\simeq} (\mathrm{R}^1 f_* \underline{\mathbf{Z}})^\vee \otimes \mathrm{R}^1 f_* \underline{\mathbf{Z}}(1) \to \underline{\mathbf{Z}}(1), \tag{1.4.1.3}$$

with the last map the canonical pairing. This pairing is compatible with base change on S, and on fibers it gives a perfect pairing

$$E_s : \mathrm{H}_1(A_s, \mathbf{Z}) \times \mathrm{H}_1(A_s, \mathbf{Z}) \to \mathbf{Z}(1). \tag{1.4.1.4}$$

Meanwhile, the top row in (1.4.1.1) gives a perfect pairing

$$(f_*\Omega^1_{A/S})^\vee \otimes (\mathrm{R}^1 f_*\mathcal{O}_A)^\vee \to \mathcal{O}_S$$

compatible with base change on S. Thus, on fibers we get a perfect pairing

$$\begin{aligned}\mathrm{H}^0(A_s,\Omega^1_{A_s})^\vee \otimes \overline{\mathrm{H}^0(A_s,\Omega^1_{A_s})^\vee} &\simeq \mathrm{H}^0(A_s,\Omega^1_{A_s})^\vee \otimes \overline{\mathrm{H}^0(A_s,\Omega^1_{A_s})}^\vee \\ &\simeq \mathrm{H}^0(A_s,\Omega^1_{A_s})^\vee \otimes \mathrm{H}^1(A_s,\mathcal{O}_{A_s})^\vee \\ &\to \mathbf{C};\end{aligned}$$

in which the second isomorphism comes from (1.4.1.2). We can view this as a perfect (right semi-linear) pairing

$$B_s : \mathrm{H}^0(A_s,\Omega^1_{A_s})^\vee \times \mathrm{H}^0(A_s,\Omega^1_{A_s})^\vee \to \mathbf{C}. \tag{1.4.1.5}$$

Using the canonical embedding

$$\int : \mathrm{H}_1(A_s,\mathbf{Z}) \hookrightarrow \mathrm{H}^0(A_s,\Omega^1_{A_s})^\vee,$$

for $s \in S$, what can we say about the diagram

$$\begin{tikzcd} \mathrm{H}^0(A_s,\Omega^1_{A_s})^\vee \times \mathrm{H}^0(A_s,\Omega^1_{A_s})^\vee \arrow[r, "B_s"] & \mathbf{C} \arrow[dr, "\overline{z}-z"] & \\ & & \mathbf{R}(1) \\ \mathrm{H}_1(A_s,\mathbf{Z}) \times \mathrm{H}_1(A_s,\mathbf{Z}) \arrow[uu, "\int\times\int"] \arrow[r, "E_s"] & \mathbf{Z}(1) \arrow[ur, hook] & \end{tikzcd} \tag{1.4.1.6}$$

relating the pairings E_s and B_s?

THEOREM 1.4.1.1. *The diagram* (1.4.1.6) *commutes.*

PROOF. Since everything is compatible with base change, we may assume that $S = \mathrm{Sp}\,\mathbf{C}$. Thus, A is a classical complex torus.

By the *definition* of the Hodge isomorphism

$$\overline{\mathrm{H}^0(A,\Omega^1_A)} \xrightarrow{\simeq} \mathrm{H}^1(A,\mathcal{O}_A)$$

in (1.4.1.2), we have a commutative diagram of groups

$$\begin{tikzcd} \mathrm{H}^1(A,\mathcal{O}_A) & \overline{\mathrm{H}^0(A,\Omega^1_A)} \arrow[l, "\simeq"'] \arrow[r, "\simeq", "\text{conj.}"'] & \mathrm{H}^0(A,\Omega^1_A) \\ \mathrm{H}^1(A,\underline{\mathbf{Z}}(1)) \arrow[u] \arrow[r, equal] & \mathrm{H}^1(A,\underline{\mathbf{Z}}(1)) \arrow[u] \arrow[r, equal] & \mathrm{H}^1(A,\underline{\mathbf{Z}}(1)) \arrow[u] \end{tikzcd}$$

where the left column is induced by $\underline{\mathbf{Z}}(1) \hookrightarrow \underline{\mathbf{C}} \hookrightarrow \mathcal{O}_A$, the middle column sends an element of $\mathrm{H}^1(A,\underline{\mathbf{Z}}(1))$ (regarded as a functional $\sigma \mapsto \varphi(\sigma) \in \mathbf{Z}(1)$ on $\mathrm{H}_1(A,\mathbf{Z})$) to $-\overline{\omega}_\varphi$ where

$$\varphi(\sigma) = \int_\sigma (\omega_\varphi - \overline{\omega}_\varphi)$$

(i.e., $\omega_\varphi - \overline{\omega}_\varphi$ is the Hodge decomposition of

$$\varphi \in \mathrm{H}^1(A,\mathbf{Z}(1)) \subseteq \mathrm{H}^1(A,\mathbf{R}(1)) = \mathrm{H}^1(A,\mathbf{C})^{\mathrm{conj}=-1}),$$

and the right column sends φ to $-\omega_\varphi$. In other words, the right-semilinear map

$$B : \mathrm{H}^0(A, \Omega^1_A)^\vee \times \mathrm{H}^0(A, \Omega^1_A)^\vee \to \mathbf{C}$$

as in (1.4.1.4) is constructed from a semilinear isomorphism on the top row of the commutative square

$$\begin{tikzcd}
\mathrm{H}^0(A, \Omega^1_A)^\vee \arrow[r, "T", "\simeq"'] & \mathrm{H}^0(A, \Omega^1_A) & -\omega_\varphi \\
\mathrm{H}_1(A, \mathbf{Z}) \arrow[u, "\int"] \arrow[r, "t", "\simeq"'] & \mathrm{H}^1(A, \underline{\mathbf{Z}}(1)) \arrow[u] & \varphi \arrow[u, mapsto]
\end{tikzcd}$$

whose bottom row defines

$$E : \mathrm{H}_1(A, \mathbf{Z}) \times \mathrm{H}_1(A, \mathbf{Z}) \xrightarrow{1\times t} \mathrm{H}_1(A, \mathbf{Z}) \times \mathrm{H}^1(A, \underline{\mathbf{Z}}(1)) \to \mathbf{Z}(1)$$

as in (1.4.1.5), where the last map here is the canonical pairing. Thus, $B(\rho_1, \rho_2) = \rho_1(T(\rho_2))$, so for $\sigma, \tau \in \mathrm{H}_1(A, \mathbf{Z})$ we have

$$B\left(\textstyle\int_\sigma, \int_\tau\right) = \int_\sigma T\left(\textstyle\int_\tau\right) = \int_\sigma -\omega_{t(\tau)}.$$

Meanwhile,

$$E(\sigma, \tau) = t(\tau)(\sigma) = \int_\sigma \omega_{t(\tau)} - \overline{\omega}_{t(\tau)} = -B\left(\textstyle\int_\sigma, \int_\tau\right) + \overline{B\left(\textstyle\int_\sigma, \int_\tau\right)}$$

which is the statement of the theorem. □

For any line bundle $\mathcal{L}$ on a classical complex torus A, we get by functoriality a unique map $\phi_{\mathcal{L}} : A \to \widehat{A}$ satisfying $\phi_{\mathcal{L}}(x) = t_x^*\mathcal{L} \otimes \mathcal{L}^{-1}$ on (geometric) points, where $t_x : A \to A$ is given by $y \mapsto y + x$. A *principal polarization* of A is such a map $\phi_{\mathcal{L}} : A \to \widehat{A}$ which is an *isomorphism* and for which $\mathcal{L}$ is *ample* (so that in particular A is an abelian variety).

Definition 1.4.1.2. A *principal polarization* of a relative complex torus is an isomorphism $A \xrightarrow{\simeq} \widehat{A}$ which induces a (classical) principal polarization on fibers. A *principally polarized relative complex torus* $A \to S$ is a relative complex torus together with a choice of principal polarization $A \xrightarrow{\simeq} \widehat{A}$.

Our aim is to give a family of principally polarized relative complex tori of fixed relative dimension which is universal for the property of being equipped with a "relative homology basis" of a special type. In the case of relative dimension 1, it will turn out that $\mathcal{E}^{\mathrm{an}} \to \mathfrak{h}$ with the structure (1.2.1.6) is such a family. The "invisibility" of the principal polarization data in the case of relative dimension 1 is due to the fact that 1-dimensional complex tori are always algebraic and admit a *unique* principal polarization, namely the canonical isomorphism $E \simeq \mathrm{Pic}^0_{E/S}$ sending $x \in E(T)$ to $\mathcal{O}(x) \otimes \mathcal{O}(e_T)^{-1} \in \mathrm{Pic}^0(E_T/T) = \underline{\mathrm{Pic}}^0_{E/S}(T)$ (this is the map from a pointed curve to its Jacobian, which is an isomorphism for genus 1 curves).

1.4.2. Hermitian pairings. Let $\varphi : A \xrightarrow{\simeq} \widehat{A}$ be an isomorphism of classical complex tori and let $E_\varphi : \mathrm{H}_1(A, \mathbf{Z}) \times \mathrm{H}_1(A, \mathbf{Z}) \to \mathbf{Z}(1)$ be the resulting perfect pairing from (1.4.1.3). By Theorem 1.4.1.1, the map

$$\mathrm{H}^0(A, \Omega^1_A)^\vee \times \mathrm{H}^0(A, \Omega^1_A)^\vee \simeq \left(\mathbf{R} \otimes \mathrm{H}_1(A, \mathbf{Z})\right) \times \left(\mathbf{R} \otimes \mathrm{H}_1(A, \mathbf{Z})\right) \xrightarrow{1\otimes E_\varphi} \mathbf{R}(1) \hookrightarrow \mathbf{C}$$

is (up to a factor of $-1/2$) the "imaginary part" of a pairing B_φ which is $\mathbf{C}$-linear on the left and semi-linear on the right. Such a B_φ is uniquely determined by E_φ. In terms of the complex structure on $\mathbf{R} \otimes \mathrm{H}_1(A, \mathbf{Z}) \simeq \mathrm{H}^0(A, \Omega^1_A)^\vee$, the existence of such a pairing B_φ given just the pairing E_φ is equivalent to the condition $E_\varphi(ix, iy) = E_\varphi(x, y)$, in which case

$$B_\varphi(x, y) = -iE_\varphi(ix, y) + E_\varphi(x, y). \tag{1.4.2.1}$$

Note that none of this depends on a choice of $i \in \mathbf{C}$. It is straightforward to *check* that the condition $B_\varphi(x, y) = \overline{B_\varphi(y, x)}$ is equivalent to the condition $E_\varphi(x, y) = -E_\varphi(y, x)$.

THEOREM 1.4.2.1. *An isomorphism $\varphi : A \xrightarrow{\simeq} \widehat{A}$ of classical complex tori has the form $\phi_{\mathcal{L}}$ for some line bundle $\mathcal{L}$ on A if and only if E_φ is skew-symmetric. In this case, $\mathcal{L}$ is ample if and only if the Hermitian pairing B_φ in* (1.4.2.1) *is positive definite.*

PROOF. A map $\varphi : A \to \widehat{A}$ amounts to the specification of a line bundle $\mathcal{M}$ on $A \times A$ an an isomorphism $\mathcal{M}|_{A \times \{0\}} \simeq \mathcal{O}_A$. Tracing through the definitions, we see that in order for φ to equal $\phi_{\mathcal{L}}$ for some $\mathcal{L}$, it is necessary and sufficient that $\mathcal{M} = m^*\mathcal{L} \otimes p_1^*\mathcal{L}^{-1} \otimes p_2^*\mathcal{L}^{-1}$, where m is the multiplication map and the p_i are the projections. In terms of the analytic calculations in [**45**, pp. 83–85], this is exactly the condition that B_φ is Hermitian, and as we saw above this in turn is the same as E_φ being skew-symmetric. More precisely, $\frac{1}{2\pi}B_\varphi$ must be the Hermitian form corresponding to $\mathcal{L}$ via the Appell-Humbert theorem [**45**, p. 20]. The key point is that [**45**, Ch 1] uses the exponential sequence

$$0 \to \underline{\mathbf{Z}} \to \mathcal{O}_X \xrightarrow{e^{2\pi i(\cdot)}} \mathcal{O}_X^\times \to 1,$$

essentially because $\mathbf{Z}$ is a ring whereas $\mathbf{Z}(1)$ is not. This plays an essential role in [**45**, pp. 16-19]. In terms of our setup, this introduces the factor of $\frac{i}{2\pi i} = \frac{1}{2\pi}$. Fortunately $\frac{1}{2\pi} > 0$, so B_φ is positive definite if and only if $\frac{1}{2\pi}B_\varphi$ is positive definite. By the theorem of Lefschetz [**45**, pp. 29-33], the ampleness of the line bundle $\mathcal{L}$ is equivalent to the positive definiteness of the associated Hermitian form $\frac{1}{2\pi}B_\varphi$, so we are done. $\square$

The analysis in the proof of Theorem 1.4.2.1 shows that for a principal polarization $\varphi : A \xrightarrow{\simeq} \widehat{A}$, the pairing

$$\mathrm{H}_1(A, \mathbf{Z}) \times \mathrm{H}_1(A, \mathbf{Z}) \xrightarrow{E_\varphi} \mathbf{Z}(1) \overset{\frac{1}{2\pi i}}{\rightrightarrows} \mathbf{Z}$$

is the *usual* $\mathbf{Z}$-valued symplectic form from the classical analytic theory (with respect to a fixed choice of $i \in \mathbf{C}$). In general, if $A \to S$ is a principally polarized relative complex torus of relative dimension g, with $\varphi : A \xrightarrow{\simeq} \widehat{A}$ the principal polarization, we conclude that

$$E_\varphi : (\mathrm{R}^1 f_* \underline{\mathbf{Z}})^\vee \otimes (\mathrm{R}^1 f_* \underline{\mathbf{Z}})^\vee \to \underline{\mathbf{Z}}(1) \tag{1.4.2.2}$$

is a skew-symmetric perfect pairing, inducing the classical pairing on fibers.

DEFINITION 1.4.2.2. Fixing $i \in \mathbf{C}$, we define the $\underline{\mathbf{Z}}$-valued perfect pairing $\frac{1}{2\pi i}E_\varphi$ to be the *symplectic form associated to φ*.

The formation of this associated symplectic form is clearly compatible with base change. If we are given a trivialization $\mathrm{R}^1 f_* \underline{\mathbf{Z}} \simeq \underline{\mathbf{Z}}^{2g}$ (which can always be found locally on S), it makes sense to require that $\frac{1}{2\pi i} E_\varphi$ have matrix with block form

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Locally on S such a trivialization of $\mathrm{R}^1 f_* \underline{\mathbf{Z}}$ can always be found, by the structure theorem for symplectic forms over principal ideal domains.

1.4.3. Moduli spaces. Fix a choice of $i \in \mathbf{C}$. For $g \geq 1$, let $F^{(g)}$ be the functor on analytic spaces defined by letting $F^{(g)}(S)$ be the set of isomorphism classes of triples

$$(f : A \to S, \quad \varphi : A \xrightarrow{\simeq} \widehat{A}, \quad \mathrm{R}^1 f_* \underline{\mathbf{Z}} \simeq \underline{\mathbf{Z}}^{2g})$$

where $A \to S$ is a relative complex torus of relative dimension g, φ is a principal polarization (so in particular $A \to S$ is a principally polarized relative complex torus), and the skew-symmetric perfect pairing

$$(\mathrm{R}^1 f_* \underline{\mathbf{Z}})^\vee \otimes (\mathrm{R}^1 f_* \underline{\mathbf{Z}})^\vee \xrightarrow{E_\varphi} \underline{\mathbf{Z}}(1) \overset{\frac{1}{2\pi i}}{\simeq} \underline{\mathbf{Z}}$$

has matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ with respect to the specified trivialization of $\mathrm{R}^1 f_* \underline{\mathbf{Z}}$. It follows from the definition of E_φ and the monicity in Lemma 1.3.5.2 (with $M = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$) that objects in $F^{(g)}(S)$ admit no non-trivial automorphisms. Thus, $F^{(g)}$ defines a sheaf of sets on any analytic space S.

From the discussion in §1.2.4, when $g = 1$ the functor $F^{(g)}$ classifies elliptic curves $f : E \to S$ equipped with an *i-permissible* trivialization $\mathrm{R}^1 f_* \underline{\mathbf{Z}} \simeq \underline{\mathbf{Z}}^2$. Our goal is to construct an explicit universal family for $F^{(g)}$, recovering $f^{\mathrm{an}} : \mathcal{E}^{\mathrm{an}} \to \mathfrak{h}$ and $\alpha^{\mathrm{an}} : \mathrm{R}^1 f_*^{\mathrm{an}} \underline{\mathbf{Z}} \simeq \underline{\mathbf{Z}}^2$ from (1.2.1.6) when $g = 1$.

We first make a convenient definition. Let $f : A \to S$ be a complex torus of relative dimension g and let

$$\mathrm{R}^1 f_* \underline{\mathbf{Z}} \simeq \underline{\mathbf{Z}}^{2g} \tag{1.4.3.1}$$

be a fixed trivialization. Shrink S so that there exists an isomorphism $f_* \Omega^1_{A/S} \simeq \mathcal{O}_S^{\oplus g}$. We then have a canonical map

$$\underline{\mathbf{Z}}^{2g} \simeq (\mathrm{R}^1 f_* \underline{\mathbf{Z}})^\vee \hookrightarrow (f_* \Omega^1_{A/S})^\vee \simeq \mathcal{O}_S^{\oplus g}$$

which we call the *period matrix.* This is motivated by the fact that on fibers this recovers the classical complex $g \times 2g$ matrix

$$\left(\int_{\sigma_j} \omega_i \right)$$

where $\{\sigma_j\}$ is the ordered homology basis specified by $(\mathrm{R}^1 f_* \underline{\mathbf{Z}})^\vee \simeq \underline{\mathbf{Z}}^{2g}$ and $\{\omega_i\}$ is the ordered basis of global 1-forms specified by $f_* \Omega^1_{A/S} \simeq \mathcal{O}_S^{\oplus g}$.

Dropping hypotheses on the global triviality of $f_* \Omega^1_{A/S}$, consider the unique isomorphism of locally constant sheaves $(\mathrm{R}^1 f_* \underline{\mathbf{Z}})^\vee \simeq \mathrm{R}^1 f_* \underline{\mathbf{Z}}(1)$ which induces the (skew-symmetric) perfect pairing

$$(\mathrm{R}^1 f_* \underline{\mathbf{Z}})^\vee \otimes (\mathrm{R}^1 f_* \underline{\mathbf{Z}})^\vee \simeq (\mathrm{R}^1 f_* \underline{\mathbf{Z}})^\vee \otimes \mathrm{R}^1 f_* \underline{\mathbf{Z}}(1) \to \underline{\mathbf{Z}}(1) \overset{\frac{1}{2\pi i}}{\simeq} \underline{\mathbf{Z}}$$

having matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

with respect to the specified trivialization (1.4.3.1). By Lemma 1.3.5.2, there is at most one way to fill in a commutative diagram of analytic spaces

(1.4.3.2)
$$\begin{tikzcd} (\mathrm{R}^1 f_* \underline{\mathbf{Z}})^\vee \arrow[r, hook] \arrow[d, "\simeq"'] & (f_* \Omega^1_{A/S})^\vee \arrow[d, dashed, "?? \,\Phi"] \\ \mathrm{R}^1 f_* \underline{\mathbf{Z}}(1) \arrow[r, hook] & \mathrm{R}^1 f_* \mathcal{O}_A \end{tikzcd}$$

in which case Φ induces an isomorphism *of complex tori* $A \simeq \widehat{A}$ over S. In such a situation, the isomorphism $A \simeq \widehat{A}$ is a principal polarization of $A \to S$ if and only if a positive definiteness condition holds on fibers (cf. Theorem 1.4.2.1).

For an example, fix $i \in \mathbf{C}$ and define the *Siegel upper half-space*

$$\mathfrak{h}_g = \{\mathfrak{z} \in M_g(\mathbf{C}) \,|\, \mathfrak{z}^t = \mathfrak{z},\ \operatorname{Im} \mathfrak{z} \text{ positive definite}\} \subseteq \mathrm{GL}_g(\mathbf{C})$$

where $M_g(\mathbf{C})$ is the space of $g \times g$ matrices over $\mathbf{C}$ and $(\cdot)^t$ denotes transpose. We give $\mathfrak{h}_g$ its canonical complex manifold structure such that $\mathfrak{h}_g \to \mathrm{GL}_g(\mathbf{C})$ is an immersion. Define a "lattice"

(1.4.3.3)
$$\begin{aligned} \mathbf{Z}^{2g} \times \mathfrak{h}_g &\to \mathbf{C}^g \times \mathfrak{h}_g \\ (m, n; \mathfrak{z}) &\mapsto (m\mathfrak{z} + n; \mathfrak{z}) \end{aligned}$$

where $m, n \in \mathbf{Z}^g$. The above discussion and the classification of principally polarized classical complex tori (cf. [**50**, §6] and [**4**, §2]) show that (1.4.3.3) induces a cocompact lattice on fibers and

$$f_g : \mathcal{A}_g \stackrel{\text{def}}{=} (\mathbf{C}^g \times \mathfrak{h}_g)/(\mathbf{Z}^{2g} \times \mathfrak{h}_g) \to \mathfrak{h}_g$$

equipped with the canonical $\alpha_g : \mathrm{R}^1 f_{g*} \underline{\mathbf{Z}} \simeq \underline{\mathbf{Z}}^{2g}$ is a relative complex torus admitting a unique principal polarization $\varphi_g : \mathcal{A}_g \xrightarrow{\simeq} \widehat{\mathcal{A}}_g$ with respect to which E_{φ_g} has matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.

Let us be precise about the construction of the holomorphic φ_g. Consider the diagram

(1.4.3.4)
$$\begin{tikzcd} (\mathrm{R}^1 f_{g*} \underline{\mathbf{Z}})^\vee \arrow[r, hook] \arrow[d, "\simeq"'] & (f_{g*} \Omega^1_{\mathcal{A}_g/\mathfrak{h}_g})^\vee \arrow[d, dashed, "??"] \\ \mathrm{R}^1 f_{g*} \underline{\mathbf{Z}}(1) \arrow[r, hook] & \mathrm{R}^1 f_{g*} \mathcal{O}_{\mathcal{A}_g} \end{tikzcd}$$

Let

(1.4.3.5)
$$\begin{tikzcd} (\mathrm{R}^1 f_{g*} \underline{\mathbf{Z}})^\vee \arrow[r, hook] \arrow[d, "\simeq"'] & \mathcal{E}_1 \arrow[d, dashed, "??"] \\ \mathrm{R}^1 f_{g*} \underline{\mathbf{Z}}(1) \arrow[r, hook] & \mathcal{E}_2 \end{tikzcd}$$

be the corresponding diagram in which the $\mathcal{E}_j$'s are the C^∞-sheaves of obtained by applying $\mathcal{C}_{\mathfrak{h}_g, \mathbf{C}} \otimes_{\mathcal{O}_{\mathfrak{h}_g}} (\cdot)$ to the right side of (1.4.3.4), where $\mathcal{C}_{\mathfrak{h}_g, \mathbf{C}}$ is the sheaf of $\mathbf{C}$-valued C^∞-functions on $\mathfrak{h}_g$. Applying $\mathcal{C}_{\mathfrak{h}_g, \mathbf{R}} \otimes_{\mathbf{Z}} (\cdot)$ to the left side of (1.4.3.5) converts the horizontal maps in (1.4.3.5) into *isomorphisms* of locally free finite rank $\mathcal{C}_{\mathfrak{h}_g,\mathbf{R}}$-modules: this is because the isomorphism $\mathcal{C}_{\mathfrak{h}_g,\mathbf{C}} \simeq \underline{\mathbf{C}} \otimes_{\underline{\mathbf{R}}} \mathcal{C}_{\mathfrak{h}_g,\mathbf{R}}$ yields

$$(\cdot) \otimes_{\mathcal{O}_s} (\mathcal{C}_{\mathfrak{h}_g,\mathbf{C}})_s \otimes_{(\mathcal{C}_{\mathfrak{h}_g,\mathbf{R}})_s} \mathbf{R} \simeq (\cdot) \otimes_{\mathcal{O}_s} \mathbf{C}.$$

Thus, we can fill in an isomorphism on the right side of (1.4.3.5).

Since the base $\mathfrak{h}_g$ is a complex manifold (and in particular is reduced), the classical theory on fibers over $\mathfrak{h}_g$ shows that this isomorphism in the right column of (1.4.3.5) is $\underline{\mathbf{C}}$-linear and takes holomorphic sections to C^∞-sections which are holomorphic in each separate variable and hence are holomorphic. That is, the isomorphism in the right column of (1.4.3.5) induces the desired isomorphism in the right column of (1.4.3.4) on the level of sheaves of holomorphic sections. This confirms the existence of a (unique) holomorphic relative polarization φ_g inducing the classical polarizations on fibers over $\mathfrak{h}_g$.

The triple $(\mathcal{A}_g, \varphi_g, \alpha_g)$ yields an element of $F^{(g)}(\mathfrak{h}_g)$. Moreover, the classical classification shows that $F^{(g)}(\mathrm{Sp}\,\mathbf{C})$ is in bijection with the fibers of $(\mathcal{A}_g, \varphi_g, \alpha_g)$ over $\mathfrak{h}_g$. When $g = 1$, note that this construction recovers $f^{\mathrm{an}} : \mathcal{E}^{\mathrm{an}} \to \mathfrak{h}$ and $\alpha^{\mathrm{an}} : \mathrm{R}^1 f^{\mathrm{an}}_* \underline{\mathbf{Z}} \simeq \underline{\mathbf{Z}}^2$. Here is the main representability result in the analytic theory. The special case $g = 1$ is all we shall need in what follows.

THEOREM 1.4.3.1. *The object $(\mathcal{A}_g, \varphi_g, \alpha_g) \in F^{(g)}(\mathfrak{h}_g)$ is universal for the functor $F^{(g)}$.*

PROOF. Let $X = (f : A \to S, \quad \varphi : A \simeq \widehat{A}, \quad \alpha : \mathrm{R}^1 f_* \underline{\mathbf{Z}} \simeq \underline{\mathbf{Z}}^{2g})$ be in $F^{(g)}(S)$. We want to prove the existence and uniqueness of a map $S \to \mathfrak{h}_g$ pulling back $(\mathcal{A}_g, \varphi_g, \alpha_g)$ to X. Since two triples $(f, \varphi, \alpha), (f, \varphi', \alpha) \in F^{(g)}(S)$ must have have $\varphi = \varphi'$, we just need to identify X with the pullback of $\mathcal{A}_g$ by a unique $S \to \mathfrak{h}_g$ such that α_g pulls back to α. Since objects classified by $F^{(g)}$ admit no non-trivial automorphisms, we can check this locally on S. In particular, we may assume that there exists a trivialization $\beta : f_* \Omega^1_{A/S} \simeq \mathcal{O}_S^{\oplus g}$.

Consider the resulting period matrix

$$\underline{\mathbf{Z}}^{2g} \simeq (\mathrm{R}^1 f_* \underline{\mathbf{Z}})^\vee \hookrightarrow (f_* \Omega^1_{A/S})^\vee \simeq \mathcal{O}_S^{\oplus g}. \tag{1.4.3.6}$$

From the classical Riemann relations on the fibers, we can *uniquely* change the trivialization β of $f_* \Omega^1_{A/S}$ so that (1.4.3.6) has matrix $(\mathfrak{z}; 1)$ for some $\mathfrak{z} \in \mathrm{GL}_g(\mathcal{O}_S)$, and $\mathrm{Im}\,\mathfrak{z}$ is automatically positive definite and symmetric on the fibers, by the classical Riemann relations [**50**, §6]. Note that this step amounts to the standardization of the symplectic form $(1/2\pi i)E_\varphi$ via the trivialization α (and in particular we have used the discussion following Theorem 1.4.2.1). As long as $\mathfrak{z}$ is symmetric, we may view $\mathfrak{z}$ as a morphism from S to $\mathrm{GL}_g(\mathbf{C})$ which factors through a map $S \to \mathfrak{h}_g$ pulling $(\mathcal{A}_g, \varphi_g, \alpha_g)$ back to X (up to isomorphism). The uniqueness of the map $S \to \mathfrak{h}_g$ then follows from the uniqueness of $\mathfrak{z}$.

It remains to show that the unique $\mathfrak{z}$ is symmetric. From the fibers we know the symmetry for reduced S. In general, it suffices (by uniqueness) to work locally on S and to realize X as the base change from a reduced S'. More precisely, we claim that the fiber conditions in Theorem 1.4.2.1 are open on the base, so it would suffice to locally lift an isomorphism $\varphi : \mathcal{A} \simeq \widehat{\mathcal{A}}$ through a closed immersion $S \hookrightarrow S'$. Moreover, by Theorem 1.1.1.14 any lift as merely a *morphism* of relative complex tori $\varphi' : \mathcal{A}' \to \widehat{\mathcal{A}'}$ over S' is automatically an *isomorphism* around S, so it would even suffice to locally lift φ as a morphism, once we verify:

LEMMA 1.4.3.2. *Let $\mathcal{A} \to S$ be a relative complex torus, $\varphi' : \mathcal{A} \simeq \widehat{\mathcal{A}}$ an isomorphism. Thus locus of $s \in S$ where φ_s is a principal polarization is open.*

PROOF. By Theorem 1.4.2.1, we want the opennes of the set of $s \in S$ for which E_{φ_s} is skew-symmetric and the resulting Hermitian B_{φ_s} is positive definite. Since the pairing E_φ of local systems on S in (1.4.2.2) induces E_{φ_s} on fibers, the openness of the skew-symmetric locus is clear. Now we may assume E_φ is skew-symmetric, so B_{φ_s} is Hermitian for all $s \in S$ and we want the openness of the set of $s \in S$ for which B_{φ_s} is positive definite. The idea for the proof is that inside of the $\mathbf{R}$-vector space of Hermitian matrices of a fixed dimension, the positive definite ones form an open subset (as the associated $\mathbf{R}$-valued quadratic form is positive definite if and only if it is positive on the *compact* unit ball). In order to apply this idea, we need to make the B_{φ_s}'s into a relative object, and although (1.4.2.1) is suggestive, we need to do a little work since i doesn't act on the local system $(\mathrm{R}^1 f_* \underline{\mathbf{Z}})^\vee$.

We will modify the argument used just below (1.4.3.5). Let $\mathcal{C}^{\mathrm{top}}_{S,\mathbf{R}}$ and

$$\mathcal{C}^{\mathrm{top}}_{S,\mathbf{C}} = \underline{\mathbf{C}} \otimes_{\underline{\mathbf{R}}} \mathcal{C}^{\mathrm{top}}_{S,\mathbf{R}}$$

denote the sheaves of *continuous* $\mathbf{R}$-valued and $\mathbf{C}$-valued functions on S. Using the injection

$$(\mathrm{R}^1 f_* \underline{\mathbf{Z}})^\vee \hookrightarrow (f_* \Omega^1_{\mathcal{A}/S})^\vee,$$

we get a map of locally free $\mathcal{C}^{\mathrm{top}}_{S,\mathbf{R}}$-modules of rank g,

$$\mathcal{C}^{\mathrm{top}}_{S,\mathbf{R}} \otimes_{\underline{\mathbf{Z}}} (\mathrm{R}^1 f_* \underline{\mathbf{Z}})^\vee \to \mathcal{C}^{\mathrm{top}}_{S,\mathbf{C}} \otimes_{\mathcal{O}_S} (f_* \Omega^1_{\mathcal{A}/S})^\vee,$$

and this is an *isomorphism* by the same argument as used below (1.4.3.5). The pairing E_φ is therefore promoted to a $\mathcal{C}^{\mathrm{top}}_{S,\mathbf{R}}$-linear map

$$\left(\mathcal{C}^{\mathrm{top}}_{S,\mathbf{R}} \otimes_{\underline{\mathbf{Z}}} (\mathrm{R}^1 f_* \underline{\mathbf{Z}})^\vee\right) \otimes_{\mathcal{C}^{\mathrm{top}}_{S,\mathbf{R}}} \left(\mathcal{C}^{\mathrm{top}}_{S,\mathbf{R}} \otimes_{\underline{\mathbf{Z}}} (\mathrm{R}^1 f_* \underline{\mathbf{Z}})^\vee\right) \to \mathcal{C}^{\mathrm{top}}_{S,\mathbf{R}}(1) \hookrightarrow \mathcal{C}^{\mathrm{top}}_{S,\mathbf{C}}$$

and hence a $\mathcal{C}^{\mathrm{top}}_{S,\mathbf{R}}$-linear map

$$E : \left(\mathcal{C}^{\mathrm{top}}_{S,\mathbf{C}} \otimes_{\mathcal{O}_S} (f_* \Omega^1_{\mathcal{A}/S})^\vee\right) \otimes_{\mathcal{C}^{\mathrm{top}}_{S,\mathbf{R}}} \left(\mathcal{C}^{\mathrm{top}}_{S,\mathbf{C}} \otimes_{\mathcal{O}_S} (f_* \Omega^1_{\mathcal{A}/S})^\vee\right) \to \mathcal{C}^{\mathrm{top}}_{S,\mathbf{C}} = \underline{\mathbf{C}} \otimes_{\underline{\mathbf{R}}} \mathcal{C}^{\mathrm{top}}_{S,\mathbf{R}}$$

which induces E_{φ_s} after passage to stalks at s and applying extension of scalars by $(\mathcal{C}^{\mathrm{top}}_{S,\mathbf{R}})_s \to \mathbf{R}$. Define $B(x,y) = -iE(ix,y) + E(x,y)$, so by (1.4.2.1) we get $B_s = B_{\varphi_s}$. Since E is skew-symmetric, B is "Hermitian". More precisely, we can view B as a *continuous* Hermitian-matrix-valued function on S. The positive definiteness at s therefore implies the positive definiteness on a neighborhood of s, as desired.

□

Continuing now with the proof of Theorem 1.4.3.1, let $\mathcal{A}'$ over S' be a lift of $\mathcal{A}$ over S (using Corollary 1.3.3.3). As we have seen, it suffices to lift φ to a morphism $\varphi' : \mathcal{A}' \to \widehat{\mathcal{A}'}$ over S', at least locally near points of S. By the universal property of the dual relative complex torus, φ corresponds to an invertible sheaf $\mathcal{L}$ on $\mathcal{A} \times_S \mathcal{A}$ with $\mathcal{L}|_{\mathcal{A} \times_S \{a\}}$ topologically trivial for all $a \in \mathcal{A}$ and $\mathcal{L}|_{\{0\} \times_S \mathcal{A}}$ trivial (the latter condition corresponds to $\varphi(0) = 0$).

Viewing $\mathcal{A} \times_S \mathcal{A}$ as a relative complex torus over S, we may in particular view $\mathcal{L}$ as an element of $\mathrm{Pic}^0(\mathcal{A} \times_S \mathcal{A}/S)$. By S'-smoothness of $\mathcal{A}' \times_{S'} \mathcal{A}'$ and working locally on S' around S, we may lift $\mathcal{L}$ to $\mathcal{L}' \in \mathrm{Pic}^0(\mathcal{A}' \times_{S'} \mathcal{A}'/S')$. To see this, we do not try to directly lift line bundles. Instead, we note that $P' = \mathrm{Pic}^0_{\mathcal{A}' \times_{S'} \mathcal{A}'/S'}$

is S'-smooth, so it suffices to show quite generally that if $X' \to S'$ is a smooth map and $S \hookrightarrow S'$ is a closed immersion, then for any $e \in X'(S)$ we can, locally on S, lift this to $e' \in X'(S')$. For $s \in S$ we can shrink S' around s in order to find an open ball B around in the origin in some $\mathbf{C}^n$ such that (over S') $S' \times B \subseteq X'$ sits as an open around $e(s) = (s, 0)$. Replacing S' by an open set meeting S in $e^{-1}(S' \times B)$, we may assume $e(S) \subseteq S' \times B$. Thus, e is given by an n-tuple of elements $f_i \in \mathrm{H}^0(S, \mathcal{O}_S)$ with f_i "small" near s. These f_i's lifts to S' near s, thereby giving rise to the desired element of $X'(S')$ lifting e. This completes the justification of the existence of the lift $\mathcal{L}'$.

Since $\mathcal{L}'|_{\mathcal{A}' \times_{S'} \{a'\}}$ is topologically trivial for all $a' \in \mathcal{A}'$ over $S \hookrightarrow S'$, by *properness* we claim that the same topological triviality holds for a' in fibers of $\mathcal{A}'$ over a neighborhood of S in S'. To see this, consider the connecting homomorphism

$$\partial : \mathrm{R}^1\pi_*(\mathcal{O}_{\mathcal{A}' \times_{S'} \mathcal{A}'}) \to \mathrm{R}^2\pi_*\underline{\mathbf{Z}}(1)$$

in which $\pi : \mathcal{A}' \times_{S'} \mathcal{A}' \to \mathcal{A}'$ is the second projection. The element $c \stackrel{\mathrm{def}}{=} \partial([\mathcal{L}'])$ is a global section of $\mathrm{R}^2\pi_*(\underline{\mathbf{Z}}(1))$ over $\mathcal{A}'$ which, by topological proper base change and the topological triviality hypotheses on $\mathcal{L}'$, vanishes in all stalks at points of $\mathcal{A}'$ over S. The locus $U \subseteq \mathcal{A}'$ where c vanishes is an open set, so the image of $\mathcal{A}' - U$ in S' is a closed set disjoint from S. The complement of this closed set in S' is an open subset V containing S such that for all $a' \in \mathcal{A}'$ over V, the restriction $\mathcal{L}'|_{\mathcal{A}' \times_{S'} \{a'\}}$ is topologically trivial. Replacing S' with V, we get an S'-map $\varphi' : \mathcal{A}' \to \widehat{\mathcal{A}'}$ lifting φ but not necessarily taking 0 to 0. Translating φ' by $-\varphi'(0)$ then gives the desired lift of φ, since the restriction of $-\varphi'(0)$ to $S \hookrightarrow S'$ is $-\varphi(0) = 0$.

$\square$

The purpose of much of our cohomological work has been to obtain Theorem 1.4.3.1 with $g = 1$, as this is the starting point for the complex analytic moduli-theoretic interpretation of modular curves and their natural construction as quotients of $\mathfrak{h}$. We could have avoided much of the preceding work if we restricted ourselves to elliptic curves over smooth bases, but the realization of the classical $\mathcal{E}^{\mathrm{an}} \to \mathfrak{h}$ as representing a functor on *arbitrary* complex analytic spaces makes the prominent role of this construction much more natural in the general theory. Theorem 1.4.3.1 will also allow us to argue quite naturally with artinian points when studying the relationship between the algebraic and analytic moduli spaces for elliptic curves, since

- $\mathrm{Sp}\, A$ ($= \mathrm{Spec}\, A$) for a finite local $\mathbf{C}$-algebra A is both a finite type $\mathbf{C}$-scheme and a complex analytic space,
- "points" with values in artin local objects are adequate to distinguish distinct morphisms in the category of analytic spaces, as well as in the category of locally finite type $\mathbf{C}$-schemes.

1.5. Geometrization of modular forms

1.5.1. Group actions on universal families. In order to give a geometric definition of modular forms, we are going to need to endow certain structures with an action of the group $\mathrm{SL}_2(\mathbf{R})$. We begin with some brief general considerations. Let $f : A \to S$ be a relative complex torus of relative dimension g. The sequence of terms of low degree in the relative Hodge to de Rham spectral sequence for f is:

$$0 \to f_*f^{-1}\mathcal{O}_S \to f_*\mathcal{O}_A \to f_*\Omega^1_{A/S} \to \mathrm{R}^1f_*(f^{-1}\mathcal{O}_S) \to \mathrm{R}^1f_*\mathcal{O}_A.$$

By Lemma 1.2.2.1(2), the map $f_*f^{-1}\mathcal{O}_S \to f_*\mathcal{O}_A$ is just the canonical isomorphism

$$\mathcal{O}_S \xrightarrow{\simeq} f_*\mathcal{O}_A.$$

Arguing as in (1.2.2.4), $\mathrm{R}^1 f_*(f^{-1}\mathcal{O}_S)$ is naturally identified with

$$\mathrm{R}^1 f_*\underline{\mathbf{Z}} \otimes_{\underline{\mathbf{Z}}} \mathcal{O}_S.$$

We get from this a left exact sequence of vector bundles

$$0 \to f_*\Omega^1_{A/S} \to \mathrm{R}^1 f_*\underline{\mathbf{Z}} \otimes_{\underline{\mathbf{Z}}} \mathcal{O}_S \to \mathrm{R}^1 f_*\mathcal{O}_A \tag{1.5.1.1}$$

compatible with base change on S, with the right map induced by the inclusion $\underline{\mathbf{Z}} \hookrightarrow \mathcal{O}_A$. On fibers, (1.5.1.1) is the sequence

$$0 \to \mathrm{H}^0(A_s, \Omega^1_{A_s}) \to \mathrm{H}^1(A_s, \underline{\mathbf{C}}) \to \mathrm{H}^1(A_s, \mathcal{O}_{A_s}) \tag{1.5.1.2}$$

with the middle map the canonical one. The sequence (1.5.1.2) is surjective on the right for dimension reasons (in other words, the Hodge to de Rham spectral sequence degenerates on fibers). Thus, (1.5.1.1) is an exact sequence of vector bundles

$$0 \to f_*\Omega^1_{A/S} \to \mathrm{R}^1 f_*\underline{\mathbf{Z}} \otimes_{\underline{\mathbf{Z}}} \mathcal{O}_S \to \mathrm{R}^1 f_*\mathcal{O}_A \to 0$$

compatible with base change on S.

Suppose now that A admits a principal polarization $\varphi : A \simeq \hat{A}$ whose associated symplectic form

$$(\mathrm{R}^1 f_*\underline{\mathbf{Z}})^\vee \otimes (\mathrm{R}^1 f_*\underline{\mathbf{Z}})^\vee \simeq (\mathrm{R}^1 f_*\underline{\mathbf{Z}})^\vee \otimes \mathrm{R}^1 f_*\underline{\mathbf{Z}}(1) \to \underline{\mathbf{Z}}(1) \overset{\frac{1}{2\pi i}}{\simeq} \underline{\mathbf{Z}}$$

from (1.4.2.2) has matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ with respect to some specified trivialization $\alpha : \mathrm{R}^1 f_*\underline{\mathbf{Z}} \simeq \underline{\mathbf{Z}}^{\oplus 2g}$. The isomorphism of vector bundles

$$(f_*\Omega^1_{A/S})^\vee = T_0(A) \overset{\varphi}{\simeq} T_0(\widehat{A}) = \mathrm{R}^1 f_*\mathcal{O}_A \tag{1.5.1.3}$$

fits into a commutative diagram of analytic spaces over S

$$\begin{tikzcd}
\underline{\mathbf{Z}}^{2g} \arrow[r, "\alpha^\vee", "\simeq"'] \arrow[d, "\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)"'] & (\mathrm{R}^1 f_*\underline{\mathbf{Z}})^\vee \arrow[r, hook] \arrow[d, "\simeq"] & (f_*\Omega^1_{A/S})^\vee \arrow[r, "\exp_{A/S}"] \arrow[d, "T_0(\varphi)"', "\simeq"] & A \arrow[d, "\varphi"', "\simeq"] \\
\underline{\mathbf{Z}}^{2g} & \mathrm{R}^1 f_*\underline{\mathbf{Z}}(1) \arrow[l, "\simeq"', "\frac{1}{2\pi i}\alpha"] \arrow[r, hook] & \mathrm{R}^1 f_*\mathcal{O}_A \arrow[r] & \widehat{A}
\end{tikzcd}$$

where the middle square is (1.4.1.1). We conclude that the maps

$$\beta : \underline{\mathbf{Z}}^{2g} \overset{\alpha^\vee}{\simeq} (\mathrm{R}^1 f_*\underline{\mathbf{Z}})^\vee \hookrightarrow (f_*\Omega^1_{A/S})^\vee$$

and

$$q : \underline{\mathbf{Z}}^{2g} \overset{\alpha^{-1}}{\simeq} \mathrm{R}^1 f_*\underline{\mathbf{Z}} \hookrightarrow \mathrm{R}^1 f_*\mathcal{O}_A \overset{T_0(\varphi)}{\simeq} (f_*\Omega^1_{A/S})^\vee$$

are related by

$$\beta = 2\pi i \cdot q \circ \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right). \tag{1.5.1.4}$$

We note in passing that when $g = 1$ and we consider the *unique* principal polarization $E \simeq \widehat{E}$ of an elliptic curve, the isomorphism $(f_*\Omega^1_{E/S})^\vee \simeq \mathrm{R}^1 f_*\mathcal{O}_E$ in (1.5.1.3) is the *same* as the map induced by the cup product pairing

$$f_*\Omega^1_{E/S} \otimes \mathrm{R}^1 f_*\mathcal{O}_E \to \mathrm{R}^1 f_*\Omega^1_{E/S} \overset{\frac{1}{2\pi i}\int}{\simeq} \mathcal{O}_E,$$

where the final isomorphism is defined using Theorem 1.2.2.2. Indeed, by base change compatibility and Corollary 1.3.3.3 we may assume that S is reduced (even smooth), so we may check this on fibers. But then the minus sign in Lemma 1.3.3.1 cancels the minus sign in Remark 1.3.3.2, giving the desired relation.

Since the above map q induces a cocompact lattice on fibers, it induces a map of sheaves of $\underline{\mathbf{R}}$-modules

$$\underline{\mathbf{R}}^{2g} \to (f_*\Omega^1_{A/S})^\vee \tag{1.5.1.5}$$

inducing an isomorphism $\mathbf{R}^{2g} \simeq \mathrm{H}^0(A_s, \Omega^1_{A_s})^\vee$ on fibers. Note, however, that (1.5.1.5) is definitely *not* an isomorphism of sheaves. The advantage of (1.5.1.5) over the initial data $\alpha : \mathrm{R}^1 f_*\underline{\mathbf{Z}} \simeq \underline{\mathbf{Z}}^{2g}$ is that we can make *real* matrices act on (1.5.1.5).

We now focus our attention on the special case $g = 1$. Let

$$(f_X : E_X \to X,\ \alpha_X : \mathrm{R}^1 f_{X*}\underline{\mathbf{Z}} \simeq \underline{\mathbf{Z}}^2)$$

be a universal object for the functor $F^{(1)}$, so this is the universal elliptic curve with an i-permissible trivialization of $\mathrm{R}^1 f_*\underline{\mathbf{Z}}$. Let $\omega_X = \omega_{E_X/X}$ as in (1.2.2.1). We can view $(E_X \to X, \alpha_X^\vee)$ as a universal family with an "i-permissible homology basis".

Recall that by (1.5.1.3) and the canonical principal polarization of an elliptic curve, there is a canonical isomorphism $\mathrm{R}^1 f_{X*}\mathcal{O}_{E_X} \simeq \omega_X^\vee = \omega_X^{-1}$. We thus have a map

$$\underline{\mathbf{R}}^2 \simeq \underline{\mathbf{R}} \otimes_{\underline{\mathbf{Z}}} \underline{\mathbf{Z}}^2 \overset{1\otimes\alpha_X^{-1}}{\simeq} \underline{\mathbf{R}} \otimes_{\underline{\mathbf{Z}}} \mathrm{R}^1 f_{X*}\underline{\mathbf{Z}} \to \mathrm{R}^1 f_{X*}\mathcal{O}_{E_X} \simeq \omega_X^{-1} \tag{1.5.1.6}$$

which is an orientation reversing isomorphism on fibers (by the anti-commutativity in the definition of "i-permissible" in §1.2.4). The map (1.5.1.6) is universal for such a structure on a holomorphic line bundle. More precisely, consider the functor associating to a complex analytic space Y the set of isomorphism classes of pairs $(\mathcal{L}, j)$ consisting of a line bundle $\mathcal{L}$ on Y and an $\underline{\mathbf{R}}$-linear sheaf map $j : \underline{\mathbf{R}}^2 \to \mathcal{L}$ inducing orientation reversing isomorphisms on fibers. (This is of course the "same" as a map of sheaves $j_0 : \underline{\mathbf{Z}}^2 \to \mathcal{L}$ which is an orientation reversing cocompact lattice on fibers; in the discussion below we will tacitly make this identification.) Note that $\mathrm{GL}_2^+(\mathbf{R})$ has a right action on such structures via

$$(\mathcal{L}, j) \to (\mathcal{L}, j \circ g);$$

in terms of $j_0 : \underline{\mathbf{Z}}^2 \to \mathcal{L}$, $g \in \mathrm{GL}_2^+(\mathbf{R})$ acts by sending $(\mathcal{L}, j_0)$ to the pair $(\mathcal{L}, j_0 \circ g)$ with $j_0 \circ g$ *defined* by

$$(j_0 \circ g)(1,0) = aj_0(1,0) + cj_0(0,1)$$
$$(j_0 \circ g)(0,1) = bj_0(1,0) + dj_0(0,1)$$

where

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Recall that $q : \underline{\mathbf{Z}}^2 \to \omega_X^{-1}$ is the natural map associated to (1.5.1.6).

THEOREM 1.5.1.1. *The pair $(X,\ q : \underline{\mathbf{Z}}^2 \to \omega_X^{-1})$ is universal for structures as above and the right $\mathrm{SL}_2(\mathbf{Z}) = \mathrm{GL}_2^+(\mathbf{Z})$-action on X arising from this universal family is related to the right $\mathrm{SL}_2(\mathbf{Z})$-action on X arising from the universal family $(E_X \to X,\ \alpha_X^\vee)$ via conjugation by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ on $\mathrm{SL}_2(\mathbf{Z})$.*

Before proving this, note that conjugation by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ on $\mathrm{SL}_2(\mathbf{Z})$ is the same as the map $g \mapsto (g^{-1})^t = (g^t)^{-1}$. Thus, if we had considered the left action of $\mathrm{SL}_2(\mathbf{Z})$ on X coming from the left $\mathrm{SL}_2(\mathbf{Z})$-action on the universal family $(E_X \to X, \alpha_X)$, this would be related to the right action arising from the (X, q) family via inversion on $\mathrm{SL}_2(\mathbf{Z})$.

PROOF. Let $(\mathcal{L}, j)$ be a pair as above on a complex analytic space Y. Let $j_0' : \underline{\mathbf{Z}}^2 \to \mathcal{L}$ denote the map obtained from $j_0 : \underline{\mathbf{Z}}^2 \to \mathcal{L}$ by the action of $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ on $\underline{\mathbf{Z}}^2$ and multiplication by $2\pi i$ on $\mathcal{L}$. By (1.5.1.4) and the above discussion, $\mathcal{L}/j_0'(\underline{\mathbf{Z}}^2)$ is an elliptic curve with a canonical i-permissible trivialization. The corresponding construction on (X, q) yields the universal pair $(E_X \to X,\ \alpha_X)$, and the result follows easily from this. $\square$

1.5.2. Explicit computation on fibers. Classically, modular forms are realized as holomorphic functions on $\mathfrak{h}$ satisfying certain transformation laws with respect to the action of $\mathrm{GL}_2^+(\mathbf{R})$. Before we geometrize this point of view, we want to study the action of $\mathrm{GL}_2^+(\mathbf{R})$ on X. In order to keep the picture clear and to treat all "cusps" on an equal footing, we prefer to work with an abstract universal X rather than the "coordinatized" $(\mathcal{E}^{\mathrm{an}} \to \mathfrak{h}, \alpha^{\mathrm{an}})$. This will greatly simplify our study of compactifications later. It will nevertheless be quite important to study various isomorphisms between X and $\mathfrak{h}$.

Before proceeding, we need to explicitly compute the map q on fibers. This uses the genus 1 case of

LEMMA 1.5.2.1. *Let Y be a compact Riemann surface. Consider the natural composite map*

$$\mathrm{H}^1(Y, \underline{\mathbf{C}}) \to \mathrm{H}^1(Y, \mathcal{O}_Y) \xrightarrow{\simeq} \mathrm{H}^0(Y, \Omega^1_Y)^\vee,$$

with the first map induced by $\underline{\mathbf{C}} \hookrightarrow \mathcal{O}_Y$ and the second induced by cup product (no sign issue) and the residue isomorphism

$$\mathrm{H}^1(Y, \Omega^1_Y) \simeq \mathrm{H}^2(Y, \underline{\mathbf{C}}) \overset{\frac{1}{2\pi i}\int}{\simeq} \mathbf{C} \tag{1.5.2.1}$$

of Theorem 1.2.2.2 (also see Corollary C1.2.7). This composite is the map

$$\eta \mapsto \frac{1}{2\pi i} \int_Y \eta \wedge (\cdot),$$

where $\mathrm{H}^1(Y, \underline{\mathbf{C}})$ is given its de Rham interpretation in terms of 1-forms.

Of course, the statement of this lemma is sensitive to the order to wedge product in the final formula.

PROOF. By Hodge theory on Y, the map

$$\mathrm{H}^0(Y, \overline{\Omega}^1_Y) \oplus \mathrm{H}^0(Y, \Omega^1_Y) \xrightarrow{\simeq} \mathrm{H}^1(Y, \underline{\mathbf{C}})$$

is an isomorphism and $\mathrm{H}^0(Y, \Omega^1_Y)$ is precisely the kernel of the map

$$\mathrm{H}^1(Y, \underline{\mathbf{C}}) \to \mathrm{H}^1(Y, \mathcal{O}_Y).$$

We are therefore just trying to compare two maps from $\mathrm{H}^0(Y, \overline{\Omega}^1_Y)$ to $\mathrm{H}^0(Y, \Omega^1_Y)^\vee$.

Our problem is to verify the commutativity of the diagram

$$\begin{tikzcd} \mathrm{H}^0(Y, \overline{\Omega}^1_Y) \otimes_{\mathbf{C}} \mathrm{H}^0(Y, \Omega^1_Y) \arrow[r, "\wedge"] \arrow[d] & \mathrm{H}^2_{DR}(Y, \mathbf{C}) \\ \mathrm{H}^1(Y, \mathcal{O}_Y) \otimes_{\mathbf{C}} \mathrm{H}^0(Y, \Omega^1_Y) \arrow[r, "\cup"'] & \mathrm{H}^1(Y, \Omega^1_Y) \arrow[u, "\simeq"'] \end{tikzcd} \tag{1.5.2.2}$$

(note the order of the wedge product in the top row). To see this commutativity, recall that the Dolbeault double complex $(\mathcal{A}^{\bullet,\bullet}, \partial, \overline{\partial})$ of $\mathbf{C}$-valued C^∞ forms of type (p, q)

$$\begin{tikzcd} \mathcal{A}^{0,1} \arrow[r, "\partial^{0,1}"] & \mathcal{A}^{1,1} \\ \mathcal{A}^{0,0} \arrow[r, "\partial^{0,0}"'] \arrow[u, "\overline{\partial}^{0,0}"] & \mathcal{A}^{1,0} \arrow[u, "\overline{\partial}^{1,0}"'] \end{tikzcd} \tag{1.5.2.3}$$

computes the Hodge to de Rham spectral sequence and has a "multiplicative" structure via wedge products.

Viewing the columns of (1.5.2.3) as (soft) acyclic resolutions of $\mathcal{O}_Y$ and Ω^1_Y respectively, we get a map of *complexes*

$$\mathcal{A}^{0,\bullet} \otimes \mathcal{A}^{1,\bullet} \xrightarrow{\wedge} \mathcal{A}^{1,\bullet}.$$

This amounts to the identity $\overline{\partial}(f\omega) = \overline{\partial}(f) \wedge \omega + f\overline{\partial}(\omega)$ for f of type $(0, 0)$ and ω of type $(1, 0)$. The complex $\mathrm{Tot}^{\oplus}(\mathcal{A}^{\bullet,\bullet})$ is the de Rham resolution of $\underline{\mathbf{C}}$, so the identification

$$\mathrm{H}^0(Y, \overline{\Omega}^1_Y) = \ker \Gamma(Y, \partial^{0,1})$$

is the same as the isomorphism

$$\mathrm{H}^0(Y, \overline{\Omega}^1_Y) \to \mathrm{H}^1(Y, \underline{\mathbf{C}}) \to \mathrm{H}^1(Y, \mathcal{O}_Y)$$

in the left side of (1.5.2.2). Similarly, the identification

$$\mathrm{H}^1(Y, \Omega^1_Y) \simeq \Gamma(Y, \mathcal{A}^{1,1}) / \operatorname{im} \Gamma(Y, \overline{\partial}^{1,0}) \to \mathrm{H}^2(\mathrm{Tot}^{\oplus} \Gamma(Y, \mathcal{A}^{\bullet,\bullet}))$$

is the same as the de Rham isomorphism

$$\mathrm{H}^1(Y, \Omega^1_Y) \simeq \mathrm{H}^2(Y, \underline{\mathbf{C}}).$$

Thus, the commutativity of (1.5.2.2) reduces to the fact that cup products in sheaf cohomology can be computed with *any* multiplicative structure on acyclic resolutions; this is proven in [**18**, Ch II, Thm 6.2]. $\square$

For a classical elliptic curve E, we conclude from Lemma 1.5.2.1 that the canonical $\mathbf{R}$-linear isomorphism

$$q_E : \mathrm{H}^1(E, \underline{\mathbf{R}}) \xrightarrow{\simeq} \mathrm{H}^1(E, \mathcal{O}_E) \simeq \mathrm{H}^0(E, \Omega^1_E)^\vee$$

induced by (1.5.2.1) is given by

$$\eta \mapsto \frac{1}{2\pi i} \int_E \eta \wedge (\cdot).$$

If $\{e_1, e_2\}$ is an $\mathbf{R}$-basis of $\mathrm{H}^1(E, \underline{\mathbf{R}})$, then the ratio

$$\frac{q_E(e_1)}{q_E(e_2)}$$

lies in $\mathbf{C}-\mathbf{R}$. By (1.5.1.4), we see that

$$\mathrm{Im}\left(\frac{q_E(e_1)}{q_E(e_2)}\right)$$

is positive if and only if q_E is orientation reversing with respect to the ordered basis $\{e_1, e_2\}$ of $\mathrm{H}^1(E, \underline{\mathbf{R}})$ and the i-orientation of the $\mathbf{C}$-vector space $\mathrm{H}^0(E, \Omega^1_E)^\vee$. In concrete terms, if $\omega \in \mathrm{H}^0(E, \Omega^1_E)^\vee$ is non-zero, then $e_1 = a\omega + \overline{a\omega}$, $e_2 = b\omega + \overline{b\omega}$ for some $\mathbf{R}$-basis $\{a, b\}$ of $\mathbf{C}$ and

$$\frac{q_E(e_1)}{q_E(e_2)} = \frac{\int_E (a\omega + \overline{a\omega}) \wedge \omega}{\int_E (b\omega + \overline{b\omega}) \wedge \omega} = \frac{\overline{a}}{\overline{b}}.$$

Now let $\{e_1, e_2\}$ be *any* oriented basis of $\mathbf{R}^2$ with respect to the orientation in which $(1,0) \wedge (0,1)$ is positive. The universal map

$$q : \underline{\mathbf{Z}}^2 \to \omega_X^{-1}$$

yields an isomorphism of ω_X^{-1} with the trivial line bundle $\mathbf{G}_a$ via the global section $-q(e_1)$. We also obtain an analytic map $X \to \mathbf{C}-\mathbf{R} \subseteq \mathbf{C}$ defined by $-q(e_2)/q(e_1)$, sending $x \in X$ to the value of $-q(e_2)/q(e_1)$ at x. (The reason for these specific choices of trivializations and maps will become clear from the special case treated below.) The orientation assumption on $\{e_1, e_2\}$ forces this map to land in $\mathfrak{h}$, and we claim that the resulting map

$$\psi_{e_1, e_2} : X \to \mathfrak{h}$$

is an isomorphism.

To see this, first consider a special case. When $e_1 = \binom{1}{0}$, $e_2 = \binom{0}{1}$ and we use the universal object

$$(E_X \to X; \alpha_X) = (\mathcal{E}^{\mathrm{an}} \to \mathfrak{h},\ \alpha^{\mathrm{an}}),$$

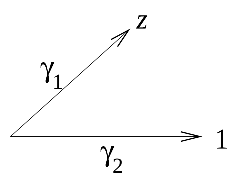
the map ψ_{e_1,e_2} is just the identity map! Indeed, over

$$z = a + ib \in \mathfrak{h}$$

with $a, b \in \mathbf{R}$ and $b > 0$, we can take $\mathcal{E}^{\mathrm{an}}_z \simeq \mathbf{C}/[1, z]$ with $\{\gamma_1, \gamma_2\}$ as shown, having dual basis

$$\left\{\frac{\mathrm{d}y}{b}, \mathrm{d}x - \frac{a}{b}\mathrm{d}y\right\}$$

in $\mathrm{H}^1(E, \underline{\mathbf{Z}}) \subseteq \mathrm{H}^1(E, \underline{\mathbf{R}})$.



Letting τ denote the parameter on $\mathbf{C}$, so

$$2\pi i \mathrm{d}\tau = 2\pi i(\mathrm{d}x + i\mathrm{d}y) = 2\pi i(\gamma_2^\vee + z\gamma_1^\vee) \in \mathrm{H}^0(E, \Omega^1_E),$$

we have (by Lemma 1.5.2.1)

$$q\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)(2\pi i\mathrm{d}\tau) = \frac{1}{2\pi i}\int_E \frac{\mathrm{d}y}{b}\wedge 2\pi i\mathrm{d}\tau = -1, \tag{1.5.2.4}$$

$$q\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)(2\pi i\mathrm{d}\tau) = \frac{1}{2\pi i}\int_E \left(\mathrm{d}x - \frac{a}{b}\mathrm{d}y\right)\wedge 2\pi i\mathrm{d}\tau = a+bi = z.$$

Here we are viewing each $q(e_j)$ as a section of the dual of the invertible ω_X. Thus

$$q\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) = -z\cdot q\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right),$$

which implies that

$$\psi_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)} : \mathfrak{h}\to\mathfrak{h}$$

is the identity.

We conclude from this that for *any* $(E_X\to X,\ \alpha_X)$, the map

$$\psi_{e_1,e_2} : X\to\mathfrak{h}$$

is an isomorphism for *some* positive basis $\{e_1,e_2\}$ of $\mathbf{Z}^2$. (Specifically, this is the case for the basis $\{e_1,e_2\}$ corresponding to $\{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\}$ on $\mathfrak{h}$.) But any other choice of (oriented) basis has the form $\{ge_1,ge_2\}$ for some $g\in\mathrm{GL}_2^+(\mathbf{R})$. Using the map

$$\underline{\mathbf{R}}^2\xrightarrow{g}\underline{\mathbf{R}}^2\xrightarrow{q}\omega_X^{-1},$$

and the universal property of X as discussed above Theorem 1.5.1.1, we conclude that there exists a unique analytic map $[g]=[g]_X : X\to X$ for which there is an isomorphism of line bundles $\omega_X^{-1}\simeq[g]^*\omega_X^{-1}$ making the diagram

$$\begin{tikzcd} \underline{\mathbf{R}}^2 \arrow[d, "q\circ g"'] \arrow[dr, "{[g]^*(q)}"] & \\ \omega_X^{-1} \arrow[r, "\simeq"] & {[g]^*\omega_X^{-1}} \end{tikzcd} \tag{1.5.2.5}$$

commutative. It follows from uniqueness that $[1]=1$, $[gh]=[h][g]$ and

$$\psi_{ge_1,ge_2} = \psi_{e_1,e_2}\circ[g].$$

In particular, $[g]$ is an automorphism of X for all $g\in\mathrm{GL}_2^+(\mathbf{R})$ and therefore ψ_{e_1,e_2} is an isomorphism for *any* choice of basis $\{e_1,e_2\}$. This $[\cdot]_X$ is exactly the *right* action of $\mathrm{GL}_2^+(\mathbf{R})$ on X coming from Theorem 1.5.1.1. Note that we do *not* generally have any relation between $[g]^*E_X$ and E_X if $g\notin\mathrm{SL}_2(\mathbf{Z})$; the point is that such g do *not* preserve the subsheaf $\mathrm{R}^1f_{X*}\underline{\mathbf{Z}}\subseteq\mathrm{R}^1f_{X*}\underline{\mathbf{R}}$.

To summarize, we have seen that for any oriented basis $\{e_1,e_2\}$ of $\mathbf{R}^2$, we get an isomorphism ψ_{e_1,e_2} of (X,q) with the data $(\mathfrak{h},q_0:\underline{\mathbf{R}}^2\to\mathbf{G}_a)$ where $q_0(a,b;z)=(-a+bz;z)$; cf. (1.5.2.4). Our next task is to get a more explicit expression for the right action of $\mathrm{GL}_2^+(\mathbf{R})$ on $\mathfrak{h}$.

THEOREM 1.5.2.2. *The isomorphism $\psi_{e_1,e_2}:X\xrightarrow{\simeq}\mathfrak{h}$ transforms the action of $g\in\mathrm{GL}_2^+(\mathbf{R})$ on X over to the automorphism*

$$z\mapsto\frac{dz-b}{-cz+a}$$

on $\mathfrak{h}$, where g has matrix $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ with respect to the basis $\{e_1,e_2\}$ of $\mathbf{R}^2$.

Proof. We must compute the *unique* automorphism $[g]$ of $\mathfrak{h}$ so that there exists a commutative diagram of sheaves on $\mathfrak{h}$

$$\begin{tikzcd} \underline{\mathbf{R}}^2 \arrow[d, "q_0"'] & \underline{\mathbf{R}}^2 \arrow[l, "\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)"'] \arrow[d, "{[g]^*(q_0)}"] \\ \mathbf{G}_a & {[g]^*\mathbf{G}_a} \arrow[l, "\simeq"'] \end{tikzcd}$$

for some automorphism of holomorphic line bundles $[g]^*\mathbf{G}_a \simeq \mathbf{G}_a$. Using the *canonical* isomorphism $[g]^*\mathbf{G}_a \simeq \mathbf{G}_a$ sending $[g]^*(f)$ to $f \circ [g]$, the map $[g]^*(q_0)$ becomes

$$(r, s; z) \mapsto (-r + s[g](z); z). \tag{1.5.2.6}$$

The condition on $[g]$ is that there exists a nowhere vanishing holomorphic function $\lambda(z)$ (i.e., an automorphism of $\mathbf{G}_a$) such that (1.5.2.6) multiplied by $\lambda(z)$ agrees with $q_0 \circ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. That is, we require

$$-(ar + bs) + (cr + ds) \cdot z = \lambda(z) \cdot (-r + s[g](z))$$

for all $r, s \in \mathbf{R}$.

In other words,

$$\begin{aligned} a - cz &= \lambda(z); \\ b - dz &= -\lambda(z)[g](z). \end{aligned}$$

This says that $\lambda(z) = a - cz$ (which is certainly nowhere zero on $\mathfrak{h}$) and

$$[g](z) = \frac{dz - b}{-cz + a}.$$

□

Keep in mind that our action of $\mathrm{GL}_2^+(\mathbf{R})$ on $\mathfrak{h}$ is presently a *right* action, which is why the above formulas do not look as classically expected for working with modular forms.

The isomorphisms $\xi_g : [g]^*\omega_X^{-1} \simeq \omega_X^{-1}$ constructed in the above proof (with respect to the trivialization $\omega_X^{-1} \simeq \mathbf{G}_a$ via $-q(e_1)$) satisfy $\xi_h \circ [h]^*(\xi_g) = \xi_{hg}$, so we get an "left" action of $\mathrm{GL}_2^+(\mathbf{R})$ on sections of ω_X^{-1} via

$$\begin{aligned} \omega_X^{-1} &\to [g]_*(\omega_X^{-1}) \\ s &\mapsto \xi_g\big([g]^*(s)\big). \end{aligned}$$

Corollary 1.5.2.3. *Choose $k \in \mathbf{Z}$. Using the isomorphism $\psi_{e_1,e_2} : X \xrightarrow{\simeq} \mathfrak{h}$ and using the global section $-q(e_1)$ to trivialize ω_X^{-1}, the action of $\mathrm{GL}_2^+(\mathbf{R})$ on $\omega_X^{\otimes k}$ goes over to an action on $\mathbf{G}_a = \mathcal{O}_{\mathfrak{h}}$ given by*

$$[g]^*(f)(z) = (-cz + a)^{-k} f\left(\frac{dz - b}{-cz + a}\right),$$

where g has matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with respect to the basis $\{e_1, e_2\}$ of $\mathbf{R}^2$.

Proof. For $k = -1$, this follows from the proof of Theorem 1.5.2.2, particularly the calculation of $\lambda(z)$ there. The general case then follows. □

Corollary 1.5.2.3 will allow us to give a geometric definition of modular forms as sections of a line bundle. However, in order to most conviently relate this to modular curves and to geometrically encode conditions at the cusps, we first need to study the Kodaira-Spencer map.

1.5.3. Definition of the Kodaira-Spencer map. We now turn to a discussion of the Kodaira-Spencer map. For any *smooth* map $f : T \to S$ of analytic spaces, the "first fundamental sequence"

$$0 \to f^*\Omega^1_S \to \Omega^1_T \to \Omega^1_{T/S} \to 0 \tag{1.5.3.1}$$

is *exact* (where $\Omega^1_Z \stackrel{\text{def}}{=} \Omega^1_{Z/\operatorname{Sp}\mathbf{C}}$). Indeed, this can be checked locally on T, and applying the faithfully flat base change functor $(\cdot) \otimes_{\mathcal{O}_{T,t}} \widehat{\mathcal{O}}_{T,t}$ gives the analogous diagram

$$0 \to \widehat{\mathcal{O}}_{T,t}\widehat{\otimes}_{\widehat{\mathcal{O}}_{S,s}}\widehat{\Omega}^1_{\widehat{\mathcal{O}}_{S,s}} \to \widehat{\Omega}^1_{\widehat{\mathcal{O}}_{T,t}} \to \widehat{\Omega}^1_{\widehat{\mathcal{O}}_{T,t}/\widehat{\mathcal{O}}_{S,s}} \to 0$$

for $s = f(t)$. Since $\widehat{\mathcal{O}}_{T,t} \simeq \widehat{\mathcal{O}}_{S,s}[[z_1, \ldots, z_n]]$, this sequence is clearly exact (the "formal first fundamental exact sequence").

Consider the special case where the base S is *smooth* and f is proper with connected 1-dimensional fibers of genus g, so (1.5.3.1) is a short exact sequence of vector bundles on T. Consider the *exact* dual sequence

$$0 \to (\Omega^1_{T/S})^\vee \to (\Omega^1_T)^\vee \to (f^*\Omega^1_S)^\vee \to 0.$$

The long exact sequence of higher direct images of this sequence gives a coboundary map

$$f_*((f^*\Omega^1_S)^\vee) \to \mathrm{R}^1 f_*((\Omega^1_{T/S})^\vee).$$

Since Ω^1_S is locally free of finite rank and f is proper and smooth with connected fibers (so $\mathcal{O}_S \simeq f_*\mathcal{O}_T$), the left side is canonically isomorphic to $(\Omega^1_S)^\vee$. We thus obtain a map

$$(\Omega^1_S)^\vee \to \mathrm{R}^1 f_*((\Omega^1_{T/S})^\vee). \tag{1.5.3.2}$$

Meanwhile, the right side of (1.5.3.2) is locally free of formation compatible with base change, thanks to the general:

LEMMA 1.5.3.1. *For any proper smooth $f : T \to S$ with connected fibers of dimension 1, the sheaves $\mathrm{R}^1 f_*((\Omega^1_{T/S})^{\otimes p})$ and $f_*((\Omega^1_{T/S})^{\otimes p})$ are locally free and commute with any base change.*

We include a proof of this well-known fact, due to lack of an adequate reference.

PROOF. Since the fibers are 1-dimensional and algebraic, by GAGA and the theory of cohomology and base change, $\mathrm{R}^p f_*$ kills coherent sheaves for all $p \geq 2$. By cohomology and base change, it is necessary and sufficient (over a fixed base S) to show that $f_*((\Omega^1_{T/S})^{\otimes p})$ is locally free. By the theorem on formal functions, it suffices to consider general local artinian S and artinian base change. All 1-dimensional compact Hausdorff analytic spaces are algebraic, so by GAGA it suffices to consider the scheme analogue. This scheme version is well-known, but for the sake of completeness we summarize the argument.

Let R be a complete noetherian regular local ring with $\mathrm{H}^0(S, \mathcal{O}_S)$ as a quotient. We claim that T lifts across $\operatorname{Spec} R$. Since $T(\mathbf{C}) \neq \emptyset$, certainly $T(S) \neq \emptyset$ by smoothness. The inverse of the invertible ideal sheaf of such a section defines an ample line bundle $\mathcal{L}$ on T. By [**26**, 5.4.5], to lift T to a proper R-scheme, it suffices to lift $(T, \mathcal{L})$ across all artinian quotients of R. Moreover, if we can find such artinian deformations as *smooth* curves, then by the openness of the flat locus and the local criterion for flatness, the resulting lifted R-scheme is automatically a proper *smooth curve* with geometrically connected fibers. (See [**28**, §12] for smoothness, connectedness, and 1-dimensionality of *all* geometric fibers). However, the deformation theory of $(T, \mathcal{L})$ is unobstructed by [**23**, Ch. 3, §7]. Thus we are indeed reduced to the setting of a *reduced* (even regular) base $\operatorname{Spec} R$. But then Grauert's theorem [**45**, Corollary 2, p. 50] and the local constancy of genus reduces us to checking that for a proper smooth connected curve T over an algebraically closed field k, the quantity $\dim_k \mathrm{H}^0(T, (\Omega^1_T)^{\otimes p})$ depends only on p and the genus g of X. For $g = 0, 1$ this is clear. For $g \geq 2$, the dimension is 0 for $p < 0$, 1 for $p = 0$, and $p(3g - 3)$ for $p \geq 1$. This concludes the verification that in our analytic situation $\mathrm{R}^q f_*((\Omega^1_{T/S})^{\otimes p})$ is well-behaved for all p, q.

□

For any $p \in \mathbf{Z}$, the cup product pairing of locally free sheaves

$$\mathrm{R}^1 f_*\big((\Omega^1_{T/S})^{\otimes p}\big) \otimes f_*\big((\Omega^1_{T/S})^{\otimes (1-p)}\big) \to \mathrm{R}^1 f_*(\Omega^1_{T/S}) \tag{1.5.3.3}$$

is perfect, as this can be checked on fibers, where it follows from Serre duality. Taking $p = -1$ yields an isomorphism

$$\mathrm{R}^1 f_*\big((\Omega^1_{T/S})^\vee\big) \simeq \mathcal{H}om_{\mathcal{O}_S}\big(f_*((\Omega^1_{T/S})^{\otimes 2}), \mathrm{R}^1 f_* \Omega^1_{T/S}\big)$$

compatible with arbitrary base change over S. Thus, the dual of the coboundary map in (1.5.3.2) can be viewed as a map

$$f_*\big((\Omega^1_{T/S})^{\otimes 2}\big) \to \Omega^1_S \otimes_{\mathcal{O}_S} \mathrm{R}^1 f_* \Omega^1_{T/S}. \tag{1.5.3.4}$$

Now use the base change compatible isomorphism

$$\mathrm{R}^1 f_* \Omega^1_{T/S} \simeq \mathcal{O}_S \tag{1.5.3.5}$$

which is the *negative* of the one from Theorem 1.2.2.2 (i.e., $-\frac{1}{2\pi i}\int_{T_s}$ on fibers). This corresponds to the negative of a natural pairing on Jacobians (cf. Remark 1.3.3.2), but it *coincides* with what one gets from Grothendieck duality in the analogous scheme-theoretic situation [**7**, B.2.2]. Moreover, for our purposes the relevance of this choice is that it will make the formula in Theorem 1.5.4.1 below have no extraneous sign. We finally arrive via (1.5.3.4) and (1.5.3.5) at the $\mathcal{O}_S$-linear map *Kodaira-Spencer map*

$$\underline{\mathrm{KS}}_{T/S} : f_*((\Omega^1_{T/S})^{\otimes 2}) \to \Omega^1_S.$$

In [**36**], Katz gives a rather general construction of a Kodaira-Spencer map which is an $\mathcal{O}_S$-linear map $(\Omega^1_S)^\vee \to \mathrm{R}^1 f_*((\Omega^1_{T/S})^\vee)$. In the general situation being considered above, Katz' construction is just the dual of the connecting homomorphism (1.5.3.2), so our Kodaira-Spencer map is essentially the dual of the one constructed by Katz. We do not need this compatibility, so we omit the essentially mechanical details of the verification.

Since the entire construction above is compatible with base change by any map $h : S' \to S$ with S' *smooth* (but the map h need not be smooth!), for the base change $f' : T' \to S'$ we get a commutative diagram

(1.5.3.6)
$$\begin{tikzcd} h^*(f_*(\Omega^1_{T/S})^{\otimes 2}) \arrow[r, "h^*(\underline{\mathrm{KS}}_{T/S})"] \arrow[d, "\simeq"'] & h^*\Omega^1_S \arrow[d] \\ f'_*(\Omega^1_{T'/S'})^{\otimes 2} \arrow[r, "\underline{\mathrm{KS}}_{T'/S'}"] & \Omega^1_{S'} \end{tikzcd}$$

This is the sense in which the Kodaira-Spencer map respects base change. In the special case where $T \to S$ has genus 1 fibers, we obtain a simplification of the Kodaira-Spencer map since the map of invertible sheaves

$$f_*(\Omega^1_{T/S})^{\otimes 2} \to f_*\big({\Omega^1_{T/S}}^{\otimes 2}\big)$$

is an isomorphism, as one checks on fibers.

Applying all of this to the universal family $E_X \to X$ of genus 1 curves over the *smooth* base X, we obtain a map

$$\underline{\mathrm{KS}}_{E_X/X} : \omega_X^{\otimes 2} \to \Omega^1_X$$

of fundamental importance in what follows.

1.5.4. Explicit computation of a Kodaira-Spencer map.

THEOREM 1.5.4.1. *The map $\underline{\mathrm{KS}}_{E_X/X}$ is an $\mathrm{SL}_2(\mathbf{R})$-equivariant isomorphism. Under $\psi_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)} : X \xrightarrow{\simeq} \mathfrak{h}$, it becomes the map*

$$\underline{\mathrm{KS}}_{\mathcal{E}^{\mathrm{an}}/\mathfrak{h}} : \omega^{\otimes 2}_{\mathcal{E}^{\mathrm{an}}/\mathfrak{h}} \to \Omega^1_{\mathfrak{h}}$$

determined by $(2\pi i \mathrm{d}\tau)^{\otimes 2} \mapsto 2\pi i \mathrm{d}z$.

REMARK 1.5.4.2. Here $\mathrm{SL}_2(\mathbf{R})$ acts on $\omega_X^{\otimes 2}$ as in Corollary 1.5.2.3 and on Ω^1_X through its action on X. Since $\mathrm{SL}_2(\mathbf{R})$ does *not* act on $E_X \to X$, this equivariance of $\underline{\mathrm{KS}}_{E_X/X}$ is *not* obvious. Also, if we had used Theorem 1.2.2.2 (rather than its negative) to define the isomorphism $\mathrm{R}^1 f_*\Omega^1_{T/S} \simeq \mathcal{O}_S$ used in the definitions of $\underline{\mathrm{KS}}_{T/S}$, then the formula in Theorem 1.5.4.1 would change by a sign.

PROOF. When ω_X^{-1} is trivialized by the global section $-q\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$, the trivialization of $\omega_X^{\otimes 2}$ goes over by $\psi_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)}$ to the trivialization $\omega^{\otimes 2}_{\mathcal{E}^{\mathrm{an}}/\mathfrak{h}} \simeq \mathbf{G}_a$ given by $(2\pi i \mathrm{d}\tau)^{\otimes 2} \mapsto 1$. (See (1.5.2.4).) It is easy to check that the unique isomorphism determined by

$$\begin{aligned} \mathbf{G}_a &\to \Omega^1_{\mathfrak{h}} \\ 1 &\mapsto 2\pi i \mathrm{d}z \end{aligned}$$

is $\mathrm{SL}_2(\mathbf{R})$-equivariant when $\mathrm{SL}_2(\mathbf{R})$ acts on $\mathbf{G}_a$ by

$$g^*(f)(z) = (-cz+a)^{-2} f\left(\frac{dz-b}{-cz+a}\right)$$

and on $\Omega^1_{\mathfrak{h}}$ by pullback along

$$z \mapsto \frac{dz-b}{-cz+a},$$

for $g = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{R})$; this uses the fact that $\det g = 1$.

By (1.5.3.6), the maps $\underline{\mathrm{KS}}_{E_X/X}$ and $\underline{\mathrm{KS}}_{\mathcal{E}^{\mathrm{an}}/\mathfrak{h}}$ are compatible via the canonical identification of $E_X \to X$ with the pullback of $\mathcal{E}^{\mathrm{an}} \to \mathfrak{h}$ along $\psi_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)} : X \xrightarrow{\simeq} \mathfrak{h}$. Thus, we see by Theorem 1.5.2.2 and Corollary 1.5.2.3 (with $k = 2$), that the entire theorem will follow if we check that

$$\underline{\mathrm{KS}}_{\mathcal{E}^{\mathrm{an}}/\mathfrak{h}}\big((2\pi i \mathrm{d}\tau)^{\otimes 2}\big) = 2\pi i \mathrm{d}z.$$

In general, in order to show that two $\mathcal{O}_{\mathfrak{h}}$-linear maps

$$\omega^{\otimes 2}_{\mathcal{E}^{\mathrm{an}}/\mathfrak{h}} \to \Omega^1_{\mathfrak{h}}$$

coincide, it suffices (since $\mathfrak{h}$ is a connected manifold) to check on a small open set in $\mathfrak{h}$. The significance of this to us is that there is a local isomorphism

$$e^{2\pi i(\cdot)} : \mathfrak{h} \to \Delta^* = \{q \in \mathbf{C} \,|\, 0 < |q| < 1\}$$

which (locally on the base) identifies $\mathcal{E}^{\mathrm{an}} \to \mathfrak{h}$ with the pullback of a rather special elliptic curve

$$f_{\underline{\mathrm{Tate}}^*} : \underline{\mathrm{Tate}}^* = (\mathbf{C}^\times \times \Delta^*)/(\mathbf{Z} \times \Delta^*) \to \Delta^*,$$

where we use the inclusion

$$\mathbf{Z} \times \Delta^* \hookrightarrow \mathbf{C}^\times \times \Delta^*$$
$$(n, q) \mapsto (q^n, q).$$

This quotient is given a canonical analytic structure of elliptic curve over Δ^* either by a variant of Lemma 1.1.2.1 or by applying $e^{2\pi i(\cdot)}$ to $\mathcal{E}^{\mathrm{an}} = (\mathbf{C} \times \mathfrak{h})/(\mathbf{Z}^2 \times \mathfrak{h})$ and working locally. (We'll see later that $\underline{\mathrm{Tate}}^*$ extends canonically to a special object $\underline{\mathrm{Tate}}$ over $\Delta = \{q \in \mathbf{C} \,|\, |q| < 1\}$.)

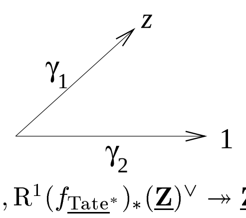
Using the parameter q, we get a canonical quotient

$$\mathrm{R}^1(f_{\underline{\mathrm{Tate}}^*})_*(\underline{\mathbf{Z}})^\vee \twoheadrightarrow \underline{\mathbf{Z}}$$

inducing

$$\mathrm{H}_1(\mathbf{C}^\times/q^{\mathbf{Z}}, \mathbf{Z}) = \mathrm{H}_1(\mathbf{C}/[1, z], \mathbf{Z}) \twoheadrightarrow \mathbf{Z}$$

via $\gamma_2 \mapsto 0$, $\gamma_1 \mapsto 1$. Note that this is well-defined since $q = e^{2\pi i z}$ determines z (mod 1), and we use the basis



One can check that $(\underline{\mathrm{Tate}}^*, \mathrm{R}^1(f_{\underline{\mathrm{Tate}}^*})_*(\underline{\mathbf{Z}})^\vee \twoheadrightarrow \underline{\mathbf{Z}})$ is the *universal* pair consisting of an elliptic curve $f : E \to S$ and a surjection $\mathrm{R}^1 f_*(\underline{\mathbf{Z}})^\vee \overset{\pi}{\twoheadrightarrow} \underline{\mathbf{Z}}$. To see this, one notes that such pairs have no non-trivial automorphisms (for this it is enough to work on the fibers) and locally one can "enhance" such a map $\pi : \mathrm{R}^1 f_*(\underline{\mathbf{Z}})^\vee \twoheadrightarrow \underline{\mathbf{Z}}$ to an isomorphism $\pi \oplus \pi' : \mathrm{R}^1 f_*(\underline{\mathbf{Z}})^\vee \simeq \underline{\mathbf{Z}}^2$ whose (inverse) dual $\alpha : \mathrm{R}^1 f_* \underline{\mathbf{Z}} \simeq \underline{\mathbf{Z}}^2$ is *i-permissible*, with π' unique up to translation by $\pi' \to \pi' + n\pi$ for $n \in \underline{\mathbf{Z}}$. This amounts to the corresponding map $S \to \mathfrak{h}$ locally being unique up to a translation $z \mapsto z + n$ on $\mathfrak{h}$, so we get a well-defined map $S \to \mathfrak{h}/(z \sim z + n) \overset{e^{2\pi i(\cdot)}}{\simeq} \Delta^*$. The universality then follows from the universality of $(\mathcal{E}^{\mathrm{an}} \to \mathfrak{h}, \alpha^{\mathrm{an}})$ and the *definition* of $\underline{\mathrm{Tate}}^*$ in terms of this.

Letting $t = e^{2\pi i\tau}$ denote the parameter on $\mathbf{C}^\times$ and $q = e^{2\pi iz}$ denote the parameter on Δ^*, $\omega^{\otimes 2}_{\underline{\mathrm{Tate}}^*/\Delta^*}$ has generator $(\mathrm{d}t/t)^{\otimes 2}$ and we want to show that

$$\underline{\mathrm{KS}}_{\underline{\mathrm{Tate}}^*/\Delta^*}\left((\mathrm{d}t/t)^{\otimes 2}\right) \stackrel{?}{=} \mathrm{d}q/q.$$

Going back to the *definition* of $\underline{\mathrm{KS}}$ in terms of the snake lemma and cup products, which can all be calculated via Čech cohomology, here is what we need to do. Let φ be a global generator of $(\Omega^1_{\Delta^*})^\vee$ sending $\mathrm{d}q/q$ to 1. Let $\mathfrak{U} = \{U_i\}$ be an ordered open covering of $\underline{\mathrm{Tate}}^*$ and $\widetilde{\varphi}_i$ a lift of

$$(f_{\underline{\mathrm{Tate}}^*})^*(\varphi)|_{U_i} \in (f_{\underline{\mathrm{Tate}}^*})^*\Omega^1_{\Delta^*})^\vee|_{U_i}$$

to $(\Omega^1_{\underline{\mathrm{Tate}}^*})^\vee|_{U_i}$. The differences

$$\widetilde{\varphi}_{ij} = \widetilde{\varphi}_j - \widetilde{\varphi}_i \in \mathrm{H}^0(U_i \cap U_j, (\Omega^1_{\underline{\mathrm{Tate}}^*/\Delta^*})^\vee)$$

for $i < j$ give an element in $\mathrm{H}^1(\mathfrak{U}, (\Omega^1_{\underline{\mathrm{Tate}}^*/\Delta^*})^\vee)$. Cupping with

$$(\mathrm{d}t/t)^{\otimes 2} \in \mathrm{H}^0\left(\mathfrak{U}, (\Omega^1_{\underline{\mathrm{Tate}}^*/\Delta^*})^{\otimes 2}\right)$$

gives an element in $\mathrm{H}^1(\mathfrak{U}, \Omega^1_{\underline{\mathrm{Tate}}^*/\Delta^*})$, and for each $q \in \Delta^*$ this in turn yields an element in $\mathrm{H}^1(\mathfrak{U}_q, \Omega^1_{\underline{\mathrm{Tate}}^*_q})$ and thus an element in $\mathrm{H}^1(\underline{\mathrm{Tate}}^*_q, \Omega^1_{\underline{\mathrm{Tate}}^*_q})$. Here $\underline{\mathrm{Tate}}^*_q = \mathbf{C}^\times/q^{\mathbf{Z}}$ is the analytic fiber of $\underline{\mathrm{Tate}}^* \to \Delta^*$ over $q \in \Delta^*$ and $\mathfrak{U}_q$ is the ordered open covering on the fiber induced by $\mathfrak{U}$. Due to the sign in the definition of (1.5.3.5), under the map

$$-\frac{1}{2\pi i}\int : \mathrm{H}^1(\underline{\mathrm{Tate}}^*_q, \Omega^1_{\underline{\mathrm{Tate}}^*_q}) \to \mathbf{C},$$

we want our Čech class to map to 1.

To carry out this calculation, we work with a very concrete open covering $\{U_0, U_1\}$ of $\underline{\mathrm{Tate}}^*$ by "relative annuli". This step is very messy to contemplate in terms of $\mathcal{E}^{\mathrm{an}} \to \mathfrak{h}$; working on $\underline{\mathrm{Tate}}^*$ will simplify this matter.

Define

$$U_0 = \left\{(t_0, q_0) \in \mathbf{C}^\times \times \Delta^* \,|\, |q_0|^{\frac{1}{2}} < |t_0| < |q_0|^{-\frac{1}{2}}\right\}$$
$$U_1 = \left\{(t_1, q_1) \in \mathbf{C}^\times \times \Delta^* \,|\, |q_1| < |t_1| < 1\right\}$$

and

$$V = \left\{(t, q) \in \mathbf{C}^\times \times \Delta^* \,|\, |q|^{1/2} < |t| < 1 \text{ or } 1 < |t| < |q|^{-1/2}\right\}.$$

Note that V is disconnected; let

$$W = \{(t, q) \,|\, 1 < |t| < |q|^{-\frac{1}{2}}\}$$
$$W' = \{(t, q) \,|\, |q|^{\frac{1}{2}} < |t| < 1\}$$

be the two components of V. See Figure 2 for the images of these sets in fiber of $\mathbf{C}^\times \times \Delta^*$ over a fixed $q \in \Delta^*$.

We want to describe $\underline{\mathrm{Tate}}^*$ as the gluing of U_0 and U_1 along V. In order to be rigorous and to follow the method of analysis we'll need to "compactify" Tate curves later, we *first* construct such a gluing as an abstract complex manifold proper and smooth over Δ^*, and *then* produce an analytic map to $\underline{\mathrm{Tate}}^*$ which we check is an isomorphism on fibers, hence an analytic isomorphism.

Motivated by the picture of annuli and $\mathbf{C}^\times/q^{\mathbf{Z}}$ on fibers over $q \in \Delta^*$, we define $\mathcal{E}$ to be the gluing of U_0 and U_1 along V, where we use the canonical inclusion

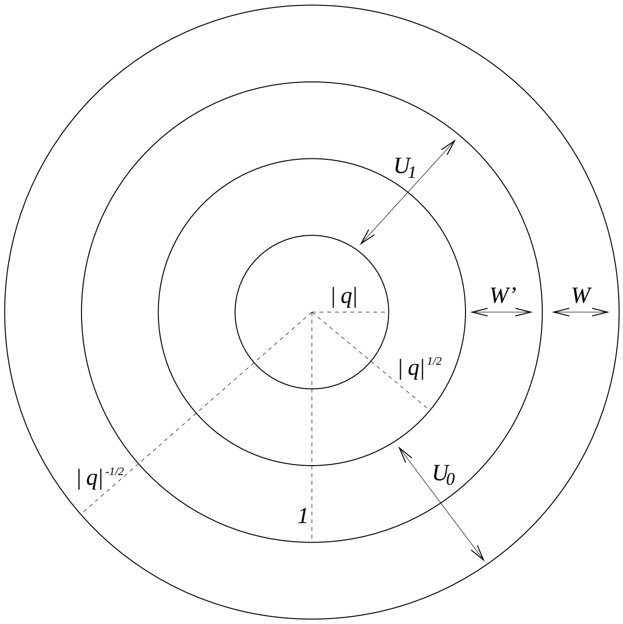


FIGURE 2

$V \hookrightarrow U_0$ sending $(t,q) \mapsto (t,q)$ and the open immersion $V \hookrightarrow U_1$ defined on connected components by

- the canonical open immersion $W' \hookrightarrow U_1$ sending $(t,q) \mapsto (t,q)$,
- the open immersion $W \hookrightarrow U_1$ sending $(t,q) \mapsto (tq,q)$.

The image of V in $U_0 \times_{\Delta^*} U_1$ is the *closed* set

$$\{(t_0,t_1,q) \,|\, t_1 = qt_0 \text{ or } t_1 = t_0\},$$

so $\mathcal{E}$ is a *Hausdorff* complex manifold, smooth over Δ^*. It is then easy to check that under $\mathcal{E} \to \Delta^*$ the preimage of a compact set is compact, and that the maps $U_j \to \mathbf{C}^\times \times \Delta^* \to \underline{\mathrm{Tate}}^*$ respect the gluing. We therefore get a commutative diagram

$$\begin{tikzcd} \mathcal{E} \arrow[rr] \arrow[dr] & & \underline{\mathrm{Tate}}^* \arrow[dl] \\ & \Delta^* & \end{tikzcd}$$

between *proper* smooth objects over Δ^*, taking the map $\Delta^* \to U_0$ defined by $q \mapsto (1,q) \in U_0$ over to the identity section of $\underline{\mathrm{Tate}}^*$. The map on fibers over Δ^* is easily checked to be an isomorphism. Thus, the map $\mathcal{E} \to \underline{\mathrm{Tate}}^*$ is an isomorphism, so we can view $\mathfrak{U} = \{U_0, U_1\}$ as an open covering of $\underline{\mathrm{Tate}}^*$.

Note that $\Omega^1_{U_j}$ has free basis $\mathrm{d}q_j/q_j$, $\mathrm{d}t_j/t_j$ over $\mathcal{O}_{U_j}$. Let

$$\eta_j = (\mathrm{d}q_j/q_j)^\vee \in \mathrm{H}^0(U_j, (\Omega^1_{U_j})^\vee),$$

which is a lift of the pullback of

$$(\mathrm{d}q/q)^\vee \in \mathrm{H}^0(\Delta^*, (\Omega^1_{\Delta^*})^\vee)$$

to a section of $(\Omega^1_{\underline{\mathrm{Tate}}^*})^\vee$ over U_j. Note that $\eta_0|_V = (\mathrm{d}q/q)^\vee$, where we regard V as an open subspace of $\underline{\mathrm{Tate}}^*$ via $V \hookrightarrow U_0$, and $\eta_1|_{W'} = (\mathrm{d}q/q)^\vee$. But $\eta_1|_W$ sends $\mathrm{d}q/q$ to 1 and sends

$$\frac{\mathrm{d}(tq)}{tq} = \frac{\mathrm{d}t}{t} + \frac{\mathrm{d}q}{q}$$

to 0, hence sends $\mathrm{d}t/t$ to -1. Since $W \cup W' = U_0 \cap U_1$ in $\underline{\mathrm{Tate}}^*$, we therefore find that

$$\eta_1 - \eta_0 \in \mathrm{H}^0\big(U_0 \cap U_1, (\Omega^1_{\underline{\mathrm{Tate}}^*/\Delta^*})^\vee\big) = \mathrm{H}^0\big(W, (\Omega^1_{W/\Delta^*})^\vee\big) \oplus \mathrm{H}^0\big(W', (\Omega^1_{W'/\Delta^*})^\vee\big)$$

is the element

$$(-(\mathrm{d}t/t)^\vee, 0).$$

Cupping this with $(\mathrm{d}t/t)^{\otimes 2}$ and passing to the fiber over $q \in \Delta^*$ yields

$$(-\mathrm{d}t/t, 0) \in \mathrm{H}^1(\mathfrak{U}_q, \Omega^1_{\underline{\mathrm{Tate}}^*_q}) = \mathrm{H}^0(W_q, \Omega^1_{W_q}) \oplus \mathrm{H}^0(W'_q, \Omega^1_{W'_q}).$$

It remains then to check that

$$\mathrm{H}^1(\mathfrak{U}_q, \Omega^1_{\underline{\mathrm{Tate}}^*_q}) \to \mathrm{H}^1(\underline{\mathrm{Tate}}^*_q, \Omega^1_{\underline{\mathrm{Tate}}^*_q}) \overset{-\frac{1}{2\pi i}\int}{\simeq} \mathbf{C}$$

sends $(-\mathrm{d}t/t, 0)$ to 1. We check this by a calculation similar to that in the proof of Corollary 1.2.3.1. Let $\{\varphi_0, 1-\varphi_0\}$ be a C^∞ partition of unity subordinate to $\{U_{0,q}, U_{1,q}\}$, with $\varphi_0 = 1$ near $|t| = 1$.

Let $\mathcal{A}^{\bullet,\bullet}$ denote the Dolbeault double complex on $\underline{\mathrm{Tate}}^*_q$. Define a map of acyclic resolutions of $\Omega^1_{\underline{\mathrm{Tate}}^*_q}$:

$$\mathcal{C}^\bullet(\mathfrak{U}_q, \Omega^1_{\underline{\mathrm{Tate}}^*_q}) \to \mathcal{A}^{1,\bullet}$$

by sending

$$(\omega_0, \omega_1) \mapsto \varphi_0\omega_0 + (1-\varphi_0)\omega_1$$

in degree 0 and

$$\eta \mapsto -\overline{\partial}\varphi_0 \wedge \eta$$

in degree 1. Then the map

$$\mathrm{H}^1(\mathfrak{U}_q, \Omega^1_{\underline{\mathrm{Tate}}^*_q}) \to \mathrm{H}^1(\underline{\mathrm{Tate}}^*_q, \Omega^1_{\underline{\mathrm{Tate}}^*_q}) \simeq \mathrm{H}^2(\underline{\mathrm{Tate}}^*_q, \underline{\mathbf{C}})$$

takes the Čech class of $(-\mathrm{d}t/t, 0)$ to the class of the *compactly supported* differential form

$$\overline{\partial}\varphi_0 \wedge \frac{\mathrm{d}t}{t} = \mathrm{d}\varphi_0 \wedge \frac{\mathrm{d}t}{t}$$

on $W_q \subseteq \underline{\mathrm{Tate}}^*_q$. In particular, over $W'_q \subseteq U_{0,q}$ and near $|t| = 1$ our form vanishes; this follows from the fact that φ_0 is identically 1 near $|t| = 1$. (See Figure 3.) We want

$$-\frac{1}{2\pi i}\int_{1 \le |t| \le |q|^{\frac{1}{2}}} \mathrm{d}\varphi_0 \wedge \frac{\mathrm{d}t}{t} = 1.$$

By Stokes' theorem, we compute

$$\frac{1}{2\pi i}\int_{1 \le |t| \le |q|^{-\frac{1}{2}}} \mathrm{d}\,\varphi_0 \wedge \frac{\mathrm{d}t}{t} = \frac{1}{2\pi i}\int_{|t| = |q|^{-\frac{1}{2}}} \varphi_0 \frac{\mathrm{d}t}{t} - \frac{1}{2\pi i}\int_{|t|=1} \varphi_0 \frac{\mathrm{d}t}{t} = -1.$$

This completes the calculation of the Kodaira-Spencer map.

$\square$

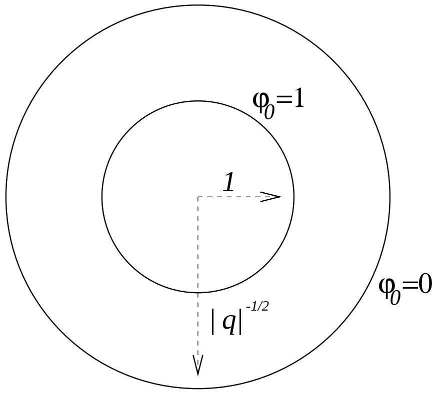


FIGURE 3

For $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{R})$, consider the isomorphism

$$\psi_{\left(\begin{smallmatrix} a \\ c \end{smallmatrix}\right), \left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right)} : X \xrightarrow{\simeq} \mathfrak{h}.$$

Recall from the discussion preceding Theorem 1.5.2.2 that we can write this map as

$$\psi_{ge_1, ge_2} = \psi_{e_1, e_2} \circ [g]$$

in terms of the standard basis $\{e_1, e_2\} = \{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)\}$.

COROLLARY 1.5.4.3. *Using the section* $-q\left(g\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\right) = [g]^* \left(-q\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\right)$ *to trivialize* ω_X^{-1}, *the pullback map* $(\psi_{ge_1, ge_2}^{-1})^* = (\psi_{e_1, e_2}^{-1})^* \circ [g]^{-1*}$ *transforms the isomorphism*

$$\underline{\mathrm{KS}}_{E_X/X} : \omega_X^{\otimes 2} \simeq \Omega_X^1$$

into the line bundle isomorphism $\mathbf{G}_a \simeq \Omega_{\mathfrak{h}}^1$ *sending* 1 *to* $2\pi i dz$.

PROOF. This corollary simply asserts that

$$(\psi_{ge_1, ge_2}^{-1})^* \circ \underline{\mathrm{KS}}_{E_X/X} \circ (\psi_{ge_1, ge_2})^* \circ (\psi_{ge_1, ge_2}^{-1})^* \left(-q\left(g\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\right)\right) = \\ (\psi_{e_1, e_2}^{-1})^* \circ \underline{\mathrm{KS}}_{E_X/X} \circ (\psi_{e_1, e_2})^* \circ (\psi_{e_1, e_2}^{-1})^* \left(-q\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\right).$$

Since

$$(\psi_{ge_1, ge_2}^{-1})^* \circ \left(-q\left(g\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\right)\right) = (\psi_{e_1, e_2}^{-1})^* \left(-q\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\right)$$

and $\psi_{ge_1, ge_2}^* = [g]^* \circ \psi_{e_1, e_2}^*$, it suffices to show that $[g]^{-1*} \underline{\mathrm{KS}}_{E_X/X} \circ [g]^* = \underline{\mathrm{KS}}_{E_X/X}$. This equality is just the $\mathrm{SL}_2(\mathbf{R})$-equivariance of $\underline{\mathrm{KS}}_{E_X/X}$. ☐

The significance of Corollary 1.5.4.3 is that it will enable us to extend the Kodaira-Spencer isomorphism $\omega_X^{\otimes 2} \simeq \Omega_X^1$ to compactifications of certain quotients of X. Before we proceed to these compactifications and use them to identify modular forms with sections of line bundles on such compactifications, we should look at our $\mathrm{SL}_2(\mathbf{Z})$-action more carefully.

Consider the action of $\mathrm{SL}_2(\mathbf{Z})$ on X via its right action on the universal object $(E_X,\ \alpha_X^\vee)$. By Theorem 1.5.1.1, together with Theorem 1.5.2.2 and Corollary 1.5.2.3, the *right* action of $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z})$ on X and $\omega_X^{\otimes k}$ is transformed by $\psi_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)} : X \xrightarrow{\simeq} \mathfrak{h}$ and $-q\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) : \omega_X^{-1} \simeq \mathbf{G}_a$ into the right action

$$z \mapsto \frac{az+c}{bz+d}, \qquad f \mapsto (bz+d)^{-k} f\left(\frac{az+c}{bz+d}\right). \tag{1.5.4.1}$$

The study of modular curves via *level* structure is really a study of trivializations of n-torsion points, which lie in *homology*, not cohomology. That is, for $f : E \to S$, instead of studying $\alpha : \mathrm{R}^1 f_* \underline{\mathbf{Z}} \simeq \underline{\mathbf{Z}}^2$, we really want to study

$$\alpha^\vee : (\underline{\mathbf{Z}/n})^2 \simeq \mathrm{R}^1 f_* (\underline{\mathbf{Z}/n})^\vee.$$

We then get a *left* action of $\mathrm{SL}_2(\mathbf{Z})$ on pairs $(E_{/S}, \alpha^\vee \bmod n)$ via $\alpha^\vee \to \alpha^\vee \circ g^t$. Since passing to the dual map is the "same" as passing to the transpose matrix, we will recover the expected classical formulas later on in the setting of modular curves, as (1.5.4.1) is merely off by a "transpose" from the classical formulas.

Before we can pass to quotients, we must check the following fact.

Lemma 1.5.4.4. *The Hodge filtration*

$$0 \to \omega_X \to \underline{\mathbf{R}}^2 \otimes_{\underline{\mathbf{R}}} \mathcal{O}_X \xrightarrow{q} \omega_X^{-1} \to 0,$$

using

$$\underline{\mathbf{R}}^2 \otimes_{\underline{\mathbf{R}}} \mathcal{O}_X \simeq \underline{\mathbf{Z}}^2 \otimes_{\underline{\mathbf{Z}}} \mathcal{O}_X \xrightarrow{\alpha_X \otimes 1} \mathrm{R}^1 f_{X*} \underline{\mathbf{Z}} \otimes_{\underline{\mathbf{Z}}} \mathcal{O}_X \simeq \mathrm{R}^1 f_{X*} \underline{\mathbf{C}} \otimes_{\underline{\mathbf{C}}} \mathcal{O}_X = \mathrm{R}^1 f_{X*}(f_X^{-1} \mathcal{O}_X),$$

is $\mathrm{SL}_2(\mathbf{R})$*-equivariant. More precisely, for* $g \in \mathrm{SL}_2(\mathbf{R})$*, the diagram*

$$\begin{tikzcd} 0 \arrow[r] & {[g]^*\omega_X} \arrow[r] \arrow[d, "\simeq"'] & \underline{\mathbf{R}}^2 \otimes_{\underline{\mathbf{R}}} \mathcal{O}_X \arrow[r, "{[g]^*(q)}"] \arrow[d, "g\otimes 1"] & {[g]^*\omega_X^{-1}} \arrow[r] & 0 \\ 0 \arrow[r] & \omega_X \arrow[r] & \underline{\mathbf{R}}^2 \otimes_{\underline{\mathbf{R}}} \mathcal{O}_X \arrow[r, "q"] & \omega_X^{-1} \arrow[r] \arrow[u, "\simeq"'] & 0 \end{tikzcd} \tag{1.5.4.2}$$

commutes, where the isomorphism in the right column is as in (1.5.2.5) *and the two outer columns are dual to each other.*

When $g \in \mathrm{SL}_2(\mathbf{Z})$*, the isomorphism* $[g]^*\omega_X \simeq \omega_X$ *in the left side of* (1.5.4.2) *is equal to the canonical isomorphism*

$$[g]^*\omega_X \simeq \omega_{[g]^*E_X/X} \simeq \omega_{E_X/X} = \omega_X.$$

Proof. Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{R})$. Using the isomorphism

$$\psi_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)} : X \xrightarrow{\simeq} \mathfrak{h}$$

and passing to the fiber over $z \in \mathfrak{h}$, the diagram (1.5.4.2) can be written as

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{H}^0(\mathcal{E}^{\mathrm{an}}_{[g](z)}, \Omega^1_{\mathcal{E}^{\mathrm{an}}_{[g](z)}}) \arrow[r] \arrow[d, "\simeq"'] & \mathrm{H}^1(\mathcal{E}^{\mathrm{an}}_{[g](z)}, \underline{\mathbf{C}}) \arrow[r] \arrow[d, "\simeq"'] & \mathrm{H}^0(\mathcal{E}^{\mathrm{an}}_{[g](z)}, \Omega^1_{\mathcal{E}^{\mathrm{an}}_{[g](z)}})^\vee \arrow[r] \arrow[d, "\simeq"] & 0 \\ 0 \arrow[r] & \mathrm{H}^0(\mathcal{E}^{\mathrm{an}}_z, \Omega^1_{\mathcal{E}^{\mathrm{an}}_z}) \arrow[r] & \mathrm{H}^1(\mathcal{E}^{\mathrm{an}}_z, \underline{\mathbf{C}}) \arrow[r] & \mathrm{H}^0(\mathcal{E}^{\mathrm{an}}_z, \Omega^1_{\mathcal{E}^{\mathrm{an}}_z})^\vee \arrow[r] & 0 \end{tikzcd}$$

where $[g](z) = \frac{dz-b}{-cz+a}$, the right column sends $(2\pi i \mathrm{d}\tau)^\vee$ to

$$\lambda(z)(2\pi i \mathrm{d}\tau)^\vee = (-cz+a)(2\pi i \mathrm{d}\tau)^\vee$$

(with τ the coordinate on $\mathbf{C}$ via the isomorphisms $\mathcal{E}_z^{\mathrm{an}} \simeq \mathbf{C}/[1,z]$ and $\mathcal{E}_{[g](z)}^{\mathrm{an}} \simeq \mathbf{C}/[1,[g](z)]$), the left column sends

$$2\pi i \mathrm{d}\tau \mapsto \left(\frac{2\pi i}{-cz+a}\right)\mathrm{d}\tau,$$

and the middle column sends $\{\gamma_1^\vee, \gamma_2^\vee\}$ to $\{a\gamma_1^\vee + c\gamma_2^\vee, b\gamma_1^\vee + d\gamma_2^\vee\}$. Also, the left horizontal maps are the canonical inclusions and the right horizontal maps are given by $\eta \mapsto \frac{1}{2\pi i}\int \eta \wedge (\cdot)$.

As we noted above (1.5.2.4), $2\pi i \mathrm{d}\tau = 2\pi i(\gamma_2^\vee + z\gamma_1^\vee)$ in the fiber over z, so the commutativity of the left square is straightforward. The commutativity of the right square is also straightforward, and was verified already in the proof of Theorem 1.5.2.2 and the discussion preceding that theorem.

By Theorem 1.5.1.1, for $g \in \mathrm{SL}_2(\mathbf{Z})$ the isomorphism

$$\mathbf{C}/[1,z] \simeq \mathcal{E}_z^{\mathrm{an}} \xrightarrow{[g]} \mathcal{E}_{[g](z)}^{\mathrm{an}} \simeq \mathbf{C}/[1,[g](z)]$$

is the unique one which takes the homology basis

$$\{\alpha_z^\vee\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), \alpha_z^\vee\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\} = \{z, 1\}$$

to the homology basis

$$\begin{aligned}\{\alpha_{[g](z)}^\vee \circ g^t\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), \alpha_{[g](z)}^\vee \circ g^t\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\} &= \{a[g](z) + b\cdot 1, c[g](z) + d\cdot 1\}\\ &= \left\{\frac{z}{-cz+a}, \frac{1}{-cz+a}\right\}\end{aligned}$$

so it must be given by multiplication by $(-cz+a)^{-1} \in \mathbf{C}$. This pulls back $2\pi i \mathrm{d}\tau$ to $(-cz+a)^{-1}(2\pi i \mathrm{d}\tau)$, as desired.

$\square$

REMARK 1.5.4.5. Since the quotients $\Gamma\backslash X$ and X/Γ are the "same" for the right action $x \mapsto x \cdot g$ and the left action $x \mapsto x \cdot g^{-1}$, Theorem 1.5.2.2 and the last part of Lemma 1.5.4.4 ensure that the quotients we will form below *do* give the usual modular curves and spaces of automorphic forms.

1.5.5. Fuchsian groups. A *Fuchsian group* is a discrete subgroup of $\mathrm{SL}_2(\mathbf{R})$. We refer to [**44**, Chapter 1] for proofs of the following facts.

1. Fuchsian groups are precisely the subgroups $\Gamma \subseteq \mathrm{SL}_2(\mathbf{R})$ whose standard (left) action on $\mathfrak{h}$ by linear fractional transformations is properly discontinuous (i.e., for all $z \in \mathfrak{h}$ there is an open set U containing z with $\gamma U \cap U = \emptyset$ when $\gamma z \neq z$).
2. If $x \in \mathbf{R} \cup \{\infty\}$ is a *cusp* of Γ (i.e., the unique fixed point of some element of Γ) then for any $g \in \mathrm{SL}_2(\mathbf{R})$ with $g(x) = \infty$ we have

$$\{\pm 1\} \cdot (g\,\mathrm{Stab}_\Gamma(x) g^{-1}) = \{\pm 1\} \cdot \left(\begin{smallmatrix}1 & h\\ 0 & 1\end{smallmatrix}\right)^{\mathbf{Z}}$$

for a unique $h > 0$ depending on g. (Conjugating g by

$$\begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix} \in \mathrm{Stab}_{\mathrm{SL}_2(\mathbf{R})}(\infty)$$

replaces h by $a^2 h$.) When $-1 \notin \Gamma$, then

$$g\,\mathrm{Stab}_\Gamma(x) g^{-1} = \left(\begin{smallmatrix}\varepsilon & h\\ 0 & \varepsilon\end{smallmatrix}\right)^{\mathbf{Z}}$$

for a unique $\varepsilon = \pm 1$ independent of g. We call x a *regular* (resp. *irregular*) cusp when $\varepsilon = 1$ (resp. $\varepsilon = -1$).

3. The union $\overline{\mathfrak{h}}_\Gamma$ of $\mathfrak{h}$ and the cusps of Γ has a natural Hausdorff topology (with respect to which the Γ-action is properly discontinuous) and $\Gamma \backslash \overline{\mathfrak{h}}_\Gamma$ has a natural structure of (possibly non-compact) Riemann surface. The isotropy groups of points $z \in \mathfrak{h}$ which are fixed by a non-scalar element of Γ are finite (z is an *elliptic point* when this group contains non-scalars).
4. $\mathrm{SL}_2(\mathbf{R})$ is unimodular, and the covolume $\mathrm{vol}(\Gamma \backslash \mathrm{SL}_2(\mathbf{R}))$ is finite if and only if $\Gamma \backslash \overline{\mathfrak{h}}_\Gamma$ is compact, in which case $\Gamma \backslash \overline{\mathfrak{h}}_\Gamma$ has only finitely many equivalence classes of cusps and elliptic points. The same properties then hold for any Γ' commensurable with Γ, and such Γ and Γ' have the same set of cusps in $\mathbf{R} \cup \{\infty\}$. This equivalence of compactness and finite covolume is a very non-trivial theorem due to Siegel; see [**44**, 1.9.1] for an elegant exposition of the proof.

We define a *Fuchsian group of the first kind* to be a Fuchsian group $\Gamma \subseteq \mathrm{SL}_2(\mathbf{R})$ for which $\mathrm{vol}(\Gamma \backslash \mathrm{SL}_2(\mathbf{R})) < \infty$. We abbreviate our terminology and simply call such Γ *Fuchsian* since no other kind of Fuchsian group will be considered here. The two most important examples for our purposes are:

- Finite index subgroups $\Gamma \subseteq \mathrm{SL}_2(\mathbf{Z})$, especially the congruence subgroups. These Γ are those which will give rise to modular curves $\Gamma \backslash \overline{\mathfrak{h}}_\Gamma$. These have as cusps $\mathbf{Q} \cup \{\infty\}$, as this is true for $\mathrm{SL}_2(\mathbf{Z})$. One can use $\mathrm{SL}_2(\mathbf{Z})$ to move any such cusp to ∞, so using only $g \in \mathrm{SL}_2(\mathbf{Z})$ in (2) above makes $h \in \mathbf{Z}^+$ *intrinsic*. This number is called the *ramification degree* of the cusp (and will have a geometric interpretation as a ramification degree later on).
- The elements of reduced norm 1 in unit groups of orders in quaternion algebras over $\mathbf{Q}$ which are split at ∞. These have an *empty* set of cusps and the quotients $\Gamma \backslash \overline{\mathfrak{h}}_\Gamma$ are called *Shimura curves*.

For technical reasons, we follow [**10**] and now consider only a Fuchsian group $\Gamma \subseteq \mathrm{SL}_2(\mathbf{R})$ which acts *freely* on X (or equivalently on $\mathfrak{h}$), in the sense that $\mathrm{Stab}_\Gamma(x) = \{1\}$ for all $x \in X$. In particular, $-1 \notin \Gamma$ and therefore the notion of regularity/irregularity makes sense for all cusps. Using *any* $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{R})$ to define

$$\psi_{\left(\begin{smallmatrix} a \\ c \end{smallmatrix}\right), \left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right)} : X \xrightarrow{\simeq} \mathfrak{h}$$

we can use the above remarks and Theorem 1.5.2.2 to endow $X \cup \mathrm{cusps}(\Gamma)$ with a natural topology and $X_\Gamma = X/\Gamma$ with the structure of an (open) Riemann surface which is compactified to $\overline{X}_\Gamma = (X \cup \mathrm{cusps}(\Gamma))/\Gamma$ by adding finitely many cusp points to X_Γ. This is all *independent* of $g \in \mathrm{SL}_2(\mathbf{R})$.

Some remarks should be made here on why we choose to assume a free action. For example, one is certainly interested in the modular curves $X_0(N)$ which correspond to the subgroup $\Gamma_0(N)$. However, from a geometric point of view these curves are not fine moduli spaces. We will essentially develop a theory suited for the geometric study of fine moduli spaces, such as the curves $X_1(N)$ (with $N \geq 5$). In practice, most structures one wants on $X_0(N)$ are obtained by suitable descent or quotient constructions beginning life on $X_1(N)$ (or some other fine moduli scheme). For example, modular forms for $\Gamma_0(N)$ can be described as those on $\Gamma_1(N)$ which satisfy an extra invariant property. The restriction to the fine moduli scheme context is not terribly significant (and somewhat simplifies the exposition). The interested reader who has worked through these fine moduli scheme cases should have no difficulty deduce whatever is desired in other cases when group actions are not free, etc. One important special case of this problem which we will treat in detail is dealing with low level. For example, Ramanujan's Δ is at level 1 but there is no fine moduli space at level 1. We will later use some ad hoc tricks to get around this sort of problem, essentially working with stacks without saying so. It should also be noted that the reason we choose to work with $\Gamma_1(N)$ rather than $\Gamma(N)$ in our main computations is because the classical Hecke rings for $\Gamma_1(N)$ have much nicer properties. On the other hand, the interested reader will have no difficulty adapting our arguments with $\Gamma_1(N)$ (for $N \geq 5$) to the case of $\Gamma(N)$ (for $N \geq 3$). We will return to this point later.

Our use of the "X" notation for $\mathfrak{h}$ is taken from [**10**], and this compels us to write X_Γ rather than Y_Γ, which might seem more traditional within the context of modular curves. We hope this causes no confusion. In our detailed study of modular curves, we will use the standard notation $Y_1(N)$ and $X_1(N)$ for $X_{\Gamma_1(N)}$ and $\overline{X}_{\Gamma_1(N)}$ respectively.

In order to make the uniqueness of the compactfications $\overline{X}_\Gamma$ canonical and to give an intrinsic definition of the topological space $X \cup \mathrm{cusps}(\Gamma)$ without the crutch of an isomorphism ψ_{e_1,e_2} between X and $\mathfrak{h}$, we first record the following general fact.

LEMMA 1.5.5.1. *Let $\Delta = \{q \in \mathbf{C} \mid |q| < 1\}$, $\Delta^* = \Delta - \{0\}$. Then any holomorphic quasi-finite map $\Delta^* \to Y$ to a compact Riemann surface uniquely extends to a holomorphic map $\Delta \to Y$.*

This lemma shows that the compact Riemann surface $\overline{X}_\Gamma$, and consequently the structure of algebraic curve on X_Γ, is unique. Note that the case $Y = \mathbf{CP}^1$ in Lemma 1.5.5.1 implies the big Picard theorem on the behavior of functions with essential singularities (and shows the need for the quasi-finiteness hypothesis). The method of proof is to reduce the general case to this special case. I am grateful to Jason Starr for suggesting the argument below.

PROOF. By the quasi-finiteness assumption, we can shrink Δ around 0 so that Δ^* does not surject onto Y. Choose $y \in Y$ not in the image of Δ^* and choose a projective embedding $Y \hookrightarrow \mathbf{CP}^n$ so that there is a hyperplane H meeting Y exactly at y. Thus, we have a quasi-finite map

$$\Delta^* \to \mathbf{CP}^n - H \simeq \mathbf{C}^{n+1}.$$

Since any non-constant map $Y \to \mathbf{CP}^1$ is finite, the coordinate functions $\Delta^* \to \mathbf{C}$ are quasi-finite. By the big Picard theorem, these functions are meromorphic at the origin. We conclude that the original map $\Delta^* \to \mathbf{CP}^n$ extends to a holomorphic map $\Delta \to \mathbf{CP}^n$, clearly factoring through the submanifold Y.

□

By Lemma 1.5.5.1, we may intrinsically define $\overline{X}_\Gamma$ to be the unique compact Riemann surface containing $X_\Gamma = X/\Gamma$ as the (open) complement of a finite set of points. With the help of an isomorphism $\psi_{e_1,e_2} : X \simeq \mathfrak{h}$, we see that for each of the finitely many points $\overline{x} \in \overline{X}_\Gamma - X_\Gamma$ and a small punctured disc $\Delta^*_{\overline{x}}$ centered at

$\overline{x}$ the preimage of $\Delta^*_{\overline{x}}$ under the universal covering map

$$\pi_\Gamma : X \to X_\Gamma$$

is a disjoint union of punctured discs $\Delta^*_{\overline{x},j}$ in X (recall that Γ acts freely on X, so π_Γ really is the universal cover of X, with fundamental group Γ via its given right action on X). We define $\mathrm{cusps}_X(\Gamma)$ to be the set of germs of such spaces $\Delta^*_{\overline{x},j}$ in X, so there is a unique topology on $X \cup \mathrm{cusps}_X(\Gamma)$ which contains X as an open subspace and looks like (1.2.1.1) near each element of $\mathrm{cusps}_X(\Gamma)$. As we noted in §1.2.1, $X \cup \mathrm{cusps}_X(\Gamma)$ is *not* locally compact, but it is Hausdorff, paracompact, and locally metrizable, hence metrizable. There is a unique continuous map

$$\overline{\pi}_\Gamma : X \cup \mathrm{cusps}_X(\Gamma) \to \overline{X}_\Gamma$$

extending the universal covering map π_Γ and the Γ-action on X uniquely extends to a $\overline{\pi}_\Gamma$-equivariant action on $X \cup \mathrm{cusps}_X(\Gamma)$. The natural map

$$(X \cup \mathrm{cusps}_X(\Gamma))/\Gamma \to \overline{X}_\Gamma$$

is a homeomorphism. By using an isomorphism with $\mathfrak{h}$, we see that the points in $\mathrm{cusps}_X(\Gamma)$ have infinite cyclic isotropy groups for this action. We will call elements of $\mathrm{cusps}_X(\Gamma)$ the *cusps for* Γ *on* X and elements in $\overline{X}_\Gamma - X_\Gamma = \overline{\pi}_\Gamma(\mathrm{cusps}_X(\Gamma))$ will be called *cusps on* $\overline{X}_\Gamma$.

Recall that since $-1 \notin \Gamma$ (i.e., Γ has trivial center), we may speak of regular and irregular cusps on $\overline{X}_\Gamma$ in the terminology introduced above. More intrinsically, for each $x \in \mathrm{cusps}_X(\Gamma)$ clearly x is *regular* precisely when its infinite cyclic isotropy group $\mathrm{Stab}_\Gamma(x)$ consists of unipotent elements. It is technically convenient to develop the theory for general Γ as above (i.e., not necessarily congruence subgroups) in order to treat all cusps on equal footing, even though we will mostly focus on $\Gamma = \Gamma_1(N)$ later on.

For an interesting example of an irregular cusp, let

$$\Gamma = \Gamma_1(4) = \{g \in \mathrm{SL}_2(\mathbf{Z}) \,|\, g \equiv \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \pmod 4\}.$$

This acts *freely* on $\mathfrak{h}$, but $\left(\begin{smallmatrix} -1 & 0 \\ 2 & -1 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{R})$ takes $\frac{1}{2}$ to ∞ and conjugates $\mathrm{Stab}_\Gamma(\frac{1}{2})$ to

$$\left(\begin{smallmatrix} -1 & 1 \\ 0 & -1 \end{smallmatrix}\right)^{\mathbf{Z}},$$

so $\frac{1}{2}$ is an irregular cusp. In our later study of a "universal elliptic curve" $E_\Gamma \to X_\Gamma$ and the problem of "compactifying" this to a suitable universal family $\overline{E}_\Gamma \to \overline{X}_\Gamma$ of "generalized elliptic curves" for $\Gamma \subseteq \mathrm{SL}_2(\mathbf{Z})$ with finite index, the only obstruction to forming the compactification will be the problem of irregularity of cusps. For example,

$$\Gamma_1(N) = \{g \in \mathrm{SL}_2(\mathbf{Z}) \,|\, g \equiv \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \pmod N\}$$

has no irregular cusps for $N > 4$, so there will be a "compactified" universal family of "generalized elliptic curves with points of order N" when $N > 4$. But for $N = 4$ this moduli compactification is not possible, even though over $X_{\Gamma_1(4)} = Y_1(4)$ there is a suitable universal elliptic curve. In other words, the $\Gamma_1(4)$-moduli problem cannot be compactified within the framework of analytic spaces (or schemes).

1.5.6. Descent via the Γ-action. In a moment we will explain the theory of descent we need in order to pass to quotients by the Γ-action on the pair (X, ω_X). First, we describe our goal. Using Lemma 1.5.4.4 we will see that the short exact sequence of $\mathcal{O}_X$-modules

$$0 \to \omega_X \to \underline{\mathbf{R}}^2 \otimes_{\underline{\mathbf{R}}} \mathcal{O}_X \to \omega_X^{-1} \to 0$$

will descend to a short exact sequence of $\mathcal{O}_{X_\Gamma}$-modules

$$0 \to \omega_\Gamma \to \mathcal{U}_\Gamma \otimes_{\underline{\mathbf{R}}} \mathcal{O}_{X_\Gamma} \to \omega_\Gamma^{-1} \to 0 \tag{1.5.6.1}$$

on X_Γ, for some local system $\mathcal{U}_\Gamma$ of 2-dimensional $\mathbf{R}$-vector spaces and (using Theorem 1.5.4.1) the isomorphism $\underline{\mathrm{KS}}_{E_X/X} : \omega_X^{\otimes 2} \xrightarrow{\simeq} \Omega_X^1$ will descend to an isomorphism $\underline{\mathrm{KS}}_\Gamma : \omega_\Gamma^{\otimes 2} \xrightarrow{\simeq} \Omega^1_{X_\Gamma}$. Note that there is generally *no* elliptic curve X_Γ relating to $\mathcal{U}_\Gamma$ or $\underline{\mathrm{KS}}_\Gamma$. However, when $\Gamma \subseteq \mathrm{SL}_2(\mathbf{Z})$, then we do have an action of $\mathrm{SL}_2(\mathbf{Z})$ on $E_X \to X$, so $E_X \to X$ will descend to an elliptic curve $f_\Gamma : E_\Gamma \to X_\Gamma$, the short exact sequence (1.5.6.1) will be canonically identified with the (relative) Hodge filtration of $E_\Gamma \to X_\Gamma$ (in particular, $\mathcal{U}_\Gamma \simeq \mathrm{R}^1 f_{\Gamma *}\underline{\mathbf{R}}$), and $\underline{\mathrm{KS}}_\Gamma$ will be *exactly* $\underline{\mathrm{KS}}_{E_\Gamma/X_\Gamma}$. Also, even without assuming that $\Gamma \subseteq \mathrm{SL}_2(\mathbf{Z})$, base change back up by $X \to X_\Gamma$ will take $\underline{\mathrm{KS}}_\Gamma$ and (1.5.6.1) over to $\underline{\mathrm{KS}}_{E_X/X}$ and the relative Hodge filtration of $E_X \to X$ (compatibly with the isomorphism $E_\Gamma \times_{X_\Gamma} X \simeq E_X$ when $\Gamma \subseteq \mathrm{SL}_2(\mathbf{Z})$).

In order to justify this type of "descent", we recall some basic facts about topological descent. Let $\pi : Z \to Y$ be a covering map of topological spaces and let G be a group of *right* Y-automorphisms of Z acting transitively on fibers and freely on Z (i.e., $\mathrm{Stab}_\Gamma(z) = \{1\}$ for all $z \in Z$). We define a *G-sheaf of sets* $\mathcal{F}$ on Z to be a sheaf of sets $\mathcal{F}$ equipped with isomorphisms $\alpha_g : \mathcal{F} \xrightarrow{\simeq} [g]_*\mathcal{F}$ such that $[g]_*(\alpha_h) \circ \alpha_g = \alpha_{hg}$ (note $[hg]_* = [g]_*[h]_*$!), or equivalently (by adjointness) isomorphisms $\alpha'_g : [g]^*\mathcal{F} \xrightarrow{\simeq} \mathcal{F}$ such that $\alpha'_g \circ [g]^*(\alpha'_h) = \alpha'_{gh}$. Here $(\cdot)^*$ denotes "topological pullback" of sheaves. Note that G "acts on the left" on G-sheaves via the α_g's.

Any G-sheaf $\mathcal{F}$ on Z gives rise to a sheaf of sets $(\pi_*\mathcal{F})^G$ on Y, where $(\pi_*\mathcal{F})^G(U)$ is the set of G-invariants in $\mathcal{F}(\pi^{-1}(U))$. Conversely, if $\mathcal{G}$ is a sheaf on Z, then $\pi^*\mathcal{G}$ has a natural structure of G-sheaf via

$$\alpha'_g : [g]^*\pi^*\mathcal{G} \simeq (\pi \circ [g])^*\mathcal{G} = \pi^*\mathcal{G}.$$

The following lemma is an easy exercise using the assumed properties of π and G.

LEMMA 1.5.6.1. *In the above situation, the functors $\pi_*(\cdot)^G$ and $\pi^*(\cdot)$ are natural quasi-inverse functors between G-sheaves of sets on Z and sheaves of sets on Y. More precisely, for any G-sheaf $\mathcal{F}$ on Z and sheaf $\mathcal{G}$ on Y, the natural composite*

$$\pi^*\left((\pi_*\mathcal{F})^G\right) \to \pi^*(\pi_*\mathcal{F}) \to \mathcal{F}$$

is an isomorphism and the natural map

$$\mathcal{G} \to \pi_*(\pi^*\mathcal{G})$$

is an injection with image $\pi_*\left((\pi^*\mathcal{G})^G\right) \subseteq \pi_*(\pi^*\mathcal{G})$. This is compatible with topological base change by any $Y' \to Y$.

Moreover, if Z is an analytic space and $\mathcal{O}_Z$ has a structure of G-sheaf of $\mathbf{C}$-algebras, then $(Y, (\pi_*\mathcal{O}_Z)^G)$ is an analytic space, uniquely characterized by the fact that π is a local analytic isomorphism. In this case, the above equivalence induces (via "module pullback" π^*) an equivalence between G-sheaves of $\mathcal{O}_Z$-modules and *sheaves of $\mathcal{O}_Y$-modules. Base change by π induces a canonical equivalence between analytic spaces over Y and "analytic spaces over Z with a right G-action"* (*over the G-action on Y*) *compatibly with fiber products over Y and Z and base change by any analytic $Y' \to Y$.*

We are particularly interested in Lemma 1.5.6.1 for local systems $\mathcal{F}$ when Z (like $\mathfrak{h}$) is contractible, or more generally when Z is path connected with trivial fundamental group, in which case $\pi : Z \to Y$ is the universal covering space and $G = \mathrm{Aut}(Z/Y)^{\mathrm{op}}$. In this situation, fixing $y_0 \in Y$ and $z_0 \in \pi^{-1}(y_0)$, there is a canonical isomorphism $G \simeq \pi_1(Y, y_0)$ defined by associating to a loop γ at y_0 the unique automorphism $[\gamma] \in G = \mathrm{Aut}(Z/Y)^{\mathrm{op}}$ such that $[\gamma](z_0) = \widetilde{\gamma}(1)$, where $\widetilde{\gamma}$ is the unique path in Z lifting γ with $\widetilde{\gamma}(0) = z_0$. Here we define the product $\gamma\gamma'$ of $\gamma, \gamma' \in \pi_1(Y, y_0)$ by first following γ' and then γ; this makes $\pi_1(Y, y_0) \xrightarrow{\simeq} G$ a homomorphism of groups. (The inverse map assigns to $g \in G$ the equivalence class in $\pi_1(Y, y_0)$ of the image by π of any path in Z going from z_0 to $g(z_0)$.)

In this situation, we recover the classical dictionary between local systems on Y and representations of $\pi_1(Y, y_0)$. More precisely, let $\mathcal{F}$ be a local system on Y, so $\pi^*\mathcal{F}$ is a G-sheaf and a local system on Z. Since Z has trivial fundamental group, the functors $M \to \underline{M}$ and $\mathcal{G} \to \mathcal{G}_{z_0}$ are quasi-inverse equivalences between abelian groups and local systems on Z (since locally constant sheaves on Z are canonically globally trivialized by transport along paths beginning at z_0). Thus, the "left" G-action on $\pi^*\mathcal{F}$ induces a *left* G-action on $(\pi^*\mathcal{F})_{z_0} \simeq \mathcal{F}_{z_0}$. By Lemma 1.5.6.1 this gives an equivalence between local systems on Z and left G-modules. Using the isomorphism $G \simeq \pi_1(Y, y_0)$, the next lemma shows that this is in fact the usual dictionary via "transport along loops at y_0".

LEMMA 1.5.6.2. *Let $\mathcal{F}$ be a local system on Y. Define an action of $\pi_1(Y, y_0)$ on $\mathcal{F}_{y_0}$ by defining $\gamma(s)$ for $s \in \mathcal{F}_{y_0}$ to be the result of transporting s along the* (*oriented*) *loop γ at y_0. This is a left action and via the isomorphism $G \simeq \pi_1(Y, y_0)$ is identical to the action of G on $(\pi^*\mathcal{F})_{z_0} \simeq \mathcal{F}_{y_0}$ as defined above.*

REMARK 1.5.6.3. If we defined multiplication in $\pi_1(Y, y_0)$ in the opposite order, the action of $\pi_1(Y, y_0)$ on $\mathcal{F}_{y_0}$ would be a right action, or equivalently in order to get a left action we would have to move *backwards* along the (oriented) loop γ.

PROOF. For $z \in Z$, the isomorphism $\pi^*\mathcal{F} \simeq \underline{(\pi^*\mathcal{F})_{z_0}} = \underline{\mathcal{F}_{y_0}}$ yields the isomorphism on stalks $(\pi^*\mathcal{F})_z \xrightarrow{\simeq} (\pi^*\mathcal{F})_{z_0}$ defined by moving along a path from z to z_0. This translates into an isomorphism $\mathcal{F}_{\pi(z)} \xrightarrow{\simeq} \mathcal{F}_{y_0}$ defined by moving along the unique homotopy class of paths from $\pi(z)$ to y_0 which lift to a path from z to z_0. In particular, if $z \in \pi^{-1}(y_0)$ and $\widetilde{\gamma}$ is a path from z_0 to z lifting a loop γ at y_0, then $(\pi^*\mathcal{F})_z \xrightarrow{\simeq} (\pi^*\mathcal{F})_{z_0}$ is the automorphism of $\mathcal{F}_{y_0}$ obtained by moving *backwards* along γ. Equivalently the isomorphism $\underline{(\pi^*\mathcal{F})_{z_0}} \xrightarrow{\simeq} \pi^*\mathcal{F}$ induces (on stalks at $z = \widetilde{\gamma}(1)$) the automorphism $\gamma : \mathcal{F}_{y_0} \xrightarrow{\simeq} \mathcal{F}_{y_0}$ defined by moving fowards along γ.

The isomorphism $\pi^*\mathcal{F} \xrightarrow{\simeq} [\gamma]_*\pi^*\mathcal{F}$ adjoint to $[\gamma]^*\pi^*\mathcal{F} \simeq \pi^*\mathcal{F}$ induces an isomorphism $(\pi^*\mathcal{F})_{z_0} \simeq (\pi^*\mathcal{F})_{[\gamma^{-1}](z_0)}$ which is just the *identity* via the identifications

$$\mathcal{F}_{\pi(z_0)} = \mathcal{F}_{y_0} = \mathcal{F}_{\pi([\gamma^{-1}](z_0))}.$$

Thus the diagram

(1.5.6.2)
$$
\begin{tikzcd}
\pi^*\mathcal{F} \arrow[r, "\simeq"] \arrow[dd, "\simeq"] & {[\gamma]_*\pi^*\mathcal{F}} \arrow[dr, "\simeq"] & \\
 & & {[\gamma]_*\big(\underline{(\pi^*\mathcal{F})_{z_0}}\big)} \\
\underline{(\pi^*\mathcal{F})_{z_0}} \arrow[r, "\gamma"] & \underline{(\pi^*\mathcal{F})_{z_0}} \arrow[ur, equal, "\text{canonical}"'] &
\end{tikzcd}
$$

induces on stalks at z_0 the diagram

$$
\begin{tikzcd}
\mathcal{F}_{y_0} \arrow[r, equal] \arrow[dd, equal] & (\pi^*\mathcal{F})_{[\gamma^{-1}](z_0)} \arrow[dr, "\simeq"] & \\
 & & (\pi^*\mathcal{F})_{z_0} \\
\mathcal{F}_{y_0} \arrow[r, "\gamma"] & \mathcal{F}_{y_0} \arrow[ur, equal] &
\end{tikzcd}
$$

where the top right map corresponds to $\mathcal{F}_{y_0} \xrightarrow{\simeq} \mathcal{F}_{y_0}$ by moving *backwards* along γ^{-1}, or equivalently by moving along γ. Thus (1.5.6.2) commutes on stalks at z_0, and therefore (by local constancy) commutes. This commutativity is exactly the content of the lemma. $\square$

With these facts in hand, we return to the study of (1.5.6.1). Using Lemma 1.5.6.1, we deduce the existence of (1.5.6.1) with all of the desired properties (including base change compatibility for the $\underline{\mathrm{KS}}$-maps). In particular, if $\Gamma \subseteq \mathrm{SL}_2(\mathbf{Z})$ then by topological proper base change we can identify $\mathcal{U}_\Gamma$ with $\mathrm{R}^1 f_{\Gamma *}\underline{\mathbf{R}}$. Using Lemma 1.5.6.2, we can make the Γ-action on $\mathcal{U}_\Gamma$ very concrete in such a case: move a C^∞ 1-form (or dually, a homology class) in the fiber $(E_\Gamma)_x$ lying over a point $x \in X_\Gamma$ along a path $\gamma \in \Gamma \simeq \pi_1(X_\Gamma, x_0)$; here we are using the fact that $E_\Gamma \to X_\Gamma$ is a C^∞ locally trivial fibration (recall that this is how we deduced local constancy of $\mathrm{R}^1 f_*\underline{\mathbf{R}} \simeq \mathrm{R}^1 f_*\underline{\mathbf{Z}} \otimes_{\underline{\mathbf{Z}}} \underline{\mathbf{R}}$ for elliptic curves $f : E \to S$ over a smooth base S.)

Now let $\Gamma \subseteq \mathrm{SL}_2(\mathbf{R})$ be Fuchsian of the first kind acting freely on $\mathfrak{h}$ (or equivalently, freely on X), but do not assume that Γ lies in $\mathrm{SL}_2(\mathbf{Z})$. We canonically extend $\Omega^1_{X_\Gamma}$ to the invertible sheaf $\Omega^1_{\overline{X}_\Gamma}$ on $\overline{X}_\Gamma$. In order to extend ω_Γ and $\underline{\mathrm{KS}}_\Gamma$ to all of $\overline{X}_\Gamma$, we need to assume that all cusps are *regular*. Before we can explain the reason for this condition, it will be convenient for local computations near cusps to introduce the following terminology. For each $x \in \mathrm{cusps}_X(\Gamma)$, we can choose one of the isomorphisms $\psi_{e_1,e_2} : X \xrightarrow{\simeq} \mathfrak{h}$ identifying a punctured neighborhood of x in $X \cup \mathrm{cusps}(\Gamma)$ with
$$U_x = \{z \in \mathfrak{h} \mid \mathrm{Im}\, z \geq N\}$$
for a large N such that $U_x \cap \gamma U_x = \emptyset$ for all $\gamma \in \Gamma$ with $\gamma(x) \neq x$.

For ease of exposition, we say "$\psi_{e_1,e_2} : x \mapsto \infty$" to describe the above situation. In terms of a cusp $\overline{x}$ on $\overline{X}_\Gamma$, we can formulate this as follows. Using the quotient map $\pi_\Gamma : X \to X_\Gamma$, for a small disk $\Delta_{\overline{x}}$ in $\overline{X}_\Gamma$ centered at $\overline{x}$, the preimage $\pi_\Gamma^{-1}(\Delta^*_{\overline{x}})$ is a disjoint union of connected components on which Γ acts transitively and we can (upon shrinking $\Delta_{\overline{x}}$ appropriately) *choose* an isomorphism $\psi_{e_1,e_2} : X \xrightarrow{\simeq} \mathfrak{h}$ sending one of these components to an upper half-space $\{z \in \mathfrak{h} \mid \operatorname{Im} z \geq N\}$. This ψ_{e_1,e_2} isomorphism specifies a choice of $x \in \overline{\pi}_\Gamma^{-1}(\overline{x})$ and is ambiguous up to applying $[\gamma]_X$ on X for some $\gamma \in \Gamma$ and applying some element of

$$\mathrm{SL}_2(\mathbf{R})_\infty = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \mid \alpha \in \mathbf{R}^\times, \beta \in \mathbf{R} \right\}$$

on $\mathfrak{h}$. Composition with the action of $[\gamma]_X$ on X replaces ψ_{e_1,e_2} with $\psi_{\gamma e_1,\gamma e_2}$ while composition with the action of $\left[\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix}\right]_{\mathfrak{h}}$ on $\mathfrak{h}$ replaces ψ_{e_1,e_2} by $\psi_{\alpha e_1, \beta e_1 + \alpha^{-1} e_2}$ (using Theorem 1.5.2.2, which ensures that

$$[g]_{\mathfrak{h}} \circ \psi_{e_1,e_2} = \psi_{e_1,e_2} \circ [g]_X = \psi_{ge_1,ge_2}$$

for $g \in \mathrm{SL}_2(\mathbf{R})$, where we recall that $[g]_{\mathfrak{h}}$ is the standard linear fractional transformation action of the matrix of $(g^{-1})^t$ with respect to the oriented basis $\{e_1, e_2\}$ of $\mathbf{R}^2$).

Consider the trivializing section $-q(e_1)^\vee$ of the invertible sheaf ω_X. We want to determine under what circumstances this descends to a trivializing section of ω_Γ near a cusp $\overline{x}$ on $\overline{X}_\Gamma$. Let U_{e_1,e_2} be the component of $\pi_\Gamma^{-1}(\Delta^*_{\overline{x}})$ taken by ψ_{e_1,e_2} to $\{z \in \mathfrak{h} \mid \operatorname{Im} z \geq N\}$. Note that for $\gamma \in \Gamma$ we have $\psi_{\gamma e_1,\gamma e_2} = \psi_{e_1,e_2} \circ [\gamma]$ so we have an isomorphism

$$[\gamma] : U_{\gamma e_1,\gamma e_2} \xrightarrow{\simeq} U_{e_1,e_2}$$

and the isomorphism $[\gamma]^*(\omega_X^{-1}) \xrightarrow{\simeq} \omega_X^{-1}$ sends $[\gamma]^*(-q(e_1))$ to $-q(\gamma e_1)$. It follows that if $q(e_1)^\vee$ on U_{e_1,e_2} *were* the pullback of some section η of $\omega_\Gamma|_{\Delta^*_{\overline{x}}}$ (which would then trivialize $\omega_\Gamma|_{\Delta^*_{\overline{x}}}$) then this section η *must* pull back to $q(\gamma e_1)^\vee$ on the connected component $U_{\gamma e_1,\gamma e_2}$, as η must be a Γ-invariant section of $\omega_X = \pi_\Gamma^* \omega_\Gamma$.

This addresses the issue of how $-q(e_1)^\vee$ behaves with respect to the Γ-action. On the other hand, $U_{e_1,e_2} = U_{\alpha e_1, \beta e_1 + \alpha^{-1} e_2}$ and $-q(\alpha e_1)^\vee = (-1/\alpha) q(e_1)^\vee$ on U_{e_1,e_2}. Thus, the question of whether $q(e_1)^\vee|_{U_{e_1,e_2}}$ *descends* to a section $\eta \in \mathrm{H}^0(\Delta^*_{\overline{x}}, \omega_\Gamma)$ is *intrinsic* to $\overline{x}$ (i.e., is independent of the choice of $x \in \overline{\pi}_\Gamma^{-1}(\overline{x})$ and $\{e_1, e_2\}$ with $\psi_{e_1,e_2} : x \mapsto \infty$). Moreover, the resulting descended section is well-defined up to $\mathbf{R}^\times$-multiple. The descended map $\underline{\mathrm{KS}}_\Gamma : \omega_\Gamma^{\otimes 2} \xrightarrow{\simeq} \Omega^1_{X_\Gamma}$ sends $\eta^{\otimes 2}$ to $\underline{\mathrm{KS}}_\Gamma(\eta^{\otimes 2}) \in \mathrm{H}^0(\Delta^*_{\overline{x}}, \Omega^1_{X_\Gamma})$ and by Corollary 1.5.4.3 the pullback of this by

$$\mathfrak{h} \xleftarrow[\simeq]{\psi_{e_1,e_2}} X \xrightarrow{\pi_\Gamma} X_\Gamma$$

is $2\pi i dz \in \mathrm{H}^0(\mathfrak{h}, \Omega^1_{\mathfrak{h}})$. In this situation, we could then extend ω_Γ across $\Delta_{\overline{x}}$ by gluing it with $\mathcal{O}_{\Delta_{\overline{x}}}$ along the trivialization $\eta : \mathcal{O}_{\Delta^*_{\overline{x}}} \simeq \omega_\Gamma|_{\Delta^*_{\overline{x}}}$ and the $\mathbf{R}^\times$-ambiguity would not affect the resulting invertible sheaf extension around $\overline{x}$, up to canonical isomorphism (the local $\underline{\mathbf{R}}^\times$ ambiguities at the cusps don't cause problems since an invertible sheaf on $\overline{X}_\Gamma$ cannot have non-trivial automorphisms inducing the identity on a non-empty open set).

Finally, observe that if $\Gamma \subseteq \mathrm{SL}_2(\mathbf{Z})$ with finite index and if we insist that $\{e_1, e_2\} \subseteq \mathbf{Z}^2 \subseteq \mathbf{R}^2$ (which is possible since $\mathrm{SL}_2(\mathbf{Z})$ acts transitively on its cusps $\mathbf{Q} \cup \{\infty\}$), then the $\mathbf{R}^\times$-ambiguity is just $\mathbf{Z}^\times$-ambiguity, so $\eta^{\otimes 2} \in \mathrm{H}^0(\Delta^*_{\overline{x}}, \omega_\Gamma^{\otimes 2})$ would be completely intrinsic to $\overline{x}$. In any event, we see that the ability to descend $q(e_1)^\vee|_{U_{e_1,e_2}}$ to $\mathrm{H}^0(\Delta^*_{\overline{x}}, \omega_\Gamma)$ for *all* cusps $\overline{x}$ on $\overline{X}_\Gamma$ would allow us to extend ω_Γ to an invertible sheaf $\overline{\omega}_\Gamma$ on $\overline{X}_\Gamma$, unique up to canonical isomorphism. We'll later see a more geometric way to *directly* construct such an $\overline{\omega}_\Gamma$ in certain favorable cases (with $\Gamma \subseteq \mathrm{SL}_2(\mathbf{Z})$).

Let's now show that the above descent question near a cusp $\overline{x}$ on $\overline{X}_\Gamma$ has an affirmative answer if and only if $\overline{x} \in \overline{X}_\Gamma$ is a *regular cusp*. Fixing $x \in \overline{\pi}_\Gamma^{-1}(\overline{x})$ and $\psi_{e_1,e_2} : x \mapsto \infty$ as above, the question is just whether or not the section $q(e_1)^\vee$ of ω_X is invariant under the subgroup of Γ that *stabilizes* U_{e_1,e_2} (rather than moving it to another component of $\pi_\Gamma^{-1}(\Delta^*_{\overline{x}})$). By Corollary 1.5.2.3, this is equivalent to the section 1 of $\mathcal{O}_\mathfrak{h}$ on $\{z \mid \operatorname{Im} z \geq N\}$ being invariant under the operation

$$[g](f)(z) = (-cz + a)f\left(\frac{dz - b}{-cz + a}\right)$$

for

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} \varepsilon & h \\ 0 & \varepsilon \end{pmatrix}^{\mathbf{Z}}.$$

Here $\varepsilon = \pm 1$ is the "measure of irregularity" of $\overline{x}$ and $\left(\begin{smallmatrix} \varepsilon & h \\ 0 & \varepsilon \end{smallmatrix}\right)^{\mathbf{Z}}$ is the subgroup of $\mathrm{SL}_2(\mathbf{R})$ corresponding to $\mathrm{Stab}_\Gamma(U_{e_1,e_2})$ via ψ_{e_1,e_2}. But for $g = \left(\begin{smallmatrix} \varepsilon & h \\ 0 & \varepsilon \end{smallmatrix}\right)$ we have $[g](1)(z) = \varepsilon \cdot 1 = \varepsilon$, so in order to perform the descent it is indeed necessary and sufficient that $\overline{x}$ be regular (i.e., $\varepsilon = 1$).

1.5.7. Definition of modular forms. We now assume that all cusps of Γ on X are *regular*. The above construction then gives us an invertible sheaf $\overline{\omega}_\Gamma$ on $\overline{X}_\Gamma$ extending ω_Γ on X_Γ, unique up to *canonical* isomorphism, and near each cusp $\overline{x} \in \overline{X}_\Gamma$ we have specified a trivializing section of $\overline{\omega}_\Gamma|_{\Delta_{\overline{x}}}$, well-defined up to $\mathbf{R}^\times$-multiple in general and even up to just $\mathbf{Z}^\times$-multiple if $\Gamma \subseteq \mathrm{SL}_2(\mathbf{Z})$ with finite index. In this latter case we also have an elliptic curve $E_\Gamma \to X_\Gamma$. Our later analysis of cusps will identify the restriction $E_\Gamma|_{\Delta^*_{\overline{x}}}$ with "$\mathbf{C}^\times/q^{h\mathbf{Z}}$," where h is the ramification degree of the cusp $\overline{x}$ (cf. §1.5.5) and the *canonical* generator of $\omega_\Gamma^{\otimes 2}|_{\Delta^*_{\overline{x}}}$ will turn out to be exactly $(\mathrm{d}t/t)^{\otimes 2}$, sent by $\underline{\mathrm{KS}}_\Gamma$ to the section $h \cdot \mathrm{d}q/q$ of $\Omega^1_{\Delta^*_{\overline{x}}}$ if q is a suitable uniformizing parameter at $\overline{x}$ on $\Delta_{\overline{x}}$. For now, we content ourselves with proving the following theorem.

THEOREM 1.5.7.1. *For any Fuchsian $\Gamma \subseteq \mathrm{SL}_2(\mathbf{R})$ with regular cusps, the isomorphism $\underline{\mathrm{KS}}_\Gamma : \omega_\Gamma^{\otimes 2} \xrightarrow{\simeq} \Omega^1_{X_\Gamma}$ extends uniquely to an isomorphism*

$$\underline{\mathrm{KS}}_\Gamma : \overline{\omega}_\Gamma^{\otimes 2} \xrightarrow{\simeq} \Omega^1_{\overline{X}_\Gamma}(\mathrm{cusps}).$$

Here, the target is the sheaf of 1-forms on $\overline{X}_\Gamma$ with at worst simple poles along the cusps $\overline{X}_\Gamma - X_\Gamma$. More precisely, for the open immersion $j : X_\Gamma \to \overline{X}_\Gamma$, the coherent subsheaves $\overline{\omega}_\Gamma^{\otimes 2}$ and $\Omega^1_{\overline{X}_\Gamma}(\mathrm{cusps})$ in

$$j_*(\omega_\Gamma^{\otimes 2}) \overset{j_*(\underline{\mathrm{KS}}_\Gamma)}{=\!=\!=} j_*(\Omega^1_{X_\Gamma})$$

coincide.

PROOF. This is a local question near each cusp, so choose a cusp $\overline{x}$ on $\overline{X}_\Gamma$ and $x \in \overline{\pi}_\Gamma^{-1}(\overline{x})$. Our above construction in terms of a choice of $\psi_{e_1,e_2} : X \simeq \mathfrak{h}$ with $\psi_{e_1,e_2} : x \mapsto \infty$ gives us a generator $\eta \in \mathrm{H}^0(\Delta^*_{\overline{x}}, \omega_\Gamma)$ of ω_Γ, well-defined up to $\mathbf{R}^\times$-multiple. We want $\underline{\mathrm{KS}}_\Gamma(\eta^{\otimes 2}) \in \mathrm{H}^0(\Delta^*_{\overline{x}}, \Omega^1_{\Delta^*_{\overline{x}}})$ to extend to a generator of the sheaf $\Omega^1_{\Delta_{\overline{x}}}(\overline{x})$ of 1-forms on $\Delta_{\overline{x}}$ with at most a simple pole at $\overline{x}$. The punctured neighborhood $\Delta^*_{\overline{x}}$ of $\overline{x}$ is identified via ψ_{e_1,e_2} with the quotient of some $\{\operatorname{Im} z \geq N\}$ by the equivalence

$$z \sim \left[\begin{pmatrix} \varepsilon & h \\ 0 & \varepsilon \end{pmatrix}\right](z) = \left[\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}\right](z) = z + h.$$

The section $\underline{\mathrm{KS}}_\Gamma(\eta^{\otimes 2})$ of X_Γ corresponds to the section $2\pi i \mathrm{d}z$ on $\Omega^1_{\mathfrak{h}}$ by Corollary 1.5.4.3. Thus, on $\Delta_{\overline{x}} = \{q_h \in \mathbf{C} \,|\, |q_h| < 1\}$ with $q_h = e^{2\pi i z/h}$, $\underline{\mathrm{KS}}_\Gamma(\eta^{\otimes 2})$ descends to the section $h \cdot \mathrm{d}q_h/q_h$ of $\Omega^1_{\Delta^*_{\overline{x}}}$. This is indeed a generator of $\Omega^1_{\Delta_{\overline{x}}}(\overline{x})$ near $\overline{x}$. □

Of course, the above proof used the rather trivial fact that $h \in \mathbf{C}^\times$. In the "integral" theory of modular curves, the generalization of Theorem 1.5.7.1 essentially requires that the ramification at cusps is in some sense *tame*. The significance of Corollary 1.5.7.1 will be realized in §1.7, and the geometric meaning of $\overline{\omega}_\Gamma$ will be discussed in §1.6. With $\overline{\omega}_\Gamma$ now available, we can finally give the long sought-after geometric definition of modular forms. First we need a lemma.

LEMMA 1.5.7.2. *For $k \in \mathbf{Z}$, the natural pullback map $f \mapsto \widetilde{f}$ defined by*

$$\mathrm{H}^0(X_\Gamma, \omega_\Gamma^{\otimes k}) \longrightarrow \mathrm{H}^0(X, \omega_X^{\otimes k}) \xrightarrow[\simeq]{\psi_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)}} \mathrm{H}^0(\mathfrak{h}, \mathcal{O}_{\mathfrak{h}}),$$

using $-q\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) : \omega_X^{-1} \simeq \mathbf{G}_a$, identifies $\mathrm{H}^0(X_\Gamma, \omega_\Gamma^{\otimes k})$ with the set of global sections $\widetilde{f} \in \mathrm{H}^0(\mathfrak{h}, \mathcal{O}_{\mathfrak{h}})$ such that

$$\widetilde{f}(z) = (\widetilde{f}|_k\gamma)(z) \overset{\text{def}}{=} \frac{1}{(cz+d)^k}\widetilde{f}\left(\frac{az+b}{cz+d}\right)$$

for all $\gamma = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \in \Gamma$ and $z \in \mathfrak{h}$. The subspace $\mathrm{H}^0(\overline{X}_\Gamma, \overline{\omega}_\Gamma^{\otimes k})$ consists of those f such that for all $x \in \mathrm{cusps}_X(\Gamma)$, $\widetilde{f}|_k\gamma$ is bounded near $i\infty$ whenever $\gamma \in \mathrm{SL}_2(\mathbf{R})$ and $\psi_{\gamma\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right),\gamma\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)} : x \mapsto \infty$. Similarly, $\mathrm{H}^0(\overline{X}_\Gamma, \overline{\omega}_\Gamma^{\otimes k}(-\mathrm{cusps}))$ is the subspace corresponding to those f such that for all $x \in \mathrm{cusps}_X(\Gamma)$, $\widetilde{f}|_k\gamma$ tends to 0 near $i\infty$ whenever $\gamma \in \mathrm{SL}_2(\mathbf{R})$ and $\psi_{\gamma\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right),\gamma\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)} : x \mapsto \infty$.

PROOF. This follows from Corollary 1.5.2.3 (which involves $f|_k\gamma^{-1}$) and the *definition* of $\overline{\omega}_\Gamma$ in terms of $-q(e_1)^\vee$ trivializations, since a holomorphic function f on $\{z \,|\, \operatorname{Im} z \geq N\}$ which is invariant under $z \mapsto z + h$ is obtained via pullback of a *holomorphic* function on Δ under the map $e^{2\pi i z/h} : \mathfrak{h} \to \Delta^*$ if and only if $|f|$ is bounded near $i\infty$. □

Combining Theorem 1.5.7.1 and Lemma 1.5.7.2, we are led to the following definition.

DEFINITION 1.5.7.3. Choose an integer $k \geq 1$ and a Fuchsian group $\Gamma \subseteq \mathrm{SL}_2(\mathbf{R})$ freely on $\mathfrak{h}$ and with regular cusps. Let $(E_X \to X, \alpha_X)$ be a universal object as above. We define the space of *modular forms of weight k for* Γ to be

$$M_k(\Gamma, \mathbf{C}) = \mathrm{H}^0(\overline{X}_\Gamma, \overline{\omega}_\Gamma^{\otimes k})$$

and the space of *cusp forms of weight k for* Γ to be

$$S_k(\Gamma, \mathbf{C}) = \mathrm{H}^0\big(\overline{X}_\Gamma, \overline{\omega}_\Gamma^{\otimes k}(-\mathrm{cusps})\big).$$

When $k \geq 2$, $\underline{\mathrm{KS}}_\Gamma$ gives us an identification

$$S_k(\Gamma, \mathbf{C}) = \mathrm{H}^0(\overline{X}_\Gamma, \overline{\omega}_\Gamma^{\otimes(k-2)} \otimes \Omega^1_{\overline{X}_\Gamma}).$$

For example, $S_2(\Gamma, \mathbf{C}) = \mathrm{H}^0(\overline{X}_\Gamma, \Omega^1_{\overline{X}_\Gamma})$. When $(E_X \to X, \alpha_X) = (\mathcal{E}^{\mathrm{an}} \to \mathfrak{h}, \alpha^{\mathrm{an}})$, this is *exactly* the classical identification of a weight 2 cusp form for Γ with the holomorphic 1-form on $\Gamma\backslash\mathfrak{h}$ descending from $2\pi i dz$ on $\mathfrak{h}$.

We'll see in Theorem 1.6.6.2 that the above definitions allows us to recover q-expansions at cusps in a geometric fashion (via Definition 1.6.6.1). Naturally, we also want to give an intrinsic geometric description of $(\overline{X}_\Gamma, \overline{\omega}_\Gamma)$ as a universal object, without the crutch of X or X_Γ. This will be taken up in the next section, for Γ a finite index subgroup of $\mathrm{SL}_2(\mathbf{Z})$.

1.6. Analytic theory of cusps

In this section we will develop an analytic theory of generalized elliptic curves in the spirit of [**14**] in order to interpret the cusps of our quotient curves via a moduli problem on analytic spaces. We first consider the more limited goal of "compactifying" the relative elliptic curve $\underline{\mathrm{Tate}}^* \to \Delta^*$ over $\Delta = \{q \,|\, |q| < 1\}$ in an essentially unique way; we will also see how this lets us understand the structure of $\overline{X}_\Gamma$ and $\overline{\omega}_\Gamma$ at the cusps when $\Gamma \subseteq \mathrm{SL}_2(\mathbf{Z})$ has finite index (with regular cusps and acting freely on $\mathfrak{h}$). This will allow us to establish a canonical "algebraicity" for $E_\Gamma \to X_\Gamma$, thereby providing the link between the algebraic and analytic theories. The essential point is that GAGA does not hold for the non-proper X_Γ, so any assertions about an "algebraic" structure on E_Γ must be throughly explored *on the analytic side* in order to be certain about the nature of such "algebraicity". The mere *existence* of an algebraic theory of modular curves actually implies rather quickly that $E_\Gamma \to X_\Gamma$ has a "canonical" algebraic structure for congruence subgroups Γ, but a satisfactory understanding of this structure on E_Γ requires some effort on the analytic side.

1.6.1. Tate curves. Throughout this section, we let $\Gamma \subseteq \mathrm{SL}_2(\mathbf{Z})$ denote a subgroup of finite index acting freely on $\mathfrak{h}$ (or equivalently, on X) and with regular cusps. We therefore have a relative elliptic curve $E_\Gamma \to X_\Gamma$, a compactification $X_\Gamma \hookrightarrow \overline{X}_\Gamma$, and an isomorphism of invertible sheaves $\overline{\omega}_\Gamma^{\otimes 2} \xrightarrow{\simeq} \Omega^1_{\overline{X}_\Gamma}$ extending

$$\underline{\mathrm{KS}}_\Gamma = \underline{\mathrm{KS}}_{E_\Gamma/X_\Gamma} : \omega_\Gamma^{\otimes 2} \xrightarrow{\simeq} \Omega^1_{X_\Gamma}.$$

Since Γ has finite index in $\mathrm{SL}_2(\mathbf{Z})$, $\mathrm{cusps}_X(\Gamma)$ is identified with $\mathbf{Q} \cup \{\infty\}$ under any isomorphism $\psi_{e_1,e_2} : X \simeq \mathfrak{h}$ with $\{e_1, e_2\}$ a basis of $\mathbf{Z}^2 \subseteq \mathbf{R}^2$. The action of $\mathrm{SL}_2(\mathbf{Z})$, rather than all of $\mathrm{SL}_2(\mathbf{R})$, is therefore sufficient to move any cusp of Γ on X "to infinity". More precisely, for any $x \in \mathrm{cusps}_X(\Gamma)$ we can find $\gamma \in \mathrm{SL}_2(\mathbf{Z})$ with $[\gamma]_X(x) = \infty_X$, where we define $\infty_X \in \mathrm{cusps}_X(\Gamma)$ to correspond to ∞ under $\psi_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)} : X \simeq \mathfrak{h}$. Recalling that the $[\cdot]_X$-action of $\mathrm{SL}_2(\mathbf{Z})$ on X is a right action, we get

$$\gamma^{-1}\,\mathrm{Stab}_\Gamma(x)\gamma = \mathrm{Stab}_{\gamma^{-1}\Gamma\gamma}(\infty_X) = \begin{pmatrix} 1 & h_x \\ 0 & 1 \end{pmatrix}^{\mathbf{Z}}$$

for a unique positive integer h_x which does not depend on γ. We must have $h_x \neq 0$ because Γ is a finite index subgroup of $\mathrm{SL}_2(\mathbf{Z})$. Recall that h_x is said to be the *ramification degree* of the (regular) cusp x.

In terms of the isomorphism $\psi_{\gamma\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right),\gamma\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)} = \psi_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)} \circ [\gamma]_X : X \xrightarrow{\simeq} \mathfrak{h}$, there is a neighborhood

$$\Delta_N = \{q \in \mathbf{C} \mid |q| < e^{-2\pi N}\} \subseteq \overline{X}_\Gamma$$

around the point $\overline{\pi}_\Gamma(x)$ (which corresponds to $q = 0$) making the diagram

$$\begin{tikzcd} U_N = \{z \in \mathfrak{h} \mid \operatorname{Im} z > N\} \arrow[r, hook] \arrow[d, "e^{2\pi i(\cdot)}"] & \mathfrak{h} \arrow[r, "\simeq"] & X \arrow[d] \\ \Delta_N^* \arrow[rr, hook] \arrow[d, hook] & & X_\Gamma \arrow[d, hook] \\ \Delta_N \arrow[rr, hook] & & \overline{X}_\Gamma \end{tikzcd} \tag{1.6.1.1}$$

commute, with the horizontal maps all open immersions.

Recall now the construction of $\underline{\mathrm{Tate}}^* \to \Delta^*$ in the proof of Theorem 1.5.4.1. We want to define a modified version $\underline{\mathrm{Tate}}_n^* \to \Delta^*$ for all $n \geq 1$ (recovering $\underline{\mathrm{Tate}}^* \to \Delta^*$ for $n = 1$) with fiber $\mathbf{C}^\times/q^{n\mathbf{Z}}$ over $q \in \Delta^*$. For this we simply use the inclusion

$$\mathbf{Z} \times \Delta^* \hookrightarrow \mathbf{C}^\times \times \Delta^*$$
$$(m, q) \mapsto (q^{mn}, q).$$

This is best thought of as the gluing of $2n$ annuli

$$\{(t,q) \in \mathbf{C}^\times \times \Delta^* \mid |q|^{j+1} < |t| < |q|^j\} \quad 0 \leq j \leq n-1$$
$$\{(t,q) \in \mathbf{C}^\times \times \Delta^* \mid |q|^{j+\frac{1}{2}} < |t| < |q|^{j-\frac{1}{2}}\} \quad 0 \leq j \leq n-1$$

in the obvious manner, generalizing the open cover $\{U_0, U_1\}$ of $\underline{\mathrm{Tate}}^*$. In fact, when $n > 1$ things are somewhat simpler as we do not even have the nuisance of disconnected overlaps between "consecutive" annuli (which we had to deal with earlier when $n = 1$).

LEMMA 1.6.1.1. *Under the inclusion $\Delta_N^* \hookrightarrow X_\Gamma$ in (1.6.1.1), the pullback of $E_\Gamma \to X_\Gamma$ is canonically isomorphic to the restriction of $\underline{\mathrm{Tate}}_h^* \to \Delta^*$ to $\Delta_N^* \hookrightarrow \Delta^*$.*

PROOF. Choose $g \in \mathrm{SL}_2(\mathbf{Z})$ with $[g](x) = \infty_X$ in $\mathrm{cusps}_\Gamma(X)$. Using the isomorphism in the top row of

$$\begin{tikzcd} {[g]^* E_X} \arrow[rr, "\simeq"] \arrow[dr] & & E_X \arrow[dl] \\ & X & \end{tikzcd}$$

and the isomorphism

$$E_X \simeq \psi^*_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)}(\mathcal{E}^{\mathrm{an}})$$

we get an isomorphism

$$\mathcal{E}^{\mathrm{an}}|_{U_N} \simeq E_X|_{\psi^{-1}_{ge_1, ge_2}(U_N)}$$

where $e_1 = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$, $e_2 = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$. Passing to the quotient by $\begin{pmatrix}1 & h\\0 & 1\end{pmatrix}^{\mathbf{Z}} = g^{-1}\,\mathrm{Stab}_\Gamma(x) g = \mathrm{Stab}_\Gamma(\infty_X)$, the elliptic curve $E_\Gamma|_{\Delta_N^*}$ is isomorphic to the descent of $\mathcal{E}^{\mathrm{an}}|_{U_N}$ under $U_N \to \Delta_N^*$ with respect to the equivalence relation induced by $(\tau, z) \sim (\tau, z + h)$ on $\mathbf{C} \times U_N$. This is obviously just the restriction of $\underline{\mathrm{Tate}}_h^* \to \Delta^*$ to $\Delta_N^* \hookrightarrow \Delta^*$.

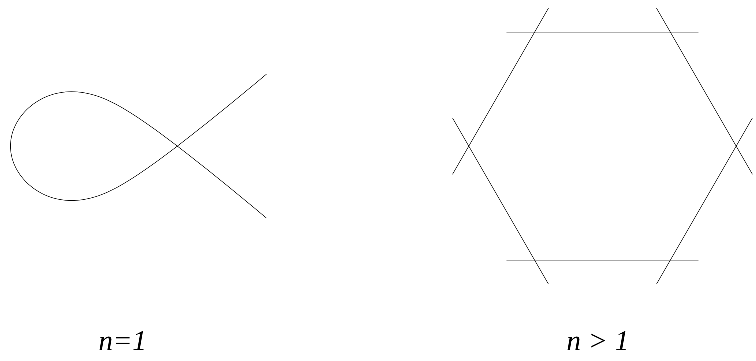


FIGURE 4. Néron polygons

This identification with $\underline{\mathrm{Tate}}_h^*$ is visibly *independent* of the choice of g, since g is determined up to an element of $\mathrm{Stab}_\Gamma(\infty_X)$ and the action of $\left(\begin{smallmatrix}1 & h\\ 0 & 1\end{smallmatrix}\right)$ on $\mathcal{E}^{\mathrm{an}}$ descends to the trivial action on $\underline{\mathrm{Tate}}_h^* \to \Delta^*$. $\square$

Since $\overline{X}_\Gamma$ is a *compact* Riemann surface, it has a unique algebraic structure. Recall (as was noted after Lemma 1.5.5.1) that via the canonical open immersion $X_\Gamma \hookrightarrow \overline{X}_\Gamma$, with finite complement, X_Γ has a *unique* algebraic structure. However, it is *not* clear that $E_\Gamma \to X_\Gamma$ is at all "algebraic". If there were a canonical proper (but *not* necessarily smooth) $\overline{E}_\Gamma \to \overline{X}_\Gamma$ extending $E_\Gamma \to X_\Gamma$ with $\overline{E}_\Gamma$ algebraic, then $\overline{E}_\Gamma \to \overline{X}_\Gamma$ is algebraic and the proper $\overline{E}_\Gamma$ over $\mathrm{Sp}(\mathbf{C})$ has a unique algebraic structure by GAGA for proper maps [**23**, Expose XII]. In this case, E_Γ would inherit a canonical algebraic structure, and likewise for the map $E_\Gamma \to X_\Gamma$. Our goal is to construct this canonical proper (in fact, even projective) $\overline{E}_\Gamma \to \overline{X}_\Gamma$.

1.6.2. Generalized elliptic curves. In order to make the construction of $\overline{E}_\Gamma \to \overline{X}_\Gamma$, by Lemma 1.6.1.1 we see that our essential task is to "uniquely compactify" Tate curves. We first introduce some definitions motivated by [**14**].

DEFINITION 1.6.2.1. A *Deligne-Mumford (DM) semistable genus* 1 *curve* over an analytic space S is a proper flat map $f : C \to S$ whose fibers are

- 1-dimensional,
- connected with arithmetic genus 1,
- have at worst ordinary double point singularities and trivial relative dualizing sheaf (as algebraic curves over $\mathrm{Spec}(\mathbf{C})$).

Equivalently [**14**, II, 1.3], the fibers of f are either smooth genus 1 curves or else Néron polygons. (The Néron 1-gon is simply the curve $y^2 = x^3 - x^2$. The Néron n-gon is the unique connected étale cover of degree n of the Néron 1-gon. See Figure 4.) A *marked DM semistable curve of genus* 1 is such an $f : C \to S$ with a choice of section $\varepsilon \in C^{\mathrm{sm}}(S)$; here C^{sm} denotes the smooth locus of C over S.

We often omit the marking ε from the notation when speaking of marked DM semistable genus 1 curves $C \to S$. Note also that $C - C^{\mathrm{sm}}$ consists precisely of the singular points in the fibers.

DEFINITION 1.6.2.2. A *generalized elliptic curve* is a marked DM semistable genus 1 curve $f : E \to S$ (with marking $\varepsilon : S \to E^{\mathrm{sm}}$) equipped with an S-map

$$m : E^{\mathrm{sm}} \times_S E \to E,$$

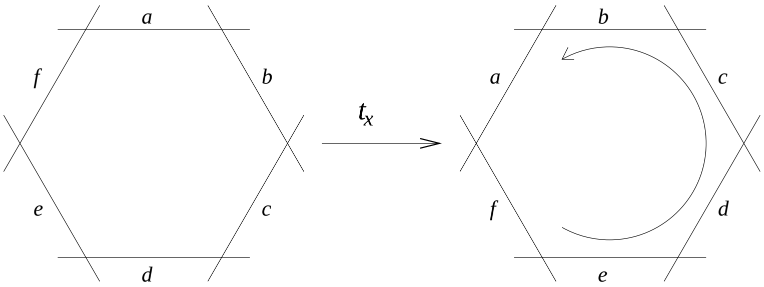


FIGURE 5. A rotation of a Néron polygon

where we require

- the map m makes E^{sm} a commutative group object over S with identity element ε,
- m is a left S-group action,
- for any $s \in S$ and $x \in E^{\mathrm{sm}}_s$, the "translation by x" map t_x on E_s acts as a "rotation" on the "diagram" of irreducible components of E_s. (See Figure 5.)

The last condition is non-vacuous if and only if E_s is an n-gon with $n \geq 2$.

Since a general S-map $E' \to E$ might *not* take E'^{sm} to E^{sm}, we won't generally speak of morphisms of generalized elliptic curves. However, an étale S-map $E' \to E$ *does* takes E'^{sm} to E^{sm}, so it makes sense to speak of *finite étale morphisms* and *isomorphisms* of generalized elliptic curves (both defined in the obvious way). The notion of base change for these structures is clear.

Beware that marked DM semistable curves of genus 1 do *not* always admit a structure of generalized elliptic curve. However, the proof of [**14**, II, 1.17] carries over essentially verbatim to the analytic case to show that for a finite étale map $E' \to E$ with E a generalized elliptic curve and $\varepsilon' \in E'^{\mathrm{sm}}(S)$ lying over $\varepsilon \in E^{\mathrm{sm}}(S)$ (so (E', ε') is a marked DM semistable curve of genus 1), there exists a unique structure of generalized elliptic curve on E' compatible with the structure of generalized elliptic curve on E. The one ingredient in the proof from [**14**] which we do not yet have in the analytic context is that marked Néron polygons over $\mathrm{Sp}(\mathbf{C})$ admit a *unique* structure of (analytic) generalized elliptic curve. This does not follow immediately from GAGA (since the generalized elliptic curve structure involves the non-proper E^{sm}) but it does follow from the next few results and is recorded in Corollary 1.6.2.5 below.

Here is an analogue of [**14**, II, 2.7(iv)].

THEOREM 1.6.2.3. *Let $E \to S$ be a marked DM semistable curve of genus 1 with irreducible fibers. Then E admits a unique structure of generalized elliptic curve over S.*

Note that each fiber of $E \to S$ as in the theorem is either an elliptic curve or a Néron 1-gon. Before proving Theorem 1.6.2.3, we first prove an "algebraicity" result for such curves.

LEMMA 1.6.2.4. *Any marked DM semistable genus 1 curve $E \to S$ with irreducible fibers is, locally on S, given by a Weierstrass equation*

$$y^2w = x^3 + axw^2 + bw^3$$

in $\mathbf{CP}^2 \times S$, with marked point $[0,1,0]$. Conversely, any $C \to S$ which is of this form locally on S is a DM semistable curve of genus 1 with irreducible fibers.

PROOF. The second statement of the lemma is trivial, so we focus on a given $f : E \to S$ with irreducible fibers. Since $\varepsilon(S)$ lies in the smooth locus of E, it is defined by an invertible ideal sheaf with inverse $\mathcal{O}(\varepsilon)$. By the classical theory on fibers (i.e., Riemann-Roch for *irreducible* reduced curves), the higher cohomology of $\mathcal{O}(n\varepsilon) = \mathcal{O}(\varepsilon)^{\otimes n}$ vanishes on fibers for all $n \geq 1$. Since $\mathcal{O}(n\varepsilon)$ is S-flat, the theory of cohomology and base change then implies that $f_*\mathcal{O}(n\varepsilon)$ is locally free on S, of formation compatible with base change. Thus (again by the classical theory on the fibers), the natural map $\mathcal{O}_S = f_*\mathcal{O}_E \to f_*\mathcal{O}(\varepsilon)$ is an isomorphism and $f_*\mathcal{O}(n\varepsilon)$ is locally free of rank n for all $n \geq 1$.

Shrinking S, $f_*\mathcal{O}(2\varepsilon)$ is free with generators $1, x$ (for some x) and $f_*\mathcal{O}(3\varepsilon)$ is free with generators $1, x, y$ (for some y). A relativization of the classical argument with a uniformizer at the origin [**57**, Ch 3, §3, Prop 3.1] thereby produces a Weierstrass relation in the locally free sheaf $f_*\mathcal{O}(6\varepsilon)$, which we can rearrange to have the form

$$y^2 = x^3 + ax + b \tag{1.6.2.1}$$

for some $a, b \in \mathrm{H}^0(S, \mathcal{O}_S)$. See §4.2.2 for an explication of the classical argument in a relative situation, written in a form which works in both the complex analytic and scheme-theoretic contexts.

The canonical map

$$E \to \mathbf{P}\big(f_*\mathcal{O}(3\varepsilon)\big) \simeq \mathbf{CP}^2 \times S$$

factors through the closed analytic subspace cut out by the homogeneous Weierstrass equation associated to (1.6.2.1). The resulting map

$$E \to \{y^2w = x^3 + axw^2 + bw^3\}$$

to a proper S-flat analytic curve in $\mathbf{CP}^2 \times S$ takes ε to $[0,1,0]$ and is an isomorphism on fibers. Thus, by properness and S-flatness (i.e., Theorem 1.1.1.14) this map is an isomorphism. □

PROOF OF THEOREM 1.6.2.3. By Lemma 1.6.2.4 we can, locally on S, identify our $E \to S$ with the analytification of an analogous algebraic situation. The existence (and uniqueness) of the generalized elliptic curve structure is known in the algebraic case [**14**, II, 2.7], so we are left with verifying uniqueness locally on S in the analytic setting. Since two maps $X \rightrightarrows Y$ are equal if and only if the corresponding maps $\mathrm{Sp}\, \mathcal{O}_{X,x}/\mathfrak{m}_x^n \rightrightarrows Y$ are equal for all $x \in X$ and $n \geq 1$, we may assume that S is a one point space.

When $E \to S$ is smooth, we can use Theorem 1.1.1.15 for this uniqueness. When E has a 1-gon (reduced) fiber, we can use the method of proof of [**14**, II, 2.7(iv)] (together with GAGA for coherent sheaves and projective morphisms) by chasing "artinian points" to conclude that the induced S-group structure on E^{sm} is *uniquely* determined.

It remains to prove that (for a one-point space S) two maps $E^{\mathrm{sm}} \times_S E \rightrightarrows E$ which coincide on $E^{\mathrm{sm}} \times_S E^{\mathrm{sm}}$ are in fact the same. By chasing artinian points (and making base change to other 1 point bases), we reduce to verifying that two maps

$E \rightrightarrows E$ which agree on E^{sm} are the same when the base is artinian. By GAGA it suffices to verify the algebraic analogue (over an artin local base). Topologically, the situation is clear because the smooth locus is (Zariski) dense in the whole space (as this can be seen on the closed fiber). It remains to check that $\mathcal{O}_E \to j_*\mathcal{O}_{E^{\mathrm{sm}}}$ is injective, for $j : E^{\mathrm{sm}} \hookrightarrow E$ the canonical map over an artin local base S. Such injectivity follows from the schematic density of E^{sm} in E [**28**, 11.10.10].

□

COROLLARY 1.6.2.5. *There is a unique analytic structure of generalized elliptic curve on the standard marked n-gon over any base S.*

PROOF. Let E denote the standard marked n-gon. When $n = 1$, we have just proved the result in a stronger form. The "identity component" $(E^{\mathrm{sm}})^0$ of E^{sm} is canonically identified with $\mathbf{G}_m \times S$ as an open subset of $\mathbf{CP}^1 \times S$ and we claim that the group law on this component is the usual one. Once this is settled, the rest goes exactly as in the algebraic case in [**14**, II, 1.15]. In order to pin down the group law on $(E^{\mathrm{sm}})^0$, we use the rather general contraction construction in Lemma 1.6.4.5 (applied to the identity section $\varepsilon : S \to E^{\mathrm{sm}}$) in order to analytically construct an S-map $c : E \to \overline{E}$ to a DM semistable genus 1 curve $\overline{E}$ with *irreducible* fibers and a section $c \circ \varepsilon \in \overline{E}^{\mathrm{sm}}(S)$. We have a canonical identification of $\overline{E}$ with the standard 1-gon in such a way that $c \circ \varepsilon$ is the standard marked section "1", and $(E^{\mathrm{sm}})^0 \to \overline{E}^{\mathrm{sm}}$ is an isomorphism. Due to the functorial properties of this contraction construction in the *analytic* category (as summarized in Lemma 1.6.4.5) it follows that there is a unique analytic action of $\overline{E}^{\mathrm{sm}}$ on $\overline{E}$ which makes c equivariant with respect to $E^{\mathrm{sm}})^0 \simeq \overline{E}^{\mathrm{sm}}$. This makes $\overline{E}$ into a generalized elliptic curve over S with identity section $c \circ \varepsilon$ and 1-gon fibers, so by uniqueness in the case $n = 1$ it follows that the group law on $\overline{E}^{\mathrm{sm}} \simeq (E^{\mathrm{sm}})^0 = \mathbf{G}_m$ *is* the usual one, as desired.

□

COROLLARY 1.6.2.6. *There exists a unique DM semistable genus* 1 *curve*

$$\underline{\mathrm{Tate}}_1 \to \Delta$$

extending $\underline{\mathrm{Tate}}^* \to \Delta^*$ *with an irreducible fiber over* $0 \in \Delta$. *This fiber is a* 1*-gon and* $\underline{\mathrm{Tate}}_1$ *is a manifold.*

PROOF. By "experimenting" with the isomorphism $\mathcal{E}^{\mathrm{an}} \simeq E^{\mathrm{alg}}$ over $\mathfrak{h}$ and using Tate's' renormalization of Weierstrass theory [**58**, Ch 5, Thm 1.1] we are led to consider holomorphic functions on Δ given by

$$\begin{aligned} s_k(q) &= \sum_{n\geq 1} \frac{n^k q^n}{1-q^n} \\ a_4(q) &= -5s_3(q) \\ a_6(q) &= -\frac{5s_3(q)+7s_5(q)}{12} \end{aligned}$$

and the relative curve $\underline{\mathrm{Tate}}_1 \hookrightarrow \mathbf{CP}^2 \times \Delta$ defined by the Weierstrass equation

$$Y^2 + XY = X^3 + a_4(q)X + a_6(q)$$

with marked point $[0, 1, 0]$. This is a DM semistable curve of genus 1 recovering $\underline{\mathrm{Tate}}^*$ over Δ^* and the fiber over 0 is a 1-gon.

To show that $\underline{\mathrm{Tate}}_1$ is a manifold, we just need to check regularity near the unique singular point $[0,0,1]$ of the fiber over $q=0$. Since $a_6(q) = -q + \cdots$, this is clear.

For uniqueness, it suffices to check more generally that if $E_1, E_2 \to \Delta$ are marked DM semistable curves of genus 1 which are smooth over Δ^* and which have irreducible fibers over 0, then an isomorphism $E_1^* \simeq E_2^*$ over Δ^* extends (necessarily uniquely) to an isomorphism $E_1 \simeq E_2$ over Δ. (This isomorphism will necessarily be compatible with the markings since this is true over Δ^*.) By Lemma 1.6.2.4 and by shrinking Δ around 0, we may assume that E_j is given by

$$y^2 = x^3 + a_j x + b_j$$

in $\mathbf{CP}^2 \times \Delta$ for holomorphic functions a_j, b_j on Δ; as always, $[0,1,0]$ is the marked point.

By the standard theory of Weierstrass coordinates (as relativized in the proof of Lemma 1.6.2.4), the isomorphism $E_1^* \xrightarrow{\simeq} E_2^*$ is given by $x \mapsto u^2 x$, $y \mapsto u^3 y$ with $a_2 = u^4 a_1$, $b_2 = u^6 b_1$ for some $u \in \mathrm{H}^0(\Delta^*, \mathcal{O}_{\Delta^*}^\times)$. We want to show that u lies in $\mathrm{H}^0(\Delta, \mathcal{O}_\Delta^\times)$.

Since $E_j^* \to \Delta^*$ is a smooth map, either $a_j \neq 0$ or $b_j \neq 0$ (for each j), so at least one of the functions

$$u^4 = \frac{a_2}{a_1}, \qquad u^6 = \frac{b_2}{b_1}$$

is meromorphic on Δ. Thus, u is meromorphic on Δ, so there are no problems with essential singularities. If $a_1(0) \neq 0$ or $b_1(0) \neq 0$, then u lies in $\mathrm{H}^0(\Delta, \mathcal{O}_\Delta)$. Likewise, if $a_2(0) \neq 0$ or $b_2(0) \neq 0$, then u^{-1} lies in $\mathrm{H}^0(\Delta, \mathcal{O}_\Delta)$. We are therefore done unless $a_1(0) = b_1(0) = 0$ *or* $a_2(0) = b_2(0) = 0$. Assume that one of these two conditions holds. Since the meromorphic j-invariants j_{E_1}, j_{E_2} are equal on Δ^*, hence equal, they behave the same way at 0. In particular, when one of them has a pole at 0, so does the other. It follows that we must have

$$a_1(0) = a_2(0) = b_1(0) = b_2(0) = 0.$$

We may now rescale the Weierstrass equations by an appropriate power of a parameter for Δ at 0 so that at least one of these values $a_j(0)$ or $b_j(0)$ is non-zero. However, having done this it is clear from the preceding argument that u is invertible at 0. $\square$

1.6.3. Compactifications of Tate curves. More generally, we have the following result. The proof of the uniqueness part will take a long time, but is essential for the well-definedness of our construction of the global compactifications over $\overline{X}_\Gamma$.

THEOREM 1.6.3.1. *For all $n \geq 1$, there exists a unique generalized elliptic curve $\underline{\mathrm{Tate}}_n \to \Delta$ extending $\underline{\mathrm{Tate}}_n^* \to \Delta^*$ with an n-gon fiber over 0 and with $\underline{\mathrm{Tate}}_n$ a manifold. Moreover, there are unique sections of $\underline{\mathrm{Tate}}_n^{\mathrm{sm}} \to \Delta$ extending the sections $\zeta \cdot q^j$ of $\underline{\mathrm{Tate}}_n^* \to \Delta^*$ for all $j \in \mathbf{Z}/n$ and $\zeta \in \mu_n(\mathbf{C})$.*

PROOF. By Theorem 1.6.2.3 and Corollary 1.6.2.6, we may assume that $n \geq 2$. We first establish the existence of $\underline{\mathrm{Tate}}_n$ by imitating Raynaud's formal scheme construction of the algebraic Tate curves in [**14**, Chapter VII]. (Raynaud's formulas lead to a sign issue, but this will not be a serious problem.)

For motivation, begin by considering the annulus

$$A_{j-\frac{1}{2}} = \left\{ t_j \in \mathbf{C}^\times \mid |q|^{-(j-\frac{1}{2})} \leq |t_j| \leq |q|^{-(j+\frac{1}{2})} \right\}$$

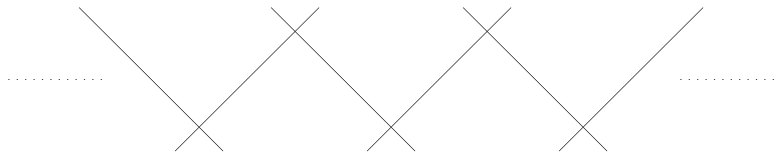


FIGURE 6. The fiber of $\overline{\mathbf{G}}_m^q$ over $q = 0$

for fixed q, $0 < |q| < 1$, and $j \in \mathbf{Z}$. We want to describe this in a manner which does not involve q^{-1}, so then we can try to extend it across Δ. We also want to describe the "overlap" of $A_{j+\frac{1}{2}}$ and $A_{j-\frac{1}{2}}$ without mentioning q^{-1}, so then we can try to define $\underline{\mathrm{Tate}}_n \to \Delta$ by "gluing annuli".

Introducing the notation $x_j = q^{-j}t_0$, $y_j = qx_{j-1}^{-1} = q^j t_0^{-1}$, we are led to *define*

$$U_{j-\frac{1}{2}} = \left\{(x_{j-1}, y_j, q) \in \mathbf{C} \times \mathbf{C} \times \Delta \,|\, x_{j-1}y_j = q\right\}$$

for all $j \in \mathbf{Z}$. We glue $U_{j-\frac{1}{2}}$ and $U_{j+\frac{1}{2}}$ along a copy of $\mathbf{G}_m$:

$$\left\{(x_{j-1}, y_j, q) \in U_{j-\frac{1}{2}} \,|\, y_j \neq 0\right\} \simeq \left\{(x_j, y_{j+1}, q) \in U_{j+\frac{1}{2}} \,|\, x_j \neq 0\right\}$$

$$\begin{array}{rcl} (x_{j-1}, y_j, q) & \longleftrightarrow & \left(y_j^{-1}, y_j q, q\right) \\ \left(qx_j, x_j^{-1}, q\right) & \longleftrightarrow & (x_j, y_{j+1}, q) \end{array}$$

One checks that the restriction $U_{j-\frac{1}{2}}^*$ of $U_{j-\frac{1}{2}}$ to Δ^* is isomorphic to $\mathbf{C}^\times \times \Delta^*$ with $\mathbf{C}^\times$ having coordinate x_{j-1}, and the gluing makes $x_{j-1} = qx_j$ for all j. Define $\overline{\mathbf{G}}_m^q$ to be the gluing of the $U_{j-\frac{1}{2}}$'s via these overlaps. The 'variable' q in the notation $\overline{\mathbf{G}}_m^q$ is purely psychological, intended to make us think of $\mathbf{C}^\times/q^{\mathbf{Z}}$ with $q \in \Delta^*$ an 'unspecified' variable parameter. Clearly $\overline{\mathbf{G}}_m^q$ is a manifold and its analytic fiber over $0 \in \Delta$ is an infinite chain of $\mathbf{CP}^1$'s indexed by $\mathbf{Z}$, with 0 on each $\mathbf{CP}^1$ glued to ∞ on the next; see Figure 6. Thus, $\overline{\mathbf{G}}_m^q \to \Delta$ is *flat* and the relative smooth locus $\mathbf{G}_m^q$ over Δ is the complement of the singular points in the fiber over $q = 0$.

Before trying to form "$\underline{\mathrm{Tate}}_n = \overline{\mathbf{G}}_m^q/q^{n\mathbf{Z}}$", we use the open cover of $\overline{\mathbf{G}}_m^q \times_\Delta \overline{\mathbf{G}}_m^q$ by $U_{j,j'} = U_{j-\frac{1}{2}} \times_\Delta U_{j'-\frac{1}{2}}$ for $j, j' \in \mathbf{Z}$ to detect separatedness over Δ. We need the diagonal in $\overline{\mathbf{G}}_m^q \times_\Delta \overline{\mathbf{G}}_m^q$ to meet each of these opens $U_{j,j'}$ in a closed set. That is, we need the graph of the overlap of $U_{j-\frac{1}{2}}$ and $U_{j'-\frac{1}{2}}$ to be *closed* in $U_{j-\frac{1}{2}} \times_\Delta U_{j'-\frac{1}{2}}$. This overlap is empty unless $j' = j + 1$, and when $j' = j + 1$ the closedness of the graph is clear. Thus, $\overline{\mathbf{G}}_m^q \to \Delta$ is separated.

We also note that the formulas from [**14**, VII, 1.3] can be used (up to shifting indices by $\frac{1}{2}$) to define a map

$$\mathbf{G}_m^q \times_\Delta \overline{\mathbf{G}}_m^q \to \overline{\mathbf{G}}_m^q$$

making $\mathbf{G}_m^q$ a commutative group object over Δ (with identity section $x_0 = 1$) acting on $\overline{\mathbf{G}}_m^q$. The relevant commutative diagrams over Δ can be checked over Δ^* (by density and Δ-flatness), where everything is clear. This group action induces the canonical action of $\mathbf{G}_m \times \mathbf{Z}$ on the infinite chain of $\mathbf{CP}^1$'s in the fiber over $q = 0$ (that is, the standard action of $\mathbf{G}_m$ on each irreducible component and the "translation" action of $\mathbf{Z}$ on the set of irreducible components).

Consider now the section $x_n = 1$ of $\mathbf{G}_m^q$. Translation by this section defines (by regarding $\mathbf{Z}$ as "$x_n^{\mathbf{Z}}$") an action over Δ by $\underline{\mathbf{Z}} \overset{\mathrm{def}}{=} \mathbf{Z} \times \Delta$ on $\overline{\mathbf{G}}_m^q$ compatible with the action $\mathbf{G}_m^q \times_\Delta \overline{\mathbf{G}}_m^q \to \overline{\mathbf{G}}_m^q$ and this takes $U_{j-\frac{1}{2}}$ into $U_{j-n-\frac{1}{2}}$. Since $n \geq 2$, $U_{j-\frac{1}{2}}$ is disjoint from its translates by all non-zero $m \in \mathbf{Z}$. Thus, the map of topological spaces

$$\pi_n : \overline{\mathbf{G}}_m^q \to \overline{\mathbf{G}}_m^q/\underline{\mathbf{Z}}$$

is a covering space map such that the action of $\underline{\mathbf{Z}}$ on $\overline{\mathbf{G}}_m^q$ induces isomorphisms between the connected components of $\pi_n^{-1}(V)$ for small opens V in $\overline{\mathbf{G}}_m^q/\underline{\mathbf{Z}}$. We conclude that $\overline{\mathbf{G}}_m^q/\underline{\mathbf{Z}}$ admits a unique analytic structure making π_n a local isomorphism and likewise for $\mathbf{G}_m^q/\underline{\mathbf{Z}}$. Also, the map $\mathbf{G}_m^q/\underline{\mathbf{Z}} \to \overline{\mathbf{G}}_m^q/\underline{\mathbf{Z}}$ is an *open* immersion, with image equal to the smooth locus over Δ, and $\overline{\mathbf{G}}_m^q/\underline{\mathbf{Z}} \to \Delta$ is flat and separated (i.e., $\overline{\mathbf{G}}_m^q/\underline{\mathbf{Z}}$ is Hausdorff).

We define $\underline{\mathrm{Tate}}_n$ to be this quotient $\overline{\mathbf{G}}_m^q/\underline{\mathbf{Z}}$. We can identify the restriction of $\underline{\mathrm{Tate}}_n$ to Δ^* with $\underline{\mathrm{Tate}}_n^*$ via the gluing data we originally gave for $\underline{\mathrm{Tate}}_n^*$. Following Raynaud, we let x_0 correspond to $1/t$, so translation by the section $x_n = 1$ goes over to translation by q^n rather than q^{-n}.

By construction, the fiber $(\underline{\mathrm{Tate}}_n^{\mathrm{sm}})_0$ is isomorphic to $\mathbf{G}_m \times \mathbf{Z}/n$ and this acts in the standard way on the "standard" n-gon fiber $(\underline{\mathrm{Tate}}_n)_0$. Using the sections $x_j = \zeta$ of $\mathbf{G}_m^q$ (for $\zeta \in \mu_n(\mathbf{C})$), we see that $\underline{\mathrm{Tate}}_n \to \Delta$ satisfies all of the desired properties in Theorem 1.6.3.1 once we check "topological properness": the preimage of a compact set under the separated map $\underline{\mathrm{Tate}}_n \to \Delta$ is compact.

Note that $U_{j-\frac{1}{2}}^* \to \underline{\mathrm{Tate}}_n^*$ is *not* an open immersion, but rather factors through the properly discontinuous action

$$(x_{j-1}, y_j, q) \sim (q^n x_{j-1}, q^{-n} y_j, q)$$

on $U_{j-\frac{1}{2}}^*$ making

$$(U_{j-\frac{1}{2}}^*/\sim) \longrightarrow \underline{\mathrm{Tate}}_n^*$$

an open immersion. (Here we are again using that $n \geq 2$.) Since $\underline{\mathrm{Tate}}_n^* \to \Delta^*$ is proper and $(\underline{\mathrm{Tate}}_n)_0$ is compact (being a finite chain of $\mathbf{CP}^1$'s), for the properness of $\underline{\mathrm{Tate}}_n \to \Delta$ it suffices to check that if $\{p_m\}$ is a sequence in $\underline{\mathrm{Tate}}_n$ with p_m over $q_m \in \Delta^*$ and q_m converges to 0 in Δ, then some subsequence of $\{p_m\}$ converges in $\underline{\mathrm{Tate}}_n$.

Choose representatives $(a_m, b_m, q_m) \in U_{j_m-\frac{1}{2}}^*$ of p_m with $a_m b_m = q_m \neq 0$ for all m. Passing to a subsequence we can find a c such that $j_m \equiv c \pmod n$ for all m. Writing $j_m - c = r_m n$, we can translate by $r_m \in \mathbf{Z}$ to assume that $(a_m, b_m, q_m) \in U_{c-\frac{1}{2}}^*$ for all m. Using the open immersion

$$U_{c-\frac{1}{2}}^*/\big((x, y, q) \sim (q^n x, q^{-n} y, q)\big) \hookrightarrow \underline{\mathrm{Tate}}_n^*,$$

we may use the equivalence relation to assume that that $|q_m|^n \leq |a_m| \leq 1$ for all m. Passing to a subsequence, we can find an r between 1 and n such that

$$|q_m|^r \leq |a_m| \leq |q_m|^{r-1}$$

for all i. The automorphism of r-fold translation by the section induced from $x_1 = q$ allows us to assume (after changing c) that $|q_m| \leq |a_m| \leq 1$ for all i. Now the fact that $a_m b_m = q_m \neq 0$ forces $|q_m| \leq |b_m| \leq 1$ for all i. Since the q_m converge to 0 in Δ, we can pass to yet another subsequence so as to assume that there is an ε, $0 < \varepsilon < 1$, such that $|q_m| \leq 1 - \varepsilon$ for all i. Thus, our sequence lies in the image of a continuous map

$$\{a \in \mathbf{C} \,|\, |a| \leq 1\} \times \{b \in \mathbf{C} \,|\, |b| \leq 1\} \times \{q \in \Delta \,|\, |q| \leq 1 - \varepsilon\} \to \underline{\mathrm{Tate}}_n.$$

Since the domain is obviously compact, this completes the proof of existence of $\underline{\mathrm{Tate}}_n$ with all expected properties.

To prove the uniqueness of our extension $\underline{\mathrm{Tate}}_n$ of $\underline{\mathrm{Tate}}_n^*$, we claim more generally that if $E_1, E_2 \to \Delta$ are generalized elliptic curves smooth over Δ with n-gon fibers over 0, then any isomorphism $E_1^* \simeq E_2^*$ over Δ^* (as generalized elliptic curves) extends to an isomorphism $E_1 \simeq E_2$ over Δ *if* E_1 and E_2 are manifolds. Note that by density and Δ-flatness, it suffices to construct an isomorphism $E_1 \simeq E_2$ merely as analytic spaces over Δ, extending the given isomorphism $E_1^* \simeq E_2^*$ over Δ^* but ignoring the group structure. When $n = 1$, we can use the method of proof of Corollary 1.6.2.6.

We want to reduce the case $n \geq 2$ to the case $n = 1$ by means of "contractions" of irreducible components of singular fibers. This requires the constructions in §1.6.4 below, and these construction are of interest for many other reasons (as we shall see).

$\square$

1.6.4. Analytic projectivizations. In order to construct the contractions that are needed in the proof of Theorem 1.6.3.1, we will need a well-behaved analytic theory of "analytic Proj" functors on certain $\mathbf{N}$-graded $\mathcal{O}_S$-algebras. Let S be an analytic space and let

$$\mathcal{A} = \bigoplus_{n \geq 0} \mathcal{A}^n$$

be a graded sheaf of $\mathcal{O}_S$-algebras which is coherent in each degree and locally finitely generated as an $\mathcal{O}_S$-algebra (i.e., locally a quotient of $\mathcal{O}_S[T_1, \dots, T_m]$ as an $\mathcal{O}_S$-algebra). We want to construct an analytic space $\operatorname{Proj} \mathcal{A} \to S$ which is proper over S and satisfies various appropriate properties to be summarized in Lemma 1.6.4.3 after the construction. The main goal of this section is Lemma 1.6.4.5.

To carry out the construction globally, one minor nuisance is that $\operatorname{Proj} \mathcal{A} \to S$ typically has no natural "universal property", so the construction must be entirely canonical. As a first step, imitating the method of [**27**], we construct an "analytic sheaf Spec".

LEMMA 1.6.4.1. *Let S be an analytic space and let $\mathcal{A}$ be a sheaf of $\mathcal{O}_S$-algebras which is locally finitely generated. Assume also that $\mathcal{A}$ is locally a direct limit of coherent $\mathcal{O}_S$-modules. Then $\mathcal{A}$ is locally the direct limit of its coherent subsheaves and the functor* $\operatorname{Hom}_{\mathcal{O}_S}(\mathcal{A}, f_*\mathcal{O}_X)$ *on objects $f : X \to S$ is represented by a space $f_{\mathcal{A}} : \operatorname{Spec} \mathcal{A} \to S$. We also have the following properties:*

- *formation of* $\operatorname{Spec} \mathcal{A} \to S$ *is naturally compatible with base change on S,*
- *a surjective map $\mathcal{A} \to \mathcal{B}$ induces a closed immersion* $\operatorname{Spec} \mathcal{B} \hookrightarrow \operatorname{Spec} \mathcal{A}$,
- *the natural map $\mathcal{A} \to f_{\mathcal{A}*}\mathcal{O}_{\operatorname{Spec} \mathcal{A}}$ is injective.*

In particular, the functor associating $\operatorname{Spec} \mathcal{A} \to S$ *to $\mathcal{A}$ is faithful.*

Finally, if S is a locally finite type $\mathbf{C}$-scheme and $\mathcal{A}$ is a quasi-coherent sheaf of $\mathcal{O}_S$-algebras which is locally finitely generated, then the natural map

$$\operatorname{Spec}(\mathcal{A}^{\mathrm{an}}) \to \operatorname{Spec}(\mathcal{A})^{\mathrm{an}}$$

is an isomorphism over S^{an}.

REMARK 1.6.4.2. In general, the functor sending $\mathcal{A}$ to $\operatorname{Spec} \mathcal{A}$ is *not* fully faithful and the map $\mathcal{A} \to f_{\mathcal{A}*}\mathcal{O}_{\operatorname{Spec} \mathcal{A}}$ is *not* an isomorphism. (For example, consider the case of $\mathcal{A} = \mathcal{O}_S[T]$.)

By the universal property of $\mathbf{C}^m$, we must have $\operatorname{Spec}\mathcal{O}_S[T_1,\dots,T_m]=\mathbf{C}^m\times S$, so the map $\operatorname{Spec}\mathcal{A}\to S$ is "affine" in the sense that it factors (locally on S) through a closed immersion into some $\mathbf{C}^m\times S$. It then follows that if $\mathcal{C}\to\mathcal{A}$ and $\mathcal{C}\to\mathcal{B}$ are maps of sheaves of algebras as above, then $\mathcal{A}\otimes_{\mathcal{C}}\mathcal{B}$ is also such a sheaf and the natural map

$$\operatorname{Spec}(\mathcal{A}\otimes_{\mathcal{C}}\mathcal{B})\to\operatorname{Spec}\mathcal{A}\times_{\operatorname{Spec}\mathcal{C}}\operatorname{Spec}\mathcal{B}$$

is an isomorphism over S.

If a is a global section of $\mathcal{A}$, then

$$\mathcal{A}_a\stackrel{\text{def}}{=}\mathcal{A}[X]/(1-Xa)$$

is another sheaf of algebras as above. By the universal property of Spec, there is a map $\operatorname{Spec}\mathcal{A}_a\to\operatorname{Spec}\mathcal{A}$ which is an open immersion onto the locus in $\operatorname{Spec}\mathcal{A}$ where the global function a is non-vanishing.

Proof of Lemma 1.6.4.1. It suffices to carry out the construction locally on S, provided we verify that what we construct is compatible with shrinking S. Thus, we may assume that $\mathcal{A}$ is a quotient of some $\mathcal{O}_S[T_1,\dots,T_m]$. Let $\mathcal{I}$ denote the kernel of the surjection

$$\mathcal{O}_S[T_1,\dots,T_m]\twoheadrightarrow\mathcal{A}.$$

We know that $\mathbf{C}^m\times S$ with its "standard coordinates" is $\operatorname{Spec}\mathcal{O}_S[T_1,\dots,T_m]$. Thus, for any $f:X\to S$, to give a map $\mathcal{A}\to f_*\mathcal{O}_X$ is the "same" as to give a map $X\to\mathbf{C}^m\times S$ for which the ideal sheaf $\pi^*\mathcal{I}\cdot\mathcal{O}_{\mathbf{C}^m\times S}$ (with $\pi:\mathbf{C}^m\times S\to S$ the usual *flat* projection) pulls back to the ideal sheaf 0 on X. We conclude that in order to define $\operatorname{Spec}\mathcal{A}$ as the closed analytic subspace of $\mathbf{C}^m\times S$ cut out by $\pi^*\mathcal{I}\cdot\mathcal{O}_{\mathbf{C}^m\times S}$, it suffices to check that $\pi^*\mathcal{I}\cdot\mathcal{O}_{\mathbf{C}^m\times S}$ is *coherent*. By the *noetherian* property of coherent sheaves on analytic spaces [**20**, 5.6/1], to show this it suffices to show that $\pi^*\mathcal{I}\cdot\mathcal{O}_{\mathbf{C}^m\times S}$ is the direct limit of its coherent subsheaves.

Let $\mathcal{E}_d\subseteq\mathcal{O}_S[T_1,\dots,T_m]$ be the free coherent subsheaf of polynomials of total degree at most d. To show that $\pi^*\mathcal{I}\cdot\mathcal{O}_{\mathbf{C}^m\times S}$ is the direct limit of its coherent subsheaves, it suffices to show that the kernel of $\mathcal{E}_d\to\mathcal{A}$ is *coherent*. To check this coherence, we may work locally on S. Writing $\mathcal{A}=\varinjlim\mathcal{A}_j$ as a limit of coherent $\mathcal{O}_S$-modules (after shrinking S), the map $\mathcal{E}_d\to\mathcal{A}$ factors through some map $\mathcal{E}_d\to\mathcal{A}_{j_0}$ and the kernels of $\mathcal{E}_d\to\mathcal{A}_j$ for $j\geq j_0$ form a direct system of coherent subsheaves of $\mathcal{E}_d$. By the noetherian property of the coherent sheaf $\mathcal{E}_d$, the kernel of the map $\mathcal{E}_d\to\mathcal{A}$ is the direct limit of the coherent kernels of the maps $\mathcal{E}_d\to\mathcal{A}_j$, and therefore is coherent as desired. It also follows that our construction is of local nature on S and that $\mathcal{A}$ is, locally on S, the direct limit of its coherent subsheaves. This completes the proof of representability and its local nature on S.

It follows from the construction that we have the desired compatibility with respect to analytification of locally finite type $\mathbf{C}$-schemes. The base change compatibility can be checked locally on S, where it follows from the construction. The injectivity of $\mathcal{A}\to f_{\mathcal{A}*}\mathcal{O}_{\operatorname{Spec}\mathcal{A}}$ and the fact that a surjection $\mathcal{A}\to\mathcal{B}$ induces a closed immersion of Spec's also follow from the construction. $\square$

We now give the analytic construction of Proj. Let S be an analytic space and let $\mathcal{A}=\oplus_{n\geq 0}\mathcal{A}^n$ be a graded sheaf of $\mathcal{O}_S$-algebras with each $\mathcal{A}^n$ coherent as an $\mathcal{O}_S$-module. Assume that $\mathcal{A}$ is locally finitely generated, so $\mathcal{A}$ is locally a quotient of $\mathcal{O}_S[T_1,\dots,T_m]$ as a graded sheaf of algebras (where the T_j are assigned suitable degrees).

Let $f \in \mathrm{H}^0(S, \mathcal{A}^n)$ be a section. Then

$$\mathcal{A}_f \overset{\mathrm{def}}{=} \mathcal{A}[X]/(1 - Xf)$$

is naturally a $\mathbf{Z}$-graded sheaf of algebras with X of degree $-n$. Let $\mathcal{A}_{(f)}$ be the degree 0 part of $\mathcal{A}_f$. It is easy to check that $\mathcal{A}_{(f)}$ satisfies the hypotheses of Lemma 1.6.4.1. Indeed, if $\mathcal{A}$ is a quotient of a suitably graded $\mathcal{O}_S[T_1, \dots, T_m]$ with $\deg T_j = d_j \geq 0$, then $\mathcal{A}_{(f)}$ is a quotient of the finitely generated $\mathcal{O}_S$-algebra

$$\mathcal{O}_S[X^{d_1} T_1^n, \dots, X^{d_m} T_m^n, X^e T_1^{e_1} \cdots T_m^{e_m} \,|\, ne = \sum e_j d_j, 0 \leq e_j < n].$$

Also, $\mathcal{A}_{(f)}$ is the cokernel of the map

$$\bigoplus_{j,n} \mathcal{A}^{jn} X^j \xrightarrow{1-Xf} \bigoplus_{j,n} \mathcal{A}^{jn} X^j.$$

Since all of the $\mathcal{A}^j$'s are coherent, it follows that $\mathcal{A}_{(f)}$ is the direct limit of its coherent subsheaves. It therefore makes sense to form the space $\mathrm{Spec}\,\mathcal{A}_{(f)}$ and this construction is visibly compatible with base change on S.

For $g \in \mathrm{H}^0(S, \mathcal{A}^m)$ there is a canonical equality of $\mathcal{O}_S$-subalgebras of $\mathcal{A}_{fg}$:

$$(\mathcal{A}_{(f)})_{\frac{g^n}{f^m}} = \mathcal{A}_{(fg)} = (\mathcal{A}_{(g)})_{\frac{f^m}{g^n}}$$

It follows that gluing $\mathrm{Spec}\,\mathcal{A}_{(f)}$ and $\mathrm{Spec}\,\mathcal{A}_{(g)}$ along the open $\mathrm{Spec}\,\mathcal{A}_{(fg)}$ (cf. Remark 1.6.4.2) satisfies triple overlap compatibilities as we vary f, g and work locally on S. To be precise, for opens $U, V \subseteq S$ and $f \in \mathrm{H}^0(U, \mathcal{A}^n)$, $g \in \mathrm{H}^0(V, \mathcal{A}^m)$, we glue $\mathrm{Spec}\,\mathcal{A}_{(f)}|_{U\cap V}$ and $\mathrm{Spec}\,\mathcal{A}_{(g)}|_{U\cap V}$ via the above recipe and this gives a globally well defined object which we *define* to be $\mathrm{Proj}(\mathcal{A})$.

From the construction, there is a natural map

$$\mathrm{Proj}(\mathcal{A}|_U) \to \mathrm{Proj}(\mathcal{A}) \times_S U$$

over open sets $U \subseteq S$. In order to see that this is an isomorphism, and in order to see more generally $\mathrm{Proj}(\mathcal{A})$ is naturally compatible with base change on S, we just have to show that if $f_1, \dots, f_r$ are sections of $\mathrm{H}^0(S, \mathcal{A}^{i_1}), \dots, \mathrm{H}^0(S, \mathcal{A}^{i_r})$ respectively which generate $\mathcal{A}$ as an $\mathcal{O}_S$-algebra, then the opens $\mathrm{Spec}\,\mathcal{A}_{(f_j)}$ in $\mathrm{Proj}(\mathcal{A})$ cover the space (so that the construction of $\mathrm{Proj}(\mathcal{A})$ can be given locally over S with just a finite amount of data). Since the hypothesis on the f_j's is preserved by localizing on S, we just have to check that $\mathrm{Spec}\,\mathcal{A}_{(g)}$ is covered by the open subsets $\mathrm{Spec}\,\mathcal{A}_{(gf_j)}$ for $g \in \mathrm{H}^0(S, \mathcal{A}^n)$ and any $n \geq 0$. This statement is purely topological, so it can be checked on fibers over S. By compatibility with analytification in Lemma 1.6.4.1 when $S = \mathrm{Sp}\,\mathbf{C}$, we are reduced to the scheme version, which is standard [**27**, 2.3.15].

We summarize the essential properties of this construction in a lemma.

LEMMA 1.6.4.3. *Let S be an analytic space, $\mathcal{A} = \oplus_{n\geq 0} \mathcal{A}^n$ a graded sheaf of $\mathcal{O}_S$-algebras with each $\mathcal{A}^n$ coherent and $\mathcal{A}$ locally finitely generated.*

1. $\mathrm{Proj}\,\mathcal{A}$ *is naturally of formation compatible with base change on S and is S-flat when $\mathcal{A}$ is $\mathcal{O}_S$-flat.*
2. *If $(S, \mathcal{A})$ is the analytification of an analogous such pair $(S_0, \mathcal{A}_0)$ over a locally finite-type $\mathbf{C}$-scheme S_0, then for any $f \in \mathrm{H}^0(S, \mathcal{A}^n)$ with $n \geq 0$, the S-isomorphisms*

$$\mathrm{Spec}\,\mathcal{A}_{(f^{\mathrm{an}})} \simeq \mathrm{Spec}(\mathcal{A}_{0,(f)})^{\mathrm{an}}$$

glue to define an isomorphism $\mathrm{Proj}(\mathcal{A}) \simeq \mathrm{Proj}(\mathcal{A}_0)^{\mathrm{an}}$ *compatible with base change on* S.

3. *If* $\mathcal{A} \to \mathcal{B}$ *is a graded map to another graded* $\mathcal{O}_S$*-algebra, locally surjective in high degrees, then there exists a unique map* $\mathrm{Proj}\,\mathcal{B} \to \mathrm{Proj}\,\mathcal{A}$ *over* S *which is compatible with base change and compatible with* (2) *and its algebraic analogue. This map is a closed immersion. In particular, locally on* S *there exists a closed immersion* $\mathrm{Proj}\,\mathcal{A} \hookrightarrow \mathbf{CP}^n \times S$*; thus,* $\mathrm{Proj}\,\mathcal{A} \to S$ *is proper.*
4. *Let* $f : X \to S$ *be proper and let* $\mathcal{L}$ *be a line bundle on* X *such that for each* $s \in S$ *there is an integer* $m_s > 0$ *for which*
$$f^* f_* \mathcal{L}^{\otimes m_s} \to \mathcal{L}^{\otimes m_s}$$
is surjective over a neighborhood of s. *Define*
$$\mathcal{A} = \bigoplus_{n \geq 0} f_* \mathcal{L}^{\otimes n}$$
and assume that $\mathcal{A}$ *is locally finitely generated. Then there exists a canonical* S*-map* $X \to \mathrm{Proj}\,\mathcal{A}$ *compatible with base change maps and analytification* (*via* (2) *and the GAGA isomorphisms*)*. When* X *is* S*-flat and* $\mathcal{L}$ *is ample on fibers, then the above hypotheses on* $\mathcal{L}$ *and* $\mathcal{A}$ *are satisfied and the map* $X \to \mathrm{Proj}\,\mathcal{A}$ *is an isomorphism.*

Note that it follows from the first three parts of the lemma that if $\mathcal{A}^{(d)}$ is the (visibly finitely generated) graded $\mathcal{O}_S$-algebra obtained from $\mathcal{A}$ by dropping all graded terms in degrees not divisible by some integer $d > 0$, then the natural isomorphisms $(\mathcal{A}^{(d)}|_U)_{(a)} \simeq (\mathcal{A}|_U)_{(a)}$ for sections $a \in \mathrm{H}^0(U, \mathcal{A}^{nd})$, $n > 0$, induce a map $\mathrm{Proj}\,\mathcal{A}^{(d)} \to \mathrm{Proj}\,\mathcal{A}$ over S which compatible with base change and is an *isomorphism*. Indeed, we may reduce to the case of an artin local base S, in which case GAGA reduces us to the well-known scheme analogue. Also, the final part of (4) is motivated by the scheme analogue [**26**, 2.3.4.1].

PROOF. We have already verified the base change compatibility. By the local criterion for flatness, the flatness in (1) can be checked on infinitesimal fibers, where it follows from (2) and the obvious scheme analogue (as the direct summand of a flat module is flat). To prove (2), we first note that the diagram

$$\begin{tikzcd}
\mathrm{Spec}\,\mathcal{A}_{(f^{\mathrm{an}})} \arrow[r, "\simeq"] & \mathrm{Spec}(\mathcal{A}_{0,(f)})^{\mathrm{an}}\mathcal{A}_{(f^{\mathrm{an}})} \\
\mathrm{Spec}\,\mathcal{A}_{((fg)^{\mathrm{an}})} \arrow[r, "\simeq"] \arrow[u] & \mathrm{Spec}(\mathcal{A}_{0,(fg)})^{\mathrm{an}} \arrow[u]
\end{tikzcd}$$

trivially commutes and is cartesian, so we get a well-defined open immersion
$$\mathrm{Proj}\,\mathcal{A} \hookrightarrow \mathrm{Proj}(\mathcal{A}_0)^{\mathrm{an}}$$
over S, compatible with base change on S. To see that this is an isomorphism we may check on the fibers. This reduces us to the case of $S = \mathrm{Sp}\,\mathbf{C}$. In this case we have $\mathrm{H}^0(\mathrm{Spec}\,\mathbf{C}, \mathcal{A}_0^n) \simeq \mathrm{H}^0(\mathrm{Sp}\,\mathbf{C}, \mathcal{A}^n)$, so the desired result is clear from the constructions.

To prove (3), we first note that uniqueness is clear since an S-map of analytic spaces over S is determined by the induced maps of infinitesimal fibers over S.

For existence, choose $b \in \mathrm{H}^0(S, \mathcal{B}^n)$ lifting to $a \in \mathrm{H}^0(S, \mathcal{A}^n)$. We claim that the composite

$$\operatorname{Spec} \mathcal{B}_{(b)} \hookrightarrow \operatorname{Spec} \mathcal{A}_{(a)} \hookrightarrow \operatorname{Proj} \mathcal{A}$$

is independent of a and is compatible with base change on S. It suffices to check this on infinitesimal fibers, where it follows from (2) and the algebraic analogue. We can now glue to define a map $\operatorname{Proj} \mathcal{B} \to \operatorname{Proj} \mathcal{A}$ of the desired type, except that we still need to show that it is a closed immersion. For this, choose $a \in \mathrm{H}^0(S, \mathcal{A}^n)$ with image $b \in \mathrm{H}^0(S, \mathcal{B}^n)$. The map $\mathcal{A}_{(a)} \to \mathcal{B}_{(b)}$ is *surjective*, so $\operatorname{Spec} \mathcal{B}_{(b)} \to \operatorname{Spec} \mathcal{A}_{(a)}$ is a closed immersion. Thus, to show that $\operatorname{Proj} \mathcal{B} \to \operatorname{Proj} \mathcal{A}$ is a closed immersion it suffices to show that

$$\begin{tikzcd} \operatorname{Spec} \mathcal{B}_{(b)} \arrow[r] \arrow[d, hook] & \operatorname{Spec} \mathcal{A}_{(a)} \arrow[d, hook] \\ \operatorname{Proj} \mathcal{B} \arrow[r] & \operatorname{Proj} \mathcal{A} \end{tikzcd}$$

is cartesian. This is a purely topological question, so it can be checked on fibers over S, where it follows from compatibility with (2) and the scheme analogue.

To prove (4), we mimic the scheme version. First suppose that $f^* f_* \mathcal{L} \to \mathcal{L}$ is surjective. For $x \in X$ and $f \in \mathrm{H}^0(X, \mathcal{L})$ generating $\mathcal{L}$ on an open U containing x, we have a canonical map

$$U \to \operatorname{Spec} \mathcal{A}_{(f)} \hookrightarrow \operatorname{Proj} \mathcal{A}$$

defined by the map $\mathcal{A}_{(f)} \to f_* \mathcal{O}_U$ sending t/f^r to g; here t is a section of $\mathcal{L}^{\otimes r}$ on a "tubular" neighborhood $f^{-1}(V)$ of $X_{f(x)}$ and $t = gf^r$ on $U \cap f^{-1}(V)$. It is easy to check that $U \to \operatorname{Proj} \mathcal{A}$ is independent of f and that this is all local over S, so it globalizes to define a map $X \to \operatorname{Proj} \mathcal{A}$ compatible with base change maps and analytification. In particular, by GAGA and the final part of Lemma 1.1.1.13 we see by comparison with the scheme case that this map is unaffected by replacing $\mathcal{L}$ with any $\mathcal{L}^{\otimes m}$ for $m > 0$. Thus, we can carry out the above construction under merely the local hypothesis of surjective maps $f^* f_* \mathcal{L}^{\otimes m_s} \to \mathcal{L}^{\otimes m_s}$ as in the statement of the lemma.

It remains to check that the hypotheses on $\mathcal{L}$ and $\mathcal{A}$ are satisfied when X is S-flat and $\mathcal{L}$ is ample on fibers, and that $X \to \operatorname{Proj} \mathcal{A}$ is an isomorphism in this case. This is essentially an exercise in cohomology and base change, together with GAGA. We argue as follows. By working locally on S and using Lemma 1.6.4.4 below, we may replace $\mathcal{L}$ by a large tensor power so that $f^* f_* \mathcal{L} \to \mathcal{L}$ is surjective and $\mathcal{A}$ is finitely generated and of formation compatible with base change. Thus, we have a well-defined S-map $X \to \operatorname{Proj}\mathcal{A}$ of formation compatible with base change, and we want to show this map is an isomorphism. By (1) it suffices to consider the infinitesimal fibers, so we can take S to be local artinian with unique point s. Since $\mathcal{L}_s$ is ample, X_s is algebraic (i.e., the analytic space attached to a proper scheme over $\operatorname{Spec}(\mathbf{C})$). As long as X is also algebraic, by GAGA $\mathcal{L}$ is also algebraic and in fact $\mathcal{L}$ is S-ample by [**26**, 4.7.1], so we are reduced to the scheme case by (2) and the scheme case is handled by [**27**, 4.5.2].

Thus, we are left with verifying that for a finite local $\mathbf{C}$-algebra R, a proper flat analytic space $f : X \to \operatorname{Sp} R$ is algebraic if there is a line bundle $\mathcal{L}$ on X for which $\mathcal{L}_0$ on $X_0 = X \times_{\operatorname{Sp} R} \operatorname{Sp} \mathbf{C}$ is ample. By analytic cohomology and base change, together with the ampleness hypothesis, we may replace $\mathcal{L}$ by a sufficiently high power so that $\mathcal{L}_0$ is very ample, $f_* \mathcal{L}$ is locally free of formation compatible with base change, and $f^*f_*\mathcal{L} \to \mathcal{L}$ is surjective. We therefore get a well-defined morphism $\iota : X \to \mathrm{Proj}(\mathrm{Sym}(f_*\mathcal{L}))$ over R whose fiber over $\mathrm{Sp}\,\mathbf{C}$ is a closed immersion. Thus, ι is proper and quasi-finite, hence finite. By Nakayama's Lemma ι is a closed immersion. But by (3) we may apply GAGA to the algebraic $\mathrm{Proj}(\mathrm{Sym}(f_*\mathcal{L}))$ to deduce the algebraicity of X.

□

Here is the lemma which was used in the proof of the final part of (4) above.

LEMMA 1.6.4.4. *Let $f : X \to S$ be a proper map, $\mathcal{L}$ an invertible sheaf on X which is ample on fibers, and $\mathcal{F}$ an S-flat coherent sheaf on X. Assume S is connected. Then the Hilbert polynomial*

$$\chi_{\mathcal{F},f,s}(n) = \sum (-1)^j \dim_{\mathbf{C}} \mathrm{H}^j(X_s, \mathcal{F}_s \otimes \mathcal{L}_s^{\otimes n})$$

is independent of s.

If X is S-flat and $\chi_f = \chi_{\mathcal{O}_X,f}$, then the graded $\mathcal{O}_S$-algebra

$$\mathcal{A} = \bigoplus_{n \geq 0} f_*\mathcal{L}^{\otimes n}$$

is locally finitely generated and for each $s \in S$ there exists an integer n_s and an open neighborhood U_s of s so that for all $n \geq n_s$,

- *$\mathrm{R}^i f_*(\mathcal{L}^{\otimes n}) = 0$ on U_s for all $i > 0$,*
- *$f_*\mathcal{L}^{\otimes n}$ is locally free of rank $\chi_f(n)$ on U_s and of formation compatible with base change on U_s,*
- *$f^*f_*\mathcal{L}^{\otimes n} \to \mathcal{L}^{\otimes n}$ is surjective*

PROOF. We first prove that $\chi_{\mathcal{F},f,s}$ is independent of $s \in S$. Working locally on S and using properness, we may find a finite set of open Stein spaces $\{U_i\}$ in X whose union contains $f^{-1}(V)$ for some open V in S around a chosen point $s \in S$. For all $s' \in V$ the coherent analytic sheaf cohomology on $X_{s'}$ can be computed with the finite Stein covering $\{(U_i)_{s'}\}$ (each open $(U_i)_{s'}$ in $X_{s'}$ is an analytic set in the Stein space U_i and hence is Stein). But cohomology of coherent analytic sheaves can be computed as Čech cohomology with respect to a Stein covering, so by replacing S with V and using the theory of cohomology and base change we conclude that there is a large integer i_0 (namely the size of the above Stein covering) so that $\mathrm{R}^i f_*$ vanishes on coherent sheaves for $i > i_0$. By descending induction from degree i_0, the S-flatness of $\mathcal{F} \otimes \mathcal{L}^{\otimes m}$ and the ampleness on fibers permits us to use cohomology and base change find a large integer m_s so that $\mathrm{R}^i f_*(\mathcal{F} \otimes \mathcal{L}^{\otimes m})_s = 0$ for all $i > 0$ and $m > m_s$. In particular, for every $m > m_s$ the coherent sheaf $f_*(\mathcal{F} \otimes \mathcal{L}^{\otimes m})$ is locally free and of formation compatible with base change near s (in a neighborhood a priori depending on m).

We conclude that for every $s \in S$ and any $m > m_s$, the locally free coherent sheaves $f_*(\mathcal{F} \otimes \mathcal{L}^{\otimes m})$ commute with base change near s and have rank $\chi_{\mathcal{F},f,s}(m)$ near s (for such m). Since the polynomial $\chi_{\mathcal{F},f,s}$ is a priori of *bounded* degree over small opens in S, it follows that $\chi_{\mathcal{F},f,s}$ is indeed independent of s.

Now assume X is S-flat. In order to prove the assertion concerning the higher direct images of $\mathcal{L}$ in this case, we may work locally near any desired $s \in S$. By the above general argument involving Stein covers, we may shrink S and find a large integer M so that $f_*\mathcal{L}^{\otimes M}$ is locally free of formation compatible with base change and $f^*f_*\mathcal{L}^{\otimes M} \to \mathcal{L}^{\otimes M}$ is surjective. A little more shrinking allows us to find a globally free sheaf $\mathcal{O}_S^{\oplus(n+1)}$ surjecting onto $f_*\mathcal{L}^{\otimes M}$, with $n = \chi_f(M) - 1$. Thus, we get a composite S-map

$$j : X \to \mathrm{Proj}(\mathrm{Sym}(f_*\mathcal{L}^{\otimes M})) \hookrightarrow \mathbf{CP}^n \times S$$

of formation compatible with base change, with $\mathcal{L}^{\otimes M} \simeq j^*\mathcal{O}(1)$ and j_s a closed immersion. The openness of the quasi-finite locus of an analytic map, together with the properness of X over S, allows us to shrink S around s so that j is quasi-finite. But j is *proper*, hence finite, and therefore (by Nakayama) is a closed immersion over a neighborhood of s.

We have now realized the S-flat X inside of $\mathbf{CP}^n \times S$, with a fixed Hilbert polynomial $P(n) = \chi_f(Mn)$ on fibers. By Lemma 2.2.3.1, we can realize the flat projective analytic $X \to S$ in $\mathbf{CP}^n \times S$ as the base change of an algebraic projective flat family in $\mathbf{P}^n_H$ via a map $S \to H^{\mathrm{an}}$, where H is a finite type $\mathbf{C}$-scheme. In particular, the scheme H is *quasi-compact*. For a fixed $0 \leq r < M$, the Hilbert polynomial of the invertible sheaf $\mathcal{L}^{\otimes r}$ to $\mathcal{L}^{\otimes M}$ is the same on all fibers over S. Writing $\mathcal{L}^{\otimes n} \simeq \mathcal{L}^{\otimes r} \otimes (\mathcal{L}^{\otimes M})^{\otimes q}$ for $q \in \mathbf{Z}$ and $0 \leq r < M$, we may use GAGA to conclude that for sufficiently large n and all $s \in S$, the higher cohomology of $\mathcal{L}_s^{\otimes n}$ vanishes and $\mathcal{L}_s^{\otimes n}$ is generated by global sections. The argument goes as follows. Our "ample" analytic situation has been obtained from a corresponding "ample" algebraic situation over a *quasi-compact* (even noetherian) scheme. The algebraic theory of cohomology and base change ensures that if $Y \to T$ is a proper finitely presented flat map to a *quasi-compact* scheme T and $\mathcal{N}$ is a T-ample invertible sheaf and $\mathcal{F}$ is a finitely presented quasi-coherent sheaf on Y, then for sufficiently large n the quasi-coherent sheaf $F_t \otimes \mathcal{N}_t^{\otimes n}$ on Y_t enjoys the following properties:

- it has vanishing higher cohomology,
- it is generated by global sections.

It is essential that T is quasi-compact because the "sufficiently large" n is determined over open affines (where higher direct images related nicely to ordinary cohomology modules) and thus we must be able to cover T by *finitely many* open affines if we are to get a global notion of "sufficiently large" n. The GAGA theorem then ensures that the analytic analogues of these fiber cohomology properties remain true for any pullback of such a setup to an analytic situation (using cohomology and sheaf theory relative to the analytic topology).

Using the analytic theory of cohomology and base change, we can bootstrap our fiber result over S to deduce the first and third assertions about higher direct images of $\mathcal{L}^{\otimes n}$'s over S. The second assertion then follows formally from the result already established about Euler characteristics on fibers. It remains to prove that $\mathcal{A}$ is locally finitely generated. We wish to prove that for each $s \in S$ there is some neighborhood U of s and some integer d so that for all $s' \in U$, every graded term of $\mathcal{A}_{s'}$ is generated by the graded terms in $\mathcal{A}_{s'}$ in degrees below d (and hence $\mathcal{A}|_U$ is generated in degrees below d). As each graded term of $\mathcal{A}$ is a coherent sheaf, so each $\mathcal{A}_{s'}$ is a *finite* $\mathcal{O}_{s'}$-module, the property of being generated in terms below degree d can be detected after passage to $\mathfrak{m}_{s'}$-adic completions. Recalling the definition of $\mathcal{A}$ as a symmetric algebra of a direct image sheaf, the analytic Theorem on Formal Functions expresses the completions of the graded terms of $\mathcal{A}$ in terms of inverse limits of corresponding direct image terms computed on infinitesimal fibers. Thus, it suffices to restrict attention to artinian S, *provided* that in such cases we prove the $\mathcal{O}_S$-algebra $\mathcal{A}$ is generated in degrees below a bound which depends *only* on n

and the above Hilbert polynomial P (as then this "universal" degree bound works after passage to the inverse limit on *all* infinitesimal fibers over the original "global" S with which we started). At the infinitesimal fiber level, GAGA ensures that our analytic situation arises from a corresponding algebraic one on an S-flat closed subscheme of $\mathbf{P}^n_S$ with Hilbert polynomial P. Thus, we are reduced to considering our universal degree problem for the infinitesimal fibers of a universal family over a quasi-compact Quot scheme. Now we get the sought-after bound by applying [**26**, 2.4.1, 3.3.1] and algebraic cohomology and base change to the universal family over a *quasi-compact* Quot scheme.

$\square$

The Proj construction will be needed for blow-ups in §1.6.5, but we can now give the first main consequence of interest for the proof of Theorem 1.6.3.1. We have verified that our analytic construction of Proj has all of the required properties to carry over, essentially verbatim, the scheme-theoretic methods used in the proof of the next result, which is [**14**, Ch IV, Prop 1.2] in the scheme case.

LEMMA 1.6.4.5. *Let $f : C \to S$ be a DM semistable curve of genus* 1 *and let $D \hookrightarrow C$ be an effective Cartier divisor which is finite flat over S and lies in C^{sm}. Assume that $D \to S$ is surjective. Then there exists a canonical construction of a map $u : C \to \overline{C}$, compatible with base change on S, where $\overline{C} \to S$ is another DM semistable curve of genus* 1 *and for all $s \in S$ the map $C_s \to \overline{C}_s$ is the contraction of all irreducible components of C_s not meeting D_s. This data $u : C \to \overline{C}$ is unique up to non-canonical isomorphism, and if $f : C \simeq C'$ is an S-isomorphism between DM semistable genus 1 curves and $u' : \overline{C} \to \overline{C}'$ is the contraction of $f(D)$, then there is a unique S-isomorphism $\overline{f} : \overline{C} \simeq \overline{C}'$ such that $\overline{f} \circ u = u' \circ f$.*

In the special case of a generalized elliptic curve $E \to S$, we can take the identity section $\varepsilon : S \to E^{\mathrm{sm}}$ as a (degree 1) divisor for which Lemma 1.6.4.5 produces a canonical "contraction" $c : E \to \overline{E}$, with $\overline{E}$ having *irreducible* fibers. Since $\varepsilon \in E^{\mathrm{sm}}(S)$ maps into $\overline{E}^{\mathrm{sm}}(S)$, we can use Theorem 1.6.2.3 to conclude that there exists a unique structure of generalized elliptic curve on $\overline{E}$ with identity section equal to $\overline{\varepsilon} = c \circ \varepsilon$.

1.6.5. Resolution of singularities. We now apply the above considerations to complete the proof of Theorem 1.6.3.1. Let $E_1, E_2 \to \Delta$ be two generalized elliptic curves and $E_1^* \simeq E_2^*$ an isomorphism over Δ^*. Assume that E_1 and E_2 are manifolds with n-gon fibers over 0. Let $\overline{E}_j$ denote the contraction of the non-identity irreducible components of fibers E_j, as above. Since the fibers of the E_j are irreducible over Δ^*, the isomorphism $E_1^* \simeq E_2^*$ is the same as an isomorphism $\overline{E}_1^* \simeq \overline{E}_2^*$. Since the fibers of the $\overline{E}_j^*$ over 0 are 1-gons, this isomorphism must extend to an isomorphism $\overline{E}_1 \simeq \overline{E}_2$ over Δ by what we saw at the end of §1.6.3. In order to enhance $\overline{E}_1 \simeq \overline{E}_2$ to a (necessarily unique) isomorphism $E_1 \simeq E_2$ over Δ, we will give a universal property for the map $E_j \to \overline{E}_j$. Quite generally, let $X \to \Delta$ be a marked DM semistable curve of genus 1, smooth over Δ^*, with X_0 an n-gon. Let $\overline{X} \to \Delta$ be its contraction with respect to the marked point, so $\overline{X}_0$ is a 1-gon. Assuming that X is a *manifold*, we claim that $X \to \overline{X}$ is the "resolution of singularities" of the possibly singular $\overline{X}$, or more precisely it is obtained by a canonical series of blow-ups at points. Once we clarify what we mean by this, it will follow formally from the universal property of analytic blow-ups in Lemma 1.6.5.1

that there exists an isomorphism $E_1 \simeq E_2$ over the isomorphism $\overline{E}_1 \simeq \overline{E}_2$, finally completing the proof of Theorem 1.6.3.1

We being the analysis of the map $X \to \overline{X}$ by first defining analytic blow-ups. The definition is given by the next lemma.

LEMMA 1.6.5.1. *Let S be an analytic space and let $\mathcal{I}$ be a coherent sheaf of ideals on S. The pullback ideal sheaf induced by $\mathcal{I}$ on $\mathrm{Bl}_{\mathcal{I}}(S) = \mathrm{Proj} \oplus_{n \geq 0} \mathcal{I}^n$ is invertible, and $\mathrm{Bl}_{\mathcal{I}}(S) \to S$ is the final object in the category of spaces over S for which the pullback ideal sheaf induced by $\mathcal{I}$ is invertible.*

The proof of the lemma follows from our construction of Proj by simply imitating the proof in the scheme case. In particular, this blow-up construction $\mathrm{Bl}_{\mathcal{I}}(S) \to S$ is compatible with *flat* base change over S and with the algebraic construction of blow-ups via analytification.

We return now to the situation of the map $X \to \overline{X}$ as above. Assume that the fiber X_0 is an n-gon with $n \geq 2$. (Otherwise $X \to \overline{X}$ is an isomorphism and there is nothing to say.) Let $\overline{x}_0$ be the unique non-smooth point of $\overline{X} \to \Delta$. The preimage of $\overline{x}_0$ in X is reduced of pure codimension 1 since this is true for the algebraic map $X_0 \to \overline{X}_0$. By the local analytic theory of dimension and irreducible decomposition, this forces the preimage of $\overline{x}_0$ to correspond to an intersection of height 1 primes in the 2-dimensional local rings $\mathcal{O}_{X,x}$ on X. Since X is a manifold, $\mathcal{O}_{X,x}$ is regular and thus a unique factorization domain. In particular, all height 1 primes of $\mathcal{O}_{X,x}$ are principal. We conclude that the preimage of $\overline{x}_0 \in \overline{X}$ under $X \to \overline{X}$ is a Cartier divisor on X.

It now follows from the universal property of blow-ups that the map $X \to \overline{X}$ factors through $\mathrm{Bl}_{\overline{x}_0}(\overline{X}) \to \overline{X}$; here by $\mathrm{Bl}_{\overline{x}_0}(\overline{X})$ we mean the blow-up of $\overline{X}$ along the coherent sheaf of ideals defining the point $\overline{x}_0$.

Since $\overline{X}$ is obtained locally from Weierstrass equations, as in Lemma 1.6.2.4, we can use the *scheme-theoretic* étale local description of ordinary double points [**17**, Chapter III, 2.7(2.9)] and the *henselian* property of local rings on analytic spaces [**20**, 2.2/2] to conclude that $\overline{X}$ is *analytically* isomorphic near $\overline{x}_0$ to an open neighborhood of $(0,0,0)$ in $\mathrm{Sp}\,\mathbf{C}[x,y,t]/(xy - t^r)$ for some $r \geq 1$. Note that this space is a manifold near (0,0,0) if and only if $r = 1$; otherwise the origin is the unique non-smooth point.

In order to compute $\mathrm{Bl}_{\overline{x}_0}(\overline{X})$ over a neighborhood of $\overline{x}_0$, we may (by compatibility with flat base change) simply work out the algebraic blow-up of

$$\mathrm{Spec}\,\mathbf{C}[x,y,t]/(xy - t^r)$$

at the origin. (Of course, after this we must also factor in that $\overline{X}_0$ is actually a 1-gon rather than two lines crossing.) If $r = 2$, then the fiber of this blow-up over 0 works out to be a chain of three projective lines, with both of the intersection points regular in the ambient space; see Figure 7. If $r > 2$, then the fiber over 0 is a chain of four projective lines: the "outer" two intersection points are regular, while the inner intersection point locally looks like the origin in $\mathrm{Sp}\,\mathbf{C}[x,y,t]/(xy - t^{r-2})$; see Figure 8.

The fact that $\overline{X}_0$ is initially a 1-gon has the additional effect of identifying the outer two of the projective lines in these chains (as $\mathrm{Bl}_{\overline{x}_0}(\overline{X}) \to \overline{X}$ is an isomorphism over $\overline{X} - \{\overline{x}_0\}$ and $\overline{X}_0 - \{\overline{x}_0\}$ is irreducible). It is clear from this how to obtain the resolution of $\overline{X}$: if initially $r = 1$, then no blow-ups are necessary. If $r = 2$, then a single blow-up yields a manifold B with B_0 a 2-gon. If $r > 2$, then a succession

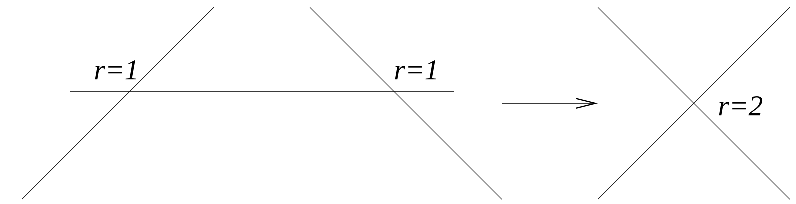


FIGURE 7. The blow-up of an ordinary double point: $r = 2$

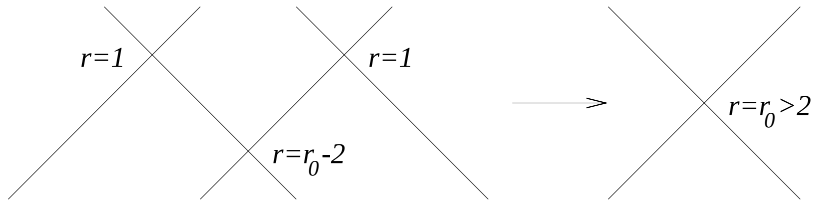


FIGURE 8. The blow-up of an ordinary double point: $r > 2$

of blow-ups yields a manifold B with B_0 an r-gon. We now see that in order to identify X and B, an essential first step is to show that this r is equal to n.

From a study of polygons over $0 \in \Delta$, we conclude just as for $X \to \overline{X}$ above that (if $r > 2$) the singular points on $\mathrm{Bl}_{\overline{x}_0}(\overline{X})$ have Cartier divisor preimage on X, so by universality the map $X \to \mathrm{Bl}_{\overline{x}_0}(\overline{X})$ factors through the next blow-up. We can keep going until the singularities of $\overline{X}$ have been resolved by $B \to \overline{X}$ as above, and we therefore obtain a map $X \to B$ over $\overline{X}$. This B has r-gon fiber over $0 \in \Delta$. We want to show that $r = n$ and that $X \to B$ is an isomorphism. Since X and B are both proper manifolds, flat of pure relative dimension 1 over Δ, and the preimages $X^* \hookrightarrow X$ and $B^* \hookrightarrow B$ of $\Delta^* \hookrightarrow \Delta$ are isomorphic dense opens, the map $X \to B$ must be surjective. Thus, visibly $r \le n$ and $X \to B$ is an isomorphism if and only if $r = n$ (since a quasi-finite map of manifolds of the same dimension is *automatically* flat by [**42**, Theorem 23.3] and $X \to B$ is finite if it is quasi-finite, by *properness* over Δ, so we may then compute the degree of the finite flat map $X \to B$ by working over B^*).

The possibility we must rule out is that $r < n$ or equivalently that $X \to B$ contracts some of the sides of X_0. In this case, $n \ge 2$ and the above analysis shows that $X \to B$ is obtained by a succession of $n - r$ blow-ups of singular points in the fiber over 0. But $B \to \Delta$ *already* locally looks like

$$\mathrm{Sp}\, \mathbf{C}[x, y] \simeq \mathrm{Sp}\, \mathbf{C}[x, y, t]/(xy - t) \to \mathrm{Sp}\, \mathbf{C}[t]$$

at each non-smooth point of B_0, so any further blow-ups introduce components of the fiber over 0 which ought to have self-intersection number -1. This would yield a contradiction if we had a good notion of intersection numbers in the analytic setting over the "open" Δ.

Lacking this, we imitate the classical intersection number argument as follows. Assuming that $X \to B$ is not an isomorphism, we consider the final blow-up

$$\varphi : X \to B'$$

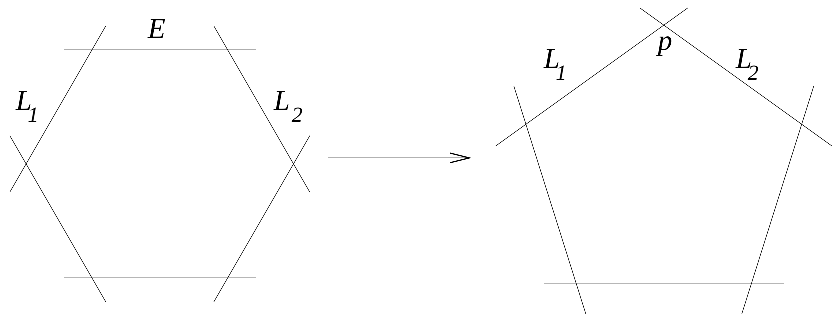


FIGURE 9. The central fiber of $X \to B'$

in the sequence of blow-ups from B to X. Let $p \in B'$ be the blow-up point for φ. The analytic fiber of B' over 0 is an $(n-1)$-gon which is analytically cut out by a single holomorphic function t. We can also choose holomorphic functions x_1 and x_2 near p which cut out the two "branch lines" L_1 and L_2 of this $(n-1)$-gon which intersect at p; we can multiply x_1 by a unit so that $x_1x_2 = t$ near p. If $n = 2$, then L_1 and L_2 only make sense locally near p. (See Figure 9.)

Passing to X, we see that $\varphi^*(t)$ analytically cuts out the entire (reduced) n-gon fiber X_0, while over a neighborhood of p each $\varphi^*(x_j)$ cuts out the pullback of L_j together with the exceptional divisor E. But then along $E^{\mathrm{sm}} \subseteq X$ we have that

$$\varphi^*(t) = \varphi^*(x_1)\varphi^*(x_2)$$

lies in the square of the ideal which cuts out E, contradicting the fact that $\varphi^*(t)$ generates the radical ideal sheaf of X_0 on X. This concludes the determination of the universal property of $X \to \overline{X}$, and thus also concludes the proof of Theorem 1.6.3.1.

Using Theorem 1.6.3.1 and Lemma 1.6.1.1, we arrive at the essential result for compactifications.

COROLLARY 1.6.5.2. *There is a unique generalized elliptic curve $\overline{E}_\Gamma \to \overline{X}_\Gamma$ extending $E_\Gamma \to X_\Gamma$ whose fiber over each cusp $x \in \overline{X}_\Gamma - X_\Gamma$ is an e_x-gon (where e_x is the ramification degree of x) and such that $\overline{E}_\Gamma$ is a manifold. Over a suitable neighborhood Δ_N of such a cusp there exists a unique isomorphism $\overline{E}_\Gamma|_{\Delta_N} \simeq \underline{\mathrm{Tate}}_{e_x}|_{\Delta_N}$ extending the canonical isomorphism $E_\Gamma|_{\Delta_N^*} \simeq \underline{\mathrm{Tate}}^*_{e_x}|_{\Delta_N^*}$.*

The significance of Corollary 1.6.5.2 is that $\overline{E}_\Gamma \to \overline{X}_\Gamma$ is projective, so $\overline{E}_\Gamma$ is algebraic and thus $E_\Gamma \to X_\Gamma$ has a *canonical* algebraic structure by GAGA. To see the projectivity of $\overline{E}_\Gamma \to \overline{X}_\Gamma$, we note that the "contraction" of all non-identity components of fibers (via Lemma 1.6.4.5) is projective over the compact Riemann surface $\overline{X}_\Gamma$ by Lemma 1.6.2.4. But by *construction* $\overline{E}_\Gamma$ is obtained from this contraction by a finite sequence of blow-ups along coherent ideal sheaves. Since the blow-up maps are defined via Proj and coherent ideal sheaves, applying GAGA compatibilities at each stage completes the proof of projectivity of $\overline{E}_\Gamma \to \overline{X}_\Gamma$.

Now that $E_\Gamma \to X_\Gamma$ can be viewed as an *algebraic* elliptic curve over the Dedekind scheme X_Γ, it makes sense to ask about the minimal regular proper model of $E_\Gamma \to X_\Gamma$ over $\overline{X}_\Gamma$. Recall that a regular curve $C \to S$ over a connected Dedekind base S (i.e., a proper map $C \to S$ with fibers of pure dimension 1 and C regular) is said to be *relatively minimal* if any birational S-morphism $C \to C'$

to another regular curve is an isomorphism. Given any regular curve $C_U \to U$ over a non-empty open $U \subseteq S$, a birational S-curve $C' \to S$ is said to be a *relatively minimal model* of $C_U \to U$ if it is relatively minimal; it is said to be the *minimal model* if all relatively minimal models of $C_U \to U$ are S-isomorphic to $C' \to S$. See [**5**] for more details.

Let $\overline{\varepsilon}$ denote the identity section of $\overline{E}_\Gamma \to \overline{X}_\Gamma$ determined by its structure of generalized elliptic curve.

THEOREM 1.6.5.3. *The algebraic curve $\overline{E}_\Gamma \to \overline{X}_\Gamma$ is the minimal regular proper model of $E_\Gamma \to X_\Gamma$ over $\overline{X}_\Gamma$, and there exists a unique isomorphism $\overline{\varepsilon}^*\Omega^1_{\overline{E}_\Gamma/\overline{X}_\Gamma} \simeq \overline{\omega}_\Gamma$ extending the isomorphism $\varepsilon^*\Omega^1_{E_\Gamma/X_\Gamma} \simeq \omega_\Gamma$.*

REMARK 1.6.5.4. Since $\overline{E}_\Gamma \to \overline{X}_\Gamma$ is a Cohen-Macaulay map, by Grothendieck duality there is a canonical relative dualizing sheaf $\omega_{\overline{E}_\Gamma/\overline{X}_\Gamma}$ on $\overline{X}_\Gamma$. This is in fact canonically isomorphic to $\overline{\varepsilon}^*\Omega^1_{\overline{E}_\Gamma/\overline{X}_\Gamma}$ and therefore provides a direct geometric construction of $\overline{\omega}_\Gamma$ without the theory of $\underline{\mathrm{KS}}_\Gamma$-maps. However, the Kodaira-Spencer maps are crucial for local calculations near the cusps.

PROOF. The Castelnuovo criterion [**5**, Thm 3.1] and our previous analysis implies that $\overline{E}_\Gamma \to \overline{X}_\Gamma$ is the minimal regular proper model of $E_\Gamma \to X_\Gamma$ over $\overline{X}_\Gamma$. Letting $j : X_\Gamma \hookrightarrow \overline{X}_\Gamma$ denote the canonical open immersion, it remains to show that $\overline{\varepsilon}^*\Omega^1_{\overline{E}_\Gamma/\overline{X}_\Gamma}$ and $\overline{\omega}_\Gamma$ *coincide* as subsheaves of $j_*\omega_\Gamma$. By the *definition* of $\overline{\omega}_\Gamma$, this amounts to checking that the section $\mathrm{d}t/t$ of

$$\varepsilon^*\Omega^1_{\underline{\mathrm{Tate}}^*_n/\Delta^*} \simeq \omega_{\underline{\mathrm{Tate}}^*_n/\Delta^*}$$

extends to a (necessarily unique) generator of the invertible sheaf $\overline{\varepsilon}^*\Omega^1_{\underline{\mathrm{Tate}}_n/\Delta}$. When $n = 1$ this is immediate from the fact that $\underline{\mathrm{Tate}}_1$ can be realized in $\mathbf{CP}^2 \times \Delta$ by Tate's Weierstrass equation. Indeed, the differential form $\mathrm{d}Y/(2Y + X)$ gives a generator of $\overline{\varepsilon}^*\Omega^1_{\underline{\mathrm{Tate}}_1/\Delta}$ which coincides over Δ^* with $\mathrm{d}t/t$ via the Tate parameterization. For $n \geq 2$ we note from the *construction* of $\underline{\mathrm{Tate}}_n$ that $\mathrm{d}x_0/x_0$ gives a global generator of $\overline{\varepsilon}^*\Omega^1_{\underline{\mathrm{Tate}}_n/\Delta}$. Over Δ^* this corresponds to $-\mathrm{d}t/t$. Indeed, one can check this equality on fibers, where it was already discussed during the identification of $(\overline{\mathbf{G}}^q_m/\underline{\mathbf{Z}})|_{\Delta^*}$ with $\underline{\mathrm{Tate}}^*_n$ in the proof of Theorem 1.6.3.1 (where we identified x_0 with $1/t$). $\square$

1.6.6. q-expansions. Let $\overline{x} \in \overline{X}_\Gamma$ be a cusp with ramification degree h and choose $x \in \overline{\pi}_\Gamma^{-1}(\overline{x})$. Choose a basis $\{e_1, e_2\}$ of $\mathbf{Z}^2$ such that $\psi_{e_1,e_2} : x \mapsto \infty$. Using this map, we get a "canonical" neighborhood $\Delta_N = \{q \in \mathbf{C} \,|\, |q| < e^{-2\pi N}\}$ of $\overline{x}$ for sufficiently large N such that $\overline{E}_\Gamma$ is isomorphic to $\underline{\mathrm{Tate}}_h$ over Δ_N. Let $(\mathrm{d}t/t)_{\overline{x}}$ denote the unique generator of $\overline{\omega}_\Gamma$ near $\overline{x}$ which restricts to the generator obtained from $\mathrm{d}t/t$ on $\underline{\mathrm{Tate}}_h|_{\Delta_N}$. This is independent of the auxiliary choice of x and $\{e_1, e_2\}$ (recall that all cusps are assumed to be regular).

DEFINITION 1.6.6.1. Let $f \in \mathrm{H}^0(\overline{X}_\Gamma, \overline{\omega}_\Gamma^{\otimes k})$ be a modular form of weight k for Γ. We define the *q-expansion* of f at a cusp $\overline{x} \in \overline{X}_\Gamma$ to be the unique holomorphic function $f_{\overline{x}}$ near $0 \in \mathbf{C}$ such that for a sufficiently small neighborhood Δ_N of $\overline{x}$ in $\overline{X}_\Gamma$ as above we have

$$f|_{\Delta_N} = f_{\overline{x}} \cdot (\mathrm{d}t/t)^{\otimes k}_{\overline{x}}$$

Equivalently, if $x \in \overline{\pi}_\Gamma^{-1}(\overline{x})$ and $\psi_{e_1,e_2} : X \xrightarrow{\simeq} \mathfrak{h}$ takes $x \mapsto \infty$ for some oriented basis $\{e_1, e_2\}$ of $\mathbf{Z}^2$, then

$$\psi_{e_1,e_2}^{-1}{}^*(f) = f_{\overline{\pi}_\Gamma(x)}\left(e^{2\pi i z/h}\right)(2\pi i \mathrm{d}\tau)^{\otimes k},$$

where $\mathrm{Stab}_\Gamma(x)$ has generator with matrix $\left(\begin{smallmatrix}1 & h\\ 0 & 1\end{smallmatrix}\right)^{\mathbf{Z}}$ with respect to the basis $\{e_1, e_2\}$ and $\mathrm{d}\tau$ is the standard generator of $\Omega^1_{\mathfrak{h}}$ (so $f_{\overline{x}} \in \mathbf{C}[\![q]\!]$ converges for all q with $|q| < 1$).

The relation with the classical notion of q-expansion at a cusp is given by the next result.

THEOREM 1.6.6.2. *Let $\overline{x} \in \overline{X}_\Gamma$ be a cusp and let $g = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z})$ be a matrix such that the left-action of $\mathrm{SL}_2(\mathbf{Z})$ on X via the moduli problem*

$$(E_X \to X,\ \alpha_X)$$

sends ∞_X into $\overline{\pi}_\Gamma^{-1}(\overline{x})$. Let $f \in \mathrm{H}^0(\overline{X}_\Gamma, \overline{\omega}_\Gamma^{\otimes k})$ be a modular form and define the holomorphic function $\widetilde{f}$ on $\mathfrak{h}$ by

$$(\psi^{-1}_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)})^*(f) = \widetilde{f}\cdot(2\pi i \mathrm{d}\tau)^{\otimes k},$$

so $\widetilde{f}$ "corresponds" to f under the trivialization of $\overline{\omega}_X^{-1}$ by $-q\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$. Then the function

$$(\widetilde{f}|_k g)(z) \stackrel{\mathrm{def}}{=} \frac{1}{(cz+d)^k}\widetilde{f}\left(\frac{az+b}{cz+d}\right)$$

agrees with the function $f_{\overline{x}}\left(e^{2\pi i z/h}\right)$, where h is the ramification degree of $\overline{x}$.

The holomorphic function $f_{\overline{\pi}_\Gamma(\infty_X)}$ on the unit disc is clearly the classical q-expansion of the modular form f in terms of the universal object $(\mathcal{E}^{\mathrm{an}} \to \mathfrak{h}, \alpha^{\mathrm{an}})$.

PROOF. Using $\alpha_X : \mathrm{R}^1 f_{X*}(\underline{\mathbf{Z}}) \simeq \underline{\mathbf{Z}}^2$, the left $\mathrm{SL}_2(\mathbf{Z})$-action on X and $\omega_X^{\otimes k}$ correspond via $\psi_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)}$ and $-q\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ to the *classical* left actions on $\mathfrak{h}$ and $\mathbf{G}_a$:

$$z \mapsto \frac{rz+s}{uz+v}, \qquad h \mapsto \frac{1}{(uz+v)^k}h\left(\frac{rz+s}{uz+v}\right)$$

for $\gamma = \left(\begin{smallmatrix}r & s\\ u & v\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z})$. This is due to Theorem 1.5.1.1, Theorem 1.5.2.2, Corollary 1.5.2.3, and the identity $\alpha_X^\vee \circ \gamma^t = (\gamma \circ \alpha_X)^\vee$ (note that conjugation by $\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$ takes γ^t to γ^{-1} and using $g = \gamma^{-1}$ in Corollary 1.5.2.3 gives the formula for $(\cdot)|_k\gamma$).

In particular, by Theorem 1.5.2.2 we have $[g^{-1}]_X(\infty_X) \in \overline{\pi}_\Gamma^{-1}(\overline{x})$, so $[g]_X(x) = \infty_X$ for some $x \in \overline{\pi}_\Gamma^{-1}(\overline{x})$. Thus

$$\psi_{g\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right),g\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)} = \psi_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)} \circ [g]_X : X \xrightarrow{\simeq} \mathfrak{h}$$

takes $x \mapsto \infty$. Letting $\rho : \mathfrak{h} \to \mathbf{C}$ denote the function $z \mapsto f_{\overline{x}}(e^{2\pi i z/h})$, by definition we have

$$\begin{aligned}\rho\cdot(2\pi i\mathrm{d}\tau)^{\otimes k} &= \psi^{-1}_{g\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right),g\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)}{}^*(f)\\ &= [g^{-1}]_{\mathfrak{h}}^* \circ \psi^{-1}_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)}{}^*(f)\\ &= [g^{-1}]_{\mathfrak{h}}^* \circ \left(\widetilde{f}\cdot(2\pi i\mathrm{d}\tau)^{\otimes k}\right)\end{aligned}$$

Since $\psi_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)}$ transforms $-q\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ into $(2\pi i d\tau)^{\vee}$, we can work out the action of $[g^{-1}]_{\mathfrak{h}}^{*}$ by Corollary 1.5.2.3 with $\{e_1, e_2\} = \{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\}$. Comparing the coefficients of $(2\pi i d\tau)^{\otimes k}$ completes the proof. $\square$

This settles the compatibility issue alluded to at the end of §1.5. This also completes our "geometrization" of the classical theory, up to discussing level structures and Hecke operators for congruence subgroups Γ. This will be addressed in Chapter 2, at which time we will more fully "complete" the analytic theory by establishing a universal property of $\overline{E}_{\Gamma} \to \overline{X}_{\Gamma}$ for certain congruence subgroups Γ.

1.7. The Eichler-Shimura isomorphism

Let $\Gamma \subseteq \mathrm{SL}_2(\mathbf{R})$ be a discrete subgroup with finite covolume, regular cusps and acting freely on $\mathfrak{h}$. From this point on, most of what we do with modular forms will hold only in weight at least 2. The reason for this is that there is an isomorphism

$$S_{k+2}(\Gamma, \mathbf{C}) \simeq \mathrm{H}^0(\overline{X}_{\Gamma}, \overline{\omega}_{\Gamma}^{\otimes k} \otimes \Omega^1_{\overline{X}_{\Gamma}})$$

for all $k \geq 0$ and there is a fundamental isomorphism due to Eichler and Shimura which relates the right side of this isomorphism with the cohomology of a certain local system of $\mathbf{C}$-vector spaces on $\overline{X}_{\Gamma}$. For $k = 0$ this will recover the classical Hodge decomposition

$$\mathrm{H}^0(\overline{X}_{\Gamma}, \Omega^1_{\overline{X}_{\Gamma}}) \oplus \overline{\mathrm{H}^0(\overline{X}_{\Gamma}, \Omega^1_{\overline{X}_{\Gamma}})} \simeq \mathrm{H}^1(X_{\Gamma}, \underline{\mathbf{C}})$$

and the fact that the map $\mathrm{H}^1_{\mathrm{c}}(\overline{X}_{\Gamma}, \underline{\mathbf{C}}) \to \mathrm{H}^1(X_{\Gamma}, \underline{\mathbf{C}})$ has image coinciding with the image of $\mathrm{H}^1(\overline{X}_{\Gamma}, \underline{\mathbf{C}}) \hookrightarrow \mathrm{H}^1(X_{\Gamma}, \underline{\mathbf{C}})$; here $\mathrm{H}^{\bullet}_{\mathrm{c}}$ denotes compactly supported cohomology (i.e., the derived functor of the left exact "compactly supported global sections" functor). The general Eichler-Shimura relation between weight $k+2$ cusp forms and local systems coming from the cohomology of the universal elliptic curve over X (and even over X_{Γ} if Γ is a subgroup of $\mathrm{SL}_2(\mathbf{Z})$ of finite index) provides the means by which étale cohomology can be used to study classical modular forms.

1.7.1. Definition of the Eichler-Shimura map. By Lemma 1.5.4.4, the Hodge filtration of $E_X \to X$

$$0 \to \omega_X \to \underline{\mathbf{R}}^2 \otimes_{\underline{\mathbf{R}}} \mathcal{O}_X \xrightarrow{q} \omega_X^{-1} \to 0$$

descends to an exact sequence

$$0 \to \omega_{\Gamma} \to \mathcal{U}_{\Gamma} \otimes_{\underline{\mathbf{R}}} \mathcal{O}_{X_{\Gamma}} \xrightarrow{q} \omega_{\Gamma}^{-1} \to 0$$

on X_{Γ}, with $\mathcal{U}_{\Gamma}$ a local system of 2-dimensional $\mathbf{R}$-vector spaces. Of course, if Γ is a finite index subgroup of $\mathrm{SL}_2(\mathbf{Z})$, then this is just the Hodge filtration of the elliptic curve $f_{\Gamma} : E_{\Gamma} \to X_{\Gamma}$, and this will be the case of interest later on (when we will replace $\mathcal{U}_{\Gamma} = \mathrm{R}^1 f_{\Gamma *}\underline{\mathbf{R}}$ with the l-adic cohomology of $E_{\Gamma} \to X_{\Gamma}$). But for this section we work with $\Gamma \subseteq \mathrm{SL}_2(\mathbf{R})$ not necessarily lying in $\mathrm{SL}_2(\mathbf{Z})$, since we will not yet need the elliptic curve $E_{\Gamma} \to X_{\Gamma}$ (which does not exist if Γ does not lie $\mathrm{SL}_2(\mathbf{Z})$).

Since the short exact sequence of holomorphic vector bundles

$$0 \to \omega_{\Gamma} \to \mathcal{U}_{\Gamma} \otimes_{\underline{\mathbf{R}}} \mathcal{O}_{X_{\Gamma}} \to \omega_{\Gamma}^{-1} \to 0$$

is locally split, the natural map on kth symmetric powers of $\mathcal{O}_{X_{\Gamma}}$-modules

$$\omega_{\Gamma}^{\otimes k} \to \mathrm{Sym}^k_{\underline{\mathbf{R}}} \mathcal{U}_{\Gamma} \otimes_{\underline{\mathbf{R}}} \mathcal{O}_{X_{\Gamma}}$$

is *injective.* We therefore have a natural injection of $\mathcal{O}_{X_\Gamma}$-modules

$$\omega_\Gamma^{\otimes k} \otimes \Omega^1_{X_\Gamma} \to \mathrm{Sym}^k_{\underline{\mathbf{R}}} \mathcal{U}_\Gamma \otimes_{\underline{\mathbf{R}}} \Omega^1_{X_\Gamma}. \tag{1.7.1.1}$$

Let us write $\mathcal{U}_\Gamma^k$ for $\mathrm{Sym}^k_{\underline{\mathbf{R}}} \mathcal{U}_\Gamma$. Tensoring the exact sequence

$$0 \to \underline{\mathbf{C}} \to \mathcal{O}_{X_\Gamma} \xrightarrow{\mathrm{d}} \Omega^1_{X_\Gamma} \to 0$$

with $\mathcal{U}_\Gamma^k$ over $\underline{\mathbf{R}}$ yields an exact sequence

$$0 \to \mathcal{U}_\Gamma^k \otimes_{\underline{\mathbf{R}}} \underline{\mathbf{C}} \to \mathcal{U}_\Gamma^k \otimes_{\underline{\mathbf{R}}} \mathcal{O}_{X_\Gamma} \to \mathcal{U}_\Gamma^k \otimes_{\underline{\mathbf{R}}} \Omega^1_{X_\Gamma} \to 0. \tag{1.7.1.2}$$

We are interested in the composite $\mathbf{C}$-linear map

$$\begin{aligned} \alpha : \mathrm{H}^0(\overline{X}_\Gamma, \overline{\omega}_\Gamma^{\otimes k} \otimes \Omega^1_{\overline{X}_\Gamma}) &\hookrightarrow \mathrm{H}^0(X_\Gamma, \omega_\Gamma^{\otimes k} \otimes \Omega^1_{X_\Gamma}) \\ &\hookrightarrow \mathrm{H}^0(X_\Gamma, \mathcal{U}_\Gamma^k \otimes_{\underline{\mathbf{R}}} \Omega^1_{X_\Gamma}) \\ &\overset{\partial}{\to} \mathrm{H}^1(X_\Gamma, \mathcal{U}_\Gamma^k \otimes_{\underline{\mathbf{R}}} \underline{\mathbf{C}}) \end{aligned}$$

where the second inclusion comes from (1.7.1.1) and the boundary map ∂ comes from the long exact cohomology sequence associated to (1.7.1.2). (This composite map will be described explicitly in terms of "polynomial-valued integrals" in Lemma 1.7.5.1.) We also define the $\mathbf{C}$-linear map

$$\overline{\alpha} : \overline{\mathrm{H}^0(\overline{X}_\Gamma, \overline{\omega}_\Gamma^{\otimes k} \otimes \Omega^1_{\overline{X}_\Gamma})} \to \widetilde{\mathrm{H}}^1(X_\Gamma, \mathcal{U}_\Gamma^k \otimes_{\underline{\mathbf{R}}} \underline{\mathbf{C}})$$

to be the extension of scalars on α by complex conjugation $\mathbf{C} \to \mathbf{C}$.

The key result is the following theorem. For any abelian sheaf $\mathcal{F}$ on X_Γ, define $\widetilde{\mathrm{H}}^1(X_\Gamma, \mathcal{F})$ to be the image of the natural map

$$\mathrm{H}^1_{\mathrm{c}}(X_\Gamma, \mathcal{F}) \to \mathrm{H}^1(X_\Gamma, \mathcal{F}).$$

THEOREM 1.7.1.1 (Eichler-Shimura). *The map α factors through the subspace*

$$\widetilde{\mathrm{H}}^1(X_\Gamma, \mathcal{U}_\Gamma^k \otimes_{\underline{\mathbf{R}}} \underline{\mathbf{C}}) \subseteq \mathrm{H}^1(X_\Gamma, \mathcal{U}_\Gamma^k \otimes_{\underline{\mathbf{R}}} \underline{\mathbf{C}})$$

and the resulting map

$$\alpha \oplus \overline{\alpha} : \mathrm{H}^0(\overline{X}_\Gamma, \overline{\omega}_\Gamma^{\otimes k} \otimes \Omega^1_{\overline{X}_\Gamma}) \oplus \overline{\mathrm{H}^0(\overline{X}_\Gamma, \overline{\omega}_\Gamma^{\otimes k} \otimes \Omega^1_{\overline{X}_\Gamma})} \to \widetilde{\mathrm{H}}^1(X_\Gamma, \mathcal{U}_\Gamma^k \otimes_{\underline{\mathbf{R}}} \underline{\mathbf{C}})$$

is an isomorphism.

The proof we will give for Theorem 1.7.1.1 is based on the proof in [**56**, §8.1–8.2]. Before beginning, we note that the map $\alpha \oplus \overline{\alpha}$ is essentially $\alpha \otimes_{\mathbf{R}} \mathbf{C}$. Thus, once the first part of Theorem 1.7.1.1 is proven, the assertion that $\alpha \oplus \overline{\alpha}$ is an isomorphism is equivalent to the two conditions:

1. the map $\mathrm{Re}(\alpha) : \mathrm{H}^0(\overline{X}_\Gamma, \overline{\omega}_\Gamma^{\otimes k} \otimes \Omega^1_{\overline{X}_\Gamma}) \to \widetilde{\mathrm{H}}^1(X_\Gamma, \mathcal{U}_\Gamma^k)$ is injective;
2. $2\dim_{\mathbf{C}} \mathrm{H}^0(\overline{X}_\Gamma, \overline{\omega}_\Gamma^{\otimes k} \otimes \Omega^1_{\overline{X}_\Gamma}) = \dim_{\mathbf{R}} \widetilde{\mathrm{H}}^1(X_\Gamma, \mathcal{U}_\Gamma^k)$.

1.7.2. First calculations. Recall that $\mathrm{R}^1 f_{X*}\underline{\mathbf{R}}$ is isomorphic to the constant sheaf $\underline{\mathbf{R}}^2$ on X and a canonical ordered basis $\{e_1, e_2\}$ of $\mathrm{R}^1 f_{X*}\underline{\mathbf{R}}$ over $\underline{\mathbf{R}}$ is determined by the universal property of X. Set $V_k = \mathrm{Sym}^k_{\underline{\mathbf{R}}} \underline{\mathbf{R}}^2$. Note that $\mathcal{U}_\Gamma^k$ is the descent of V_k via Γ.

In order to proof Theorem 1.7.1.1, we need to compute the following composite map, which amounts to the first two steps in the definition of α:

$$
\begin{aligned}
(1.7.2.1)\qquad S_{k+2}(\Gamma,\mathbf{C}) &\overset{\mathrm{KS}_\Gamma}{\longrightarrow} \mathrm{H}^0(\overline{X}_\Gamma, \overline{\omega}_\Gamma^{\otimes k}\otimes\Omega^1_{\overline{X}_\Gamma}) \\
&\hookrightarrow \mathrm{H}^0(X_\Gamma, \omega_\Gamma^{\otimes k}\otimes\Omega^1_{X_\Gamma}) \\
&\hookrightarrow \mathrm{H}^0(X_\Gamma, \mathcal{U}_\Gamma^{k}\otimes_{\underline{\mathbf{R}}}\Omega^1_{X_\Gamma}) \\
&\simeq \mathrm{H}^0(X, V_k\otimes_{\underline{\mathbf{R}}}\Omega^1_{X})^\Gamma
\end{aligned}
$$

Here, the left action of $\Gamma\subseteq \mathrm{SL}_2(\mathbf{R})$ on V_k is the canonical one and the left action on Ω^1_X is, as always, via pullback by the right $[\cdot]_X$-action on X.

Lemma 1.7.2.1. *Via the isomorphism $\psi_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)} : X \overset{\simeq}{\longrightarrow} \mathfrak{h}$, the composite map in the diagram* (1.7.2.1) *becomes:*

$$
(1.7.2.2)\qquad S_{k+2}(\Gamma,\mathbf{C}) \to \mathrm{H}^0(\mathfrak{h}, \mathrm{Sym}^k\,\underline{\mathbf{R}}^2\otimes_{\underline{\mathbf{R}}}\Omega^1_{\mathfrak{h}})^\Gamma = \mathrm{H}^0(\mathfrak{h}, \mathrm{Sym}^k\,\underline{\mathbf{C}}^2\otimes_{\underline{\mathbf{C}}}\Omega^1_{\mathfrak{h}})^\Gamma
$$

and is given by

$$
(1.7.2.3)\qquad f\mapsto \omega_f \overset{\mathrm{def}}{=} (2\pi i)^{k+1}\cdot (ze_1+e_2)^k\cdot \widetilde{f}\mathrm{d}z,
$$

where $\widetilde{f}\in \mathrm{H}^0(\mathfrak{h},\mathcal{O}_{\mathfrak{h}})$ corresponds to f under $\psi_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)}$ and the trivialization of $\overline{\omega}_X^{-1}$ by $-q\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$.

The Γ-action the right side in (1.7.2.2) involves $g\in\Gamma$ acting on $\Omega^1_{\mathfrak{h}}$ through pullback by the classical linear fractional action of g^{-1} on $\mathfrak{h}$ and involves

$$\Gamma\subseteq \mathrm{SL}_2(\mathbf{R})\subseteq \mathrm{SL}_2(\mathbf{C})$$

acting in the canonical manner on $\mathrm{Sym}^k(\underline{\mathbf{C}}^2)$.

Proof. The calculation of the Kodaira-Spencer map in Theorem 1.5.4.1 shows that (1.7.2.2) sends

$$\widetilde{f}\cdot(2\pi i\mathrm{d}\tau)^{\otimes(k+2)}\mapsto \big(a_1(z)e_1+a_2(z)e_2\big)^k\cdot\widetilde{f}\cdot(2\pi i\mathrm{d}z),$$

where

$$a_1(z)e_1+a_2(z)e_2\in \mathrm{H}^1(\mathcal{E}_z^{\mathrm{an}},\underline{\mathbf{R}})\otimes_{\mathbf{R}}\mathbf{C}\simeq \mathrm{H}^1(\mathcal{E}_z^{\mathrm{an}},\underline{\mathbf{C}})\simeq\mathbf{C}^2$$

is the image of $2\pi i\mathrm{d}\tau$ under the canonical map $\mathrm{H}^0(\mathcal{E}_z^{\mathrm{an}},\Omega^1_{\mathcal{E}_z^{\mathrm{an}}})\to \mathrm{H}^1(\mathcal{E}_z^{\mathrm{an}},\underline{\mathbf{C}})$ identifying a holomorphic 1-form with the corresponding de Rham cohomology class. By the definition of this identification and the definitions of e_1, e_2, we see that $a_j(z)$ is the period of $2\pi i\mathrm{d}\tau$ against the homology generator γ_j on $\mathcal{E}_z^{\mathrm{an}}$:

$$a_j(z)=\int_{\gamma_j}2\pi i\mathrm{d}\tau.$$

Thus $a_1(z)=2\pi i z$ and $a_2(z)=2\pi i$. Collecting powers of $2\pi i$ now gives (1.7.2.3). □

As an explicit "check" on the Γ-invariance of the formula in Lemma 1.7.2.1, we calculate the pullback action of $[g]_{\mathfrak{h}}$ on $\mathrm{H}^0(\mathfrak{h},\mathrm{Sym}^k\underline{\mathbf{C}}^2\otimes_{\underline{\mathbf{C}}}\Omega^1_{\mathfrak{h}})$ for $g=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in$

$\mathrm{SL}_2(\mathbf{R})$. This sends the right side of (1.7.2.3) to

$$\begin{aligned} & (2\pi i)^{k+1}([g](z)\cdot ge_1+ge_2)^k\cdot[g]^*(\widetilde{f}\mathrm{d}z) \\ = \quad & (2\pi i)^{k+1}\left(\frac{dz-b}{-cz+a}(ae_1+ce_2)+(be_1+de_2)\right)^k\widetilde{f}\left(\frac{dz-b}{-cz+a}\right)\frac{\mathrm{d}z}{(-cz+a)^2} \\ = \quad & (2\pi i)^{k+1}(ze_1+e_2)^k(\widetilde{f}|_{k+2}g^{-1})\mathrm{d}z \end{aligned}$$

since $ad-bc=1$. When $g\in\Gamma$ this is equal to

$$(2\pi i)^{k+1}(ze_1+e_2)^k\widetilde{f}\mathrm{d}z$$

since $f\in S_{k+2}(\Gamma,\mathbf{C})$. This reconfirms the Γ-invariance in the lemma, but the calculation with $g\in\mathrm{SL}_2(\mathbf{R})$ will be useful later.

Let $\mathcal{U}_\Gamma^k(\mathbf{C})=\mathcal{U}_\Gamma^k\otimes_{\underline{\mathbf{R}}}\underline{\mathbf{C}}$. Using Lemma 1.7.2.1, the down-to-earth content of the first part of Theorem 1.7.1.1 is that if $\omega\in\mathrm{H}^0(X_\Gamma,\mathcal{U}_\Gamma^k(\mathbf{C})\otimes_{\underline{\mathbf{C}}}\Omega^1_{X_\Gamma})$ is the descent of the Γ-invariant form

$$(2\pi i)^{k+1}(ze_1+e_2)^k\widetilde{f}\mathrm{d}z\in\mathrm{H}^0(\mathfrak{h},\mathrm{Sym}^k\,\underline{\mathbf{C}}\otimes_{\underline{\mathbf{C}}}\Omega^1_{\mathfrak{h}}),$$

then

$$\partial:\mathrm{H}^0(X_\Gamma,\mathcal{U}_\Gamma^k(\mathbf{C})\otimes_{\underline{\mathbf{C}}}\Omega^1_{X_\Gamma})\to\mathrm{H}^1(X_\Gamma,\mathcal{U}_\Gamma^k(\mathbf{C}))$$

sends ω into the image $\widetilde{\mathrm{H}}^1(X_\Gamma,\mathcal{U}_\Gamma^k(\mathbf{C}))$ of

$$\mathrm{H}^1_c(X_\Gamma,\mathcal{U}_\Gamma^k(\mathbf{C}))\to\mathrm{H}^1(X_\Gamma,\mathcal{U}_\Gamma^k(\mathbf{C})).$$

To compute this image subspace we note that for a compact set $K\subseteq X_\Gamma$ there is an exact sequence

$$\mathrm{H}^0(X_\Gamma,\mathcal{F})\to\mathrm{H}^0(X_\Gamma-K,\mathcal{F})\to\mathrm{H}^1_K(X_\Gamma,\mathcal{F})\to\mathrm{H}^1(X_\Gamma,\mathcal{F})\to\mathrm{H}^1(X_\Gamma-K,\mathcal{F})$$

for any abelian sheaf $\mathcal{F}$ on X_Γ. We are interested in the exact sequence

$$\mathrm{H}^1_K(X_\Gamma,\mathcal{F})\to\mathrm{H}^1(X_\Gamma,\mathcal{F})\to\mathrm{H}^1(X_\Gamma-K,\mathcal{F}).$$

Passing to the direct limit over increasing compact K on the left side gives the canonical map $\mathrm{H}^1_c(X_\Gamma,\mathcal{F})\to\mathrm{H}^1(X_\Gamma,\mathcal{F})$, so we get an exact sequence

$$0\to\widetilde{\mathrm{H}}^1(X_\Gamma,\mathcal{F})\to\mathrm{H}^1(X_\Gamma,\mathcal{F})\to\varinjlim_K\mathrm{H}^1(X_\Gamma-K,\mathcal{F}). \tag{1.7.2.4}$$

In fact, one sees easily that a cofinal system of such K is given by those for which $X_\Gamma-K$ is a union of small punctured disks around the cusps, so that we can take the last direct limit to only be over these K. We provisionally call such K *nice compacts*.

In order to work with (1.7.2.4) we will need to know that when $\mathcal{F}$ is a local system on X_Γ (for example, $\mathcal{F}=\mathcal{U}_\Gamma^k(\mathbf{C})$), the restriction maps

$$\mathrm{H}^1(X_\Gamma-K,\mathcal{F})\to\mathrm{H}^1(X_\Gamma-K',\mathcal{F})$$

are isomorphisms for nice compacts $K\subseteq K'$ as above. In other words, if $\Delta^*_a\hookrightarrow\Delta^*$ is the inclusion

$$\{q\,|\,0<|q|<a\}\hookrightarrow\{q\,|\,0<|q|<1\}$$

for some a, $0<a<1$, then we want the restriction map

$$\mathrm{H}^1(\Delta^*,\mathcal{F})\to\mathrm{H}^1(\Delta^*_a,\mathcal{F})$$

to be an isomorphism for local systems $\mathcal{F}$ on Δ^*. Since $\pi_1(\Delta^*_a)\to\pi_1(\Delta^*)$ is an isomorphism, we essentially need to identify sheaf cohomology for local systems with group cohomology of the fundamental group, compatibly with restriction maps.

Since $e^{2\pi i(\cdot)} : \mathfrak{h} \to \Delta^*$ is the universal cover, for this we may use the final part of Lemma 1.7.3.2(2) below.

1.7.3. Local systems and π_1-modules. For any connected, locally path connected, and locally simply connected topological space Y with a fixed base point y_0, let $\widetilde{(\cdot)}$ denote the functor associating a local system to a module for the fundamental group $\pi_1(Y, y_0)$. By §1.5.6 and a universal δ-functor argument, there are unique δ-functorial maps

$$\mathrm{H}^i(\pi_1(Y, y_0), \cdot) \to \mathrm{H}^i(Y, \widetilde{\cdot})$$

extending the canonical isomorphism in degree 0.

LEMMA 1.7.3.1. *Assume that the universal cover $\widetilde{Y}$ of Y has vanishing higher sheaf cohomology for constant sheaves. Then for any $\pi_1(Y, y_0)$-module M, the maps*

$$\mathrm{H}^i(\pi_1(Y, y_0), M) \to \mathrm{H}^i(Y, \widetilde{M})$$

are isomorphisms for all i.

PROOF. Since this is clear for $i = 0$, we just need to show that $\mathrm{H}^i(Y, \widetilde{\cdot})$ is an erasable δ-functor on $\pi_1(Y, y_0)$-modules. Let M be an injective $\pi_1(Y, y_0)$-module. We will show that $\mathrm{H}^i(Y, \widetilde{M}) = 0$ for all $i > 0$.

For this, let $p : \widetilde{Y} \to Y$ be a universal cover. By hypothesis, $\mathrm{H}^i(\widetilde{Y}, p^*\widetilde{M}) = 0$ for all $i > 0$, since $p^*\widetilde{M}$ is *constant.* By [**45**, Appendix, p. 22] it follows that

$$\mathrm{H}^j(Y, \widetilde{M}) \simeq \mathrm{H}^j\big(\pi_1(Y, y_0), \mathrm{H}^0(\widetilde{Y}, p^*\widetilde{M})\big) = \mathrm{H}^j(\pi_1(Y, y_0), M)$$

for all j. This vanishes for $j > 0$ since M is an injective $\pi_1(Y, y_0)$-module. $\square$

LEMMA 1.7.3.2. *Let $f : Y' \to Y$ be a continuous map of topological spaces which are connected, locally path connected, and locally simply connected. Fix a base point $y_0' \in Y'$ and set $y_0 = f(y_0') \in Y$.*

1. *For any $\pi_1(Y, y_0)$-module M, the local system $f^*\widetilde{M}$ has stalk $(f^*\widetilde{M})_{y_0'} \simeq \widetilde{M}_{y_0} = M$ equal to the $\pi_1(Y', y_0')$-module ${}_fM$ underlying M via $\pi_1(f)$.*
2. *The diagram*

$$\begin{array}{ccc} \mathrm{H}^i(\pi_1(Y, y_0), M) & \longrightarrow & \mathrm{H}^i(\pi_1(Y', y_0'), {}_fM) \\ \downarrow & & \downarrow \\ \mathrm{H}^i(Y, \widetilde{M}) & \longrightarrow & \mathrm{H}^i(Y', f^*\widetilde{M}) \end{array}$$

commutes, where the right column is defined using (1).

In particular, if the universal covers $\widetilde{Y'}, \widetilde{Y}$ of Y', Y have vanishing higher sheaf cohomology for constant sheaves and the map $\pi_1(Y', y_0') \to \pi_1(Y, y_0)$ is an isomorphism, then $\mathrm{H}^i(Y, \mathcal{F}) \to \mathrm{H}^i(Y', f^*\mathcal{F})$ is an isomorphism for all local systems $\mathcal{F}$ on Y.

PROOF. Since the π_1-action can be described via "transport along a loop" (cf. Lemma 1.5.6.2), (1) is clear. The commutativity of the diagram in (2) follows by universality from the case $i = 0$, which is clear. The last statement follows easily from (2) and Lemma 1.7.3.1. $\square$

We conclude this discussion by considering the hypothesis of Lemma 1.7.3.1. Let us briefly explain why a contractible Hausdorff space Z has $\mathrm{H}^i(Z, \underline{M}) = 0$ for $i > 0$ if Z is either paracompact with a base of contractible opens (cf. Remark 1.2.1.3), locally compact, or metrizable. In the first case, the comparison theorem with singular cohomology applies, and higher singular cohomology vanishes on contractible spaces. When Z is a (not necessarily contractible) Hausdorff space which is locally compact or metrizable, we can establish a similar "homotopy invariance". More precisely, for any two homotopic maps $f_0, f_1 : Z \to Z'$, we claim that the induced maps $\mathrm{H}^i(Z', \underline{M}) \to \mathrm{H}^i(Z, \underline{M})$ coincide for all i. (In particular, if either of the f_i factors through a point, then these maps are zero for all $i > 0$.) This amounts to showing that the standard inclusions $j_0, j_1 : Z \to Z \times [0,1]$ induce the same pullback maps on cohomology of constant sheaves, so it suffices to show that the projection $\pi : Z \times [0,1] \to Z$ induces maps $\mathrm{H}^i(Z, \underline{M}) \to \mathrm{H}^i(Z \times [0,1], \underline{M})$ which are isomorphisms for all i (for then j_0^* and j_1^* are both inverses to this and thus coincide.)

For this last assertion we can apply topological proper base change to π to conclude that the canonical map $\underline{M} \to \pi_* \underline{M}$ is an isomorphism, so $\pi^* \underline{M} \to \pi^* \pi_* \underline{M} \to \underline{M}$ is the canonical "identity." (Keep in mind that not all of these $\underline{M}$'s live on the same space.) Thus, our map of interest is just the composite

$$\mathrm{H}^i(Z, \pi_* \underline{M}) \to \mathrm{H}^i(Z \times [0,1], \pi^* \pi_* \underline{M}) \to \mathrm{H}^i(Z \times [0,1], \underline{M}).$$

By [**26**, 0,12.1.7], this is *exactly* an edge map in the spectral sequence

$$E_2^{ij} = \mathrm{H}^i(Z, \mathrm{R}^j \pi_* \underline{M}) \Rightarrow \mathrm{H}^i(Z \times [0,1], \underline{M})$$

so it suffices to show that $\mathrm{R}^j \pi_* \underline{M} = 0$ for all $j > 0$. For this we use that, by proper base change again, $(\mathrm{R}^j \pi_* \underline{M})_z \simeq \mathrm{H}^j([0,1], \underline{M}) = 0$ for $j > 0$ since $[0,1]$ is a contractible paracompact Hausdorff space.

1.7.4. Proof that the Eichler-Shimura map has compactly supported image. We return now to the proof of Theorem 1.7.1.1. By Lemma 1.7.3.2, the transition maps in the direct limit in (1.7.2.4) are isomorphisms. Thus, we have an exact sequence

$$0 \to \widetilde{\mathrm{H}}^1(X_\Gamma, \mathcal{F}) \to \mathrm{H}^1(X_\Gamma, \mathcal{F}) \to \bigoplus_{\overline{x} \in \overline{X}_\Gamma - X_\Gamma} \mathrm{H}^1(\Delta^*_{\overline{x}}, \mathcal{F}) \tag{1.7.4.1}$$

for any local system $\mathcal{F}$ on X_Γ, where $\Delta^*_{\overline{x}}$ is a small punctured disk around a cusp $\overline{x}$ on $\overline{X}_\Gamma$.

We are now in a position to prove the first part of Theorem 1.7.1.1. For $f \in S_{k+2}(\Gamma, \mathbf{C})$, by (1.7.4.1) we want to show that ω_f is killed by the composite

$$\mathrm{H}^0(X_\Gamma, \mathcal{U}^k_\Gamma(\mathbf{C}) \otimes_{\underline{\mathbf{C}}} \Omega^1_{X_\Gamma}) \xrightarrow{\partial} \mathrm{H}^1(X_\Gamma, \mathcal{U}^k_\Gamma(\mathbf{C})) \to \mathrm{H}^1(\Delta^*_{\overline{x}}, \mathcal{U}^k_\Gamma(\mathbf{C}))$$

for each $\overline{x} \in \overline{X}_\Gamma - X_\Gamma$. Equivalently, we want to show that $\omega_f|_{\Delta^*_{\overline{x}}}$ is killed by the coboundary map

$$\mathrm{H}^0(\Delta^*_{\overline{x}}, \mathcal{U}^k_\Gamma(\mathbf{C}) \otimes_{\underline{\mathbf{C}}} \Omega^1_{X_\Gamma}) \to \mathrm{H}^1(\Delta^*_{\overline{x}}, \mathcal{U}^k_\Gamma(\mathbf{C})).$$

In other words, by (1.7.1.2) we need to show that $\omega_f|_{\Delta^*_{\overline{x}}}$ is in the image of the map

$$\mathrm{H}^0(1 \otimes \mathrm{d}) : \mathrm{H}^0(\Delta^*_{\overline{x}}, \mathcal{U}^k_\Gamma(\mathbf{C}) \otimes_{\underline{\mathbf{C}}} \mathcal{O}_{X_\Gamma}) \to \mathrm{H}^0(\Delta^*_{\overline{x}}, \mathcal{U}^k_\Gamma(\mathbf{C}) \otimes_{\underline{\mathbf{C}}} \Omega^1_{X_\Gamma}). \tag{1.7.4.2}$$

Let σ be a path in $\Delta^*_{\overline{x}}$ and let ω be a C^∞ differential form on $\Delta^*_{\overline{x}}$ with values in a local system V of $\mathbf{C}$-vector spaces. Since local systems on $[0,1]$ are always constant in a canonical way (via contraction to the initial point 0), one can define the integral

$$\int_\sigma \omega \in V_{\sigma(0)} \in \mathrm{H}^0(\Delta_x^*, V).$$

To prove that $\omega_f|_{\Delta_{\overline{x}}^*}$ lifts to $\mathcal{U}_\Gamma^k(\mathbf{C}) \otimes_{\underline{\mathbf{C}}} \mathcal{O}_{\Delta_{\overline{x}}^*}$ it will suffice to show that $\int_\sigma \omega_f = 0$ for a loop σ generating $\pi_1(\Delta_{\overline{x}}^*)$. Indeed, given this we can obtain a well-defined "integral" of ω_f from a base point of $\Delta_{\overline{x}}^*$ to any other points of $\Delta_{\overline{x}}^*$ (thanks to the assumed absence of monodromy) and this "integral" will map to ω_f under (1.7.4.2).

By Lemma 1.5.7.2, with $\gamma \in \mathrm{SL}_2(\mathbf{R})$ as defined there relative to a choice of $x \in \overline{\pi}_\Gamma^{-1}(\overline{x})$ in $\mathrm{cusps}_X(\Gamma)$, we find that

$$\int_\sigma \omega_f = \int_{a+ib}^{a+h+ib} (ze_1 + e_2)^k (\widetilde{f}|_{k+2})\mathrm{d}z \in \mathrm{Sym}^2(\mathbf{C}^2) \tag{1.7.4.3}$$

for any $a, b \in \mathbf{R}$, $b > 0$; here, $h > 0$ is determined by

$$\mathrm{Stab}_{\gamma^{-1}\Gamma\gamma}(\infty) = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}^{\mathbf{Z}}$$

(recall that the classical linear fractional transformation on $\mathfrak{h}$ by $\gamma^{-1} \in \mathrm{SL}_2(\mathbf{R})$ is what moves $\psi_{\binom{1}{0},\binom{0}{1}}(x)$ to ∞). In (1.7.4.3) the "individual" integrals

$$\int_{a+ib}^{a+h+ib} z^j (\widetilde{f}|_{k+2}\gamma)\mathrm{d}z \in \mathbf{C}$$

for $0 \le j \le k$ depend very much on a and b, whereas the integral of interest in (1.7.4.3) is of course independent of these choices.

To prove the vanishing of (1.7.4.3), we may therefore fix a and let b go to infinity, so it suffices to show that for fixed $j \ge 0$, we have

$$\lim_{b\to\infty} \int_{a+ib}^{a+h+ib} z^j (\widetilde{f}|_{k+2}\gamma)\mathrm{d}z = 0. \tag{1.7.4.4}$$

But f is a *cusp* form, so by Lemma 1.5.7.2 we see that $\widetilde{f}|_{k+2}\gamma$ has exponential decay near $i\infty$. Thus, (1.7.4.4) vanishes. This completes the proof of the first part of Theorem 1.7.1.1.

1.7.5. Parabolic cohomology. We now know that we do indeed have a map

$$\alpha : S_{k+2}(\Gamma, \mathbf{C}) \to \widetilde{\mathrm{H}}^1(X_\Gamma, \mathcal{U}_\Gamma^k(\mathbf{C})).$$

It remains to prove that $\mathrm{Re}(\alpha)$ is injective and that

$$2 \dim_{\mathbf{C}} S_{k+2}(\Gamma, \mathbf{C}) = \dim_{\mathbf{R}} \widetilde{\mathrm{H}}^1(X_\Gamma, \mathcal{U}_\Gamma^k).$$

For each $x \in \mathrm{cusps}_\Gamma(X)$, let $\Gamma_x \subseteq \Gamma$ denote the stabilizer $\mathrm{Stab}_\Gamma(x)$; Γ-conjugate cusps yield conjugate stabilizers Γ_x. For a Γ-module M, define the *parabolic cohomology group* $\mathrm{H}^1_{\mathrm{P}}(\Gamma, M)$ to be the kernel of the map

$$\mathrm{H}^1(\Gamma, M) \to \bigoplus_{x \in \mathrm{cusps}_X(\Gamma)} \mathrm{H}^1(\Gamma_x, M).$$

Of course, it suffices to consider in the direct sum a finite set of cusps representing $\overline{X}_\Gamma - X_\Gamma$ since conjugation by Γ acts trivially on $\mathrm{H}^1(\Gamma, M)$.

Fix $\overline{x}_0 \in X_\Gamma$, so $\Gamma \simeq \pi_1(X_\Gamma, \overline{x}_0)$. Using Lemma 1.7.3.1 and the exact sequence (1.7.4.1), we conclude that there is a canonical isomorphism

$$\mathrm{H}^1(X_\Gamma, \mathcal{U}_\Gamma^k(\mathbf{C})) \simeq \mathrm{H}^1(\Gamma, V_k \otimes_{\mathbf{R}} \mathbf{C})$$

and this takes $\widetilde{\mathrm{H}}^1(X_\Gamma, \mathcal{U}_\Gamma^k(\mathbf{C}))$ over to $\mathrm{H}^1_{\mathrm{P}}(\Gamma, V_k \otimes_{\mathbf{R}} \mathbf{C})$. Here, $V_k = \mathrm{Sym}^k_{\mathbf{R}}(\mathbf{R}^2)$ with Γ-action induced by the standard left-action of $\Gamma \subseteq \mathrm{SL}_2(\mathbf{R})$ on $\mathbf{R}^2$; we identify V_k with the stalk of $\mathcal{U}_\Gamma^k$ at $\overline{x}_0$. We want to compute the map

$$\alpha : S_{k+2}(\Gamma, \mathbf{C}) \to \mathrm{H}^1_{\mathrm{P}}(\Gamma, V_k \otimes_{\mathbf{R}} \mathbf{C}).$$

Strictly speaking, we do not need to consider this group cohomology formulation, but we include some discussion of it in order to retain the connection with the "classical" picture. In particular, the following lemma will not be used at all in our proof of Theorem 1.7.1.1 but it is useful in other contexts.

LEMMA 1.7.5.1. *Fix $x_0 \in X$ with image $\overline{x}_0 \in X_\Gamma$. Then*

$$\alpha : S_{k+2}(\Gamma, \mathbf{C}) \to \mathrm{H}^1_{\mathrm{P}}(\Gamma, V_k \otimes_{\mathbf{R}} \mathbf{C}) \subseteq \mathrm{H}^1(\Gamma, V_k \otimes_{\mathbf{R}} \mathbf{C})$$

sends f to the cohomology class of the 1*-cocycle*

$$\gamma \mapsto \int_{\gamma^{-1}} \omega_f \in \mathcal{U}_\Gamma^k(\mathbf{C})_{\overline{x}_0} = V_k \otimes_{\mathbf{R}} \mathbf{C}$$

where we use the isomorphism $\Gamma \simeq \pi_1(X_\Gamma, \overline{x}_0)$ arising from the choice of x_0.

REMARK 1.7.5.2. Using $\psi_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)} : X \xrightarrow{\simeq} \mathfrak{h}$ and defining $z_0 = \psi_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)}(x_0)$, we have

$$\int_{\gamma^{-1}} \omega_f = \int_{z_0}^{[\gamma^{-1}](z_0)} (2\pi i)^{k+1} (ze_1 + e_2)^k \widetilde{f} \mathrm{d}z.$$

By Theorem 1.5.2.2, these are *exactly* the period integrals considered by Eichler and Shimura. When $k = 0$, we have $\int_{\gamma^{-1}} \omega_f = -\int_\gamma \omega_f$, so α is the classical identification of a weight 2 cusp form with a holomorphic 1-form on an (open) modular curve, up to a factor of $-2\pi i$.

PROOF. Let $\mathcal{F}$ be an arbitrary $\underline{\mathbf{C}}$-module on X_Γ and let $\underline{\mathbf{C}} \to \mathcal{A}^\bullet$ be the $\mathbf{C}$-valued C^∞ de Rham complex. Since $\mathcal{F} \to \mathcal{F} \otimes_{\underline{\mathbf{C}}} \mathcal{A}^\bullet$ is a fine resolution, it computes the δ-functor $\mathrm{H}^\bullet(X_\Gamma, \mathcal{F})$ on $\underline{\mathbf{C}}$-modules. We are interested in the case of degrees 0 and 1, which are computed by the complex

$$0 \to \mathrm{H}^0(X_\Gamma, \mathcal{F} \otimes_{\underline{\mathbf{C}}} \mathcal{A}^0) \to \mathrm{H}^0(X_\Gamma, \mathcal{F} \otimes_{\underline{\mathbf{C}}} \mathcal{A}^1)^{\mathrm{d}=0} \to 0.$$

Recalling the definition of α in terms of a coboundary map and using Lemma 1.7.2.1, we see that we need to understand the diagram

$$\begin{tikzcd} \mathrm{H}^0(X_\Gamma, \mathcal{F} \otimes_{\underline{\mathbf{C}}} \Omega^1_{X_\Gamma}) \arrow[r] \arrow[dr, "\partial"'] & \mathrm{H}^0(X_\Gamma, \mathcal{F} \otimes_{\underline{\mathbf{C}}} \mathcal{A}^1)^{\mathrm{d}=0} \arrow[d, two heads] \\ & \mathrm{H}^1(X_\Gamma, \mathcal{F}) \end{tikzcd} \tag{1.7.5.1}$$

where the horizontal map is the natural map and ∂ is the coboundary map arising from the exact sequence

$$0 \to \mathcal{F} \to \mathcal{F} \otimes_{\underline{\mathbf{C}}} \mathcal{O}_{X_\Gamma} \to \mathcal{F} \otimes_{\underline{\mathbf{C}}} \Omega^1_{X_\Gamma} \to 0$$

obtained by applying the exact functor $\mathcal{F} \otimes_{\underline{\mathbf{C}}} (\cdot)$ to the exact holomorphic de Rham complex on X_Γ.

We claim that (1.7.5.1) commutes. Indeed,

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d, equal] & \mathcal{F} \otimes_{\underline{\mathbf{C}}} \mathcal{A}^0 \arrow[r] & \mathcal{F} \otimes_{\underline{\mathbf{C}}} \mathcal{A}^1 \arrow[r] & \mathcal{F} \otimes_{\underline{\mathbf{C}}} \mathcal{A}^2 \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{F} \otimes_{\underline{\mathbf{C}}} \mathcal{O}_{X_\Gamma} \arrow[r] \arrow[u] & \mathcal{F} \otimes_{\underline{\mathbf{C}}} \Omega^1_{X_\Gamma} \arrow[r] \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

is a commutative diagram of resolutions with the top resolution acyclic; the commutativity of (1.7.5.1) follows.

Now suppose that $\mathcal{F}$ is a local system and consider the diagram of horizontal chain complexes

$$\begin{tikzcd}
\mathrm{H}^0(X_\Gamma, \mathcal{F} \otimes_{\underline{\mathbf{C}}} \mathcal{A}^0) \arrow[r] \arrow[d] & \mathrm{H}^0(X_\Gamma, \mathcal{F} \otimes_{\underline{\mathbf{C}}} \mathcal{A}^1)^{\mathrm{d}=0} \arrow[r] \arrow[d, "\beta"] & 0 \\
\mathcal{F}_{\overline{x}_0} \arrow[r] & Z^1(\Gamma, \mathcal{F}_{\overline{x}_0}) \arrow[r] & 0
\end{tikzcd} \tag{1.7.5.2}$$

(with Z^1 the 1-cocycles), where the left map is induced by the map $\mathcal{A}^0_{\overline{x}_0} \to \mathbf{C}$ given by $\varphi \mapsto \varphi(\overline{x}_0)$ and the right map is defined $\beta(\omega) = \int_{\gamma^{-1}} \omega \in \mathcal{F}_{x_0}$, where $\int_{\gamma^{-1}} \omega$ is defined by canonically trivializing the restriction of $\mathcal{F}$ to the path γ^{-1} via transport to the initial point $\overline{x}_0$.

Note that $\int_{\gamma^{-1}} \omega$ only depends on γ up to homotopy since $\mathrm{d}\omega = 0$, and $\beta(\omega)$ is a 1-cocycle since

$$\int_{(\gamma_1\gamma_2)^{-1}} \omega = \int_{\gamma_2^{-1}\gamma_1^{-1}} \omega = \gamma_1 \left(\int_{\gamma_2^{-1}} \omega \right) + \int_{\gamma_1} \omega.$$

Indeed, using the *definition* of this integration, when integrating along $(\gamma_2\gamma_1)^{-1}$ we are always transporting back to neighborhoods of the initial point $\overline{x}_0$. Thus, after going along γ_1^{-1} we are computing the integral along γ_2^{-1} by starting with the transport *backwards* along γ_1^{-1} of the "local system" part of

$$\omega \in \mathrm{H}^0(X_\Gamma, \mathcal{F} \otimes_{\underline{\mathbf{C}}} \mathcal{A}^1)$$

near $\overline{x}_0$. By Lemma 1.5.6.2, this amounts to applying the action of γ_1 to $\mathcal{F}_{\overline{x}_0}$. Thus, the map β makes sense.

In a similar manner, we compute that for $\varphi \in \mathrm{H}^0(X_\Gamma, \mathcal{F} \otimes_{\underline{\mathbf{C}}} \mathcal{A}^0)$, we have

$$\int_{\gamma^{-1}} \mathrm{d}\varphi = \gamma\big(\varphi(\overline{x}_0)\big) - \varphi(\overline{x}_0).$$

This shows that (1.7.5.2) commutes, so it defines a δ-functorial map

$$\mathrm{H}^*(X_\Gamma, \cdot) \to \mathrm{H}^*(\Gamma, \cdot)$$

in degrees 0 and 1 between functors on $\mathbf{C}[\pi_1(X, \overline{x}_0)]$-modules, giving the usual map in degree 0. By Lemma 1.7.3.1, both sides are erasable and so the map *must* be the canonical isomorphism (or rather, its inverse). This construction based on (1.7.5.2) allows us to explicitly compute the map $\mathrm{H}^1(X_\Gamma, \mathcal{F}) \xrightarrow{\simeq} \mathrm{H}^1(\Gamma, \mathcal{F}_{\overline{x}_0})$ for local systems of $\underline{\mathbf{C}}$-modules $\mathcal{F}$ on X_Γ. Applying this and (1.7.5.1) with $\mathcal{F} = \mathcal{U}^k_\Gamma(\mathbf{C})$, the definition of β completes the proof. $\square$

1.7.6. Injectivity of the Eichler-Shimura map. In the group cohomology language of Lemma 1.7.5.1, the injectivity of $\mathrm{Re}(\alpha)$ amounts to the assertion that for $f \in S_{k+2}(\Gamma, \mathbf{C})$, the vanishing of the class of the cocycle

$$\gamma \mapsto \int_{z_0}^{\gamma(z_0)} \mathrm{Re}\big((2\pi i)^{k+1}(ze_1+e_2)^k \widetilde{f} \mathrm{d}z\big) \in \mathrm{Sym}^k_{\mathbf{R}}(\mathbf{R}^2) = V_k$$

in $\mathrm{H}^1_{\mathrm{P}}(\Gamma, V_k) \subseteq \mathrm{H}^1(\Gamma, V_k)$ forces $f = 0$. We prove this in the equivalent form of the next lemma.

LEMMA 1.7.6.1. *Let $f \in S_{k+2}(\Gamma, \mathbf{C})$ be a cusp form. If $\mathrm{Re}(\omega_f) \in \mathrm{H}^1(X_\Gamma, \mathcal{U}^k_\Gamma)$ vanishes, then $f = 0$.*

PROOF. We use the fact (analogous to the proof of Lemma 1.7.5.1) that the fine C^∞ de Rham resolution of $\underline{\mathbf{R}}$ can be used to δ-functorially compute $\mathrm{H}^*_{\mathrm{c}}(X_\Gamma, \cdot)$ and $\mathrm{H}^*(X_\Gamma, \cdot)$ on local systems of $\underline{\mathbf{R}}$-modules. The wedge product pairing on such resolutions computes cup product pairings such as

$$\mathrm{H}^i_{\mathrm{c}}(X_\Gamma, \mathcal{F}) \otimes_{\mathbf{R}} \mathrm{H}^j(X_\Gamma, \mathcal{G}) \to \mathrm{H}^{i+j}_{\mathrm{c}}(X_\Gamma, \mathcal{F} \otimes_{\underline{\mathbf{R}}} \mathcal{G}),$$

by [**18**, Ch II, Thm 6.2].

Consider the non-degenerate pairing

$$\langle \cdot, \cdot \rangle : V_1 \otimes_{\mathbf{R}} V_1 \to \mathbf{R} \tag{1.7.6.1}$$

defined by

$$(ue_1 + ve_2) \otimes (re_1 + se_2) \to \det \begin{pmatrix} u & r \\ v & s \end{pmatrix}$$

whose extension of scalars to $\mathbf{C}$ is given by the same formula. See (2.3.3.1)ff for the geometric meaning of this pairing. This pairing is $\mathrm{SL}_2(\mathbf{R})$-equivariant (i.e., $\langle \gamma x, \gamma x' \rangle = \langle x, x' \rangle$ for all $\gamma \in \mathrm{SL}_2(\mathbf{R})$). We therefore get an induced $\mathrm{SL}_2(\mathbf{R})$-equivariant perfect pairing on kth symmetric powers which descends to a perfect pairing

$$T_\Gamma : \mathcal{U}^k_\Gamma \otimes_{\underline{\mathbf{R}}} \mathcal{U}^k_\Gamma \to \underline{\mathbf{R}} \tag{1.7.6.2}$$

on X_Γ. We use this to define a pairing

$$\begin{aligned} (\cdot, \cdot)_\Gamma : \mathrm{H}^1_{\mathrm{c}}(X_\Gamma, \mathcal{U}^k_\Gamma) \otimes_{\mathbf{R}} \mathrm{H}^1(X_\Gamma, \mathcal{U}^k_\Gamma) &\xrightarrow{\cup} \mathrm{H}^2_{\mathrm{c}}(X_\Gamma, \mathcal{U}^k_\Gamma \otimes_{\mathbf{R}} \mathcal{U}^k_\Gamma) \\ &\xrightarrow{T_\Gamma} \mathrm{H}^2_{\mathrm{c}}(X_\Gamma, \underline{\mathbf{R}}) \\ &\xrightarrow{\int_{X_\Gamma}} \underline{\mathbf{R}} \end{aligned}$$

where $\int_{X_\Gamma}$ is computed with respect to the orientation $i\mathrm{d}z \wedge \mathrm{d}\overline{z} = 2\mathrm{d}x\mathrm{d}y$. Since T_Γ is perfect, it follows from "Poincaré duality for local systems" that $(\cdot, \cdot)_\Gamma$ is perfect, but we will not need this yet; we will discuss it in the proof of Lemma 1.7.7.1 below.

For arbitrary $g \in S_{k+2}(\Gamma, \mathbf{C})$, since $\mathrm{Re}(\alpha)$ has image inside of $\widetilde{\mathrm{H}}^1(X_\Gamma, \mathcal{U}^k_\Gamma)$ we can write (non-uniquely) $\mathrm{Re}(\omega_g) = \eta + \mathrm{d}\theta$ for a compactly supported $\mathcal{U}^k_\Gamma$-valued C^∞ 1-form η and a $\mathcal{U}^k_\Gamma$-valued C^∞ function θ. We will prove the theorem by computing the pairing $(\eta, \mathrm{Re}(\omega_f))_\Gamma \in \mathbf{R}$ for arbitrary $f \in S_{k+2}(\Gamma, \mathbf{C})$ (and we will then see what happens when f satisfies the hypothesis of the lemma).

Using the de Rham resolutions as above,

$$(\eta, \mathrm{Re}(\omega_f))_\Gamma = \int_{X_\Gamma} T_\Gamma\big(\eta \wedge \mathrm{Re}(\omega_f)\big).$$

Here an below, we abuse notation and also write T_Γ for the induced map on de Rham resolutions. Letting K run through compacts in X_Γ with the support of η contained in K and with $X_\Gamma - K$ a disjoint union of discs around cusps, we have

$$\begin{aligned}\big(\eta, \mathrm{Re}(\omega_f)\big)_\Gamma &= \lim_K \int_K T_\Gamma\big(\eta \wedge \mathrm{Re}(\omega_f)\big)\\ &= \lim_K \left(\int_K T_\Gamma\big(\mathrm{Re}(\omega_g)\wedge \mathrm{Re}(\omega_f)\big) - \int_K T_\Gamma\big(\mathrm{d}\theta \wedge \mathrm{Re}(\omega_f)\big)\right).\end{aligned}$$

Since $\mathrm{d}(\theta\cdot \mathrm{Re}(\omega_f)) = \mathrm{d}\theta \wedge \mathrm{Re}(\omega_f)$, we have $T_\Gamma(\mathrm{d}\theta\wedge\mathrm{Re}(\omega_f)) = \mathrm{d}\big(T_\Gamma(\theta\,\mathrm{Re}(\omega_f))\big)$. Thus,

$$(1.7.6.3)\quad \big(\eta, \mathrm{Re}(\omega_f)\big)_\Gamma = \lim_K \left(\int_K T_\Gamma\big(\mathrm{Re}(\omega_g)\wedge \mathrm{Re}(\omega_f)\big) - \int_{\partial K} T_\Gamma\big(\theta \cdot \mathrm{Re}(\omega_f)\big)\right).$$

We now consider the first term on the right side of (1.7.6.3). For typographical reasons, we introduce the notation $G = \widetilde{g}$, $F = \widetilde{f} \in \mathrm{H}^0(\mathfrak{h}, \mathcal{O}_\mathfrak{h})$. We compute via the explicit formulas for ω_g and ω_f in (1.7.2.3) that, up to some non-zero universal factors of 2, π and i depending only on k, $T_\Gamma(\mathrm{Re}(\omega_g)\wedge\mathrm{Re}(\omega_f))$ is the descent to X_Γ of the Γ-invariant 2-form

$$\left(\begin{pmatrix} z & \overline{z}\\ 1 & 1\end{pmatrix}^k G\overline{F} - \begin{pmatrix} \overline{z} & z\\ 1 & 1\end{pmatrix}^k \overline{G}F\right)\mathrm{d}z\mathrm{d}\overline{z}$$

on $\mathfrak{h} \simeq X$ (the isomorphism being $\psi_{\binom{1}{0},\binom{0}{1}}$, as usual). Up to a universal non-zero constant, this equals

$$\begin{cases}\mathrm{Re}(G\overline{F}y^k\mathrm{d}x\mathrm{d}y) & k \text{ odd};\\ \mathrm{Im}(G\overline{F}y^k\mathrm{d}x\mathrm{d}y) & k \text{ even}.\end{cases}$$

Thus

$$\lim_K \int_K T_\Gamma\big(\mathrm{Re}(\omega_g)\wedge\mathrm{Re}(\omega_f)\big)$$

exists and is nothing other than a universal non-zero constant multiple (depending on k) of the real (resp. imaginary) part of the classical Petersson product $\langle G, F\rangle_\Gamma$ when k is odd (resp. even). Recall that the convergence of $\langle G, F\rangle_\Gamma$ is a consequence of the exponential decay of cusps forms near cusps, and $\langle F, F\rangle_\Gamma = 0$ if and only if $f = 0$.

Suppose that we knew that the term

$$(1.7.6.4)\qquad \lim_K \int_{\partial K} T_\Gamma\big(\theta\cdot\mathrm{Re}(\omega_f)\big)$$

in (1.7.6.3) always vanishes. By choosing $g = f$ (resp. $g = if$) when k is odd (resp. even), we then see from (1.7.6.3) that the vanishing of $\mathrm{Re}(\omega_f) \in \mathrm{H}^1(X_\Gamma, \mathcal{U}^k_\Gamma)$ forces $\left\langle \widetilde{f}, \widetilde{f}\right\rangle_\Gamma$ to vanish, and this forces f to vanish. Thus, it remains to show that (1.7.6.4) vanishes.

This calculation is very much like the one at the end of the proof of the first part of Theorem 1.7.1.1. In order to study $\lim_K \int_{\partial K} T_\Gamma(\theta\cdot\mathrm{Re}(\omega_f))$, we may choose a cusp and study a small neighborhood of $\Delta = \{q \mid |q| < e^{-2\pi N}\}$ in $\overline{X}_\Gamma$. If $\Delta_r \subseteq \Delta$ is the disk of radius $r \ll 1$ centered at 0, then we want

$$\lim_{r\to 0}\int_{\partial\Delta_r} T_\Gamma\big(\theta\cdot\mathrm{Re}(\omega_f)\big) = 0.$$

Pulling back to $\mathfrak{h}$ and using an element $\gamma \in \mathrm{SL}_2(\mathbf{R})$ which brings our chosen cusp to $i\infty$ under $\psi_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)}$ and $[\cdot]_{\mathfrak{h}}$, we want (after eliminating some non-zero constants)

$$\lim_{b\to\infty}\int_{a+ib}^{a+ib+h}\left\langle \theta(\gamma(z)), \mathrm{Re}((\widetilde{f}|_{k+2}\gamma)\cdot(ze_1+e_2)^k)\right\rangle_k \mathrm{d}z = 0 \tag{1.7.6.5}$$

for fixed $a \in \mathbf{R}$, where

$$\langle\cdot,\cdot\rangle_k : V_k \otimes_{\mathbf{R}} V_k \to \mathbf{R}$$

is the pairing induced by $\langle\cdot,\cdot\rangle$ from (1.7.6.1) on kth symmetric powers and

$$|(\widetilde{f}|_{k+2}\gamma)(z)|$$

decays like $e^{-\mathrm{Im}\, z}$ as $\mathrm{Im}(z) \to \infty$. To obtain (1.7.6.5), we just need the $\mathcal{U}_\Gamma^k$-valued C^∞ function $\theta \circ \gamma$ to be "bounded" on the set $\{x+iy \,|\, a \le x \le a+h, y \ge N\}$.

But due to the compactness of the support of η on X_Γ and the defining property

$$\mathrm{Re}(\omega_g) = \eta + \mathrm{d}\theta,$$

for $y \ge N$ with N sufficiently large (independent of a and h) we have

$$\theta\circ\gamma(x+iy) = \theta\circ\gamma(x+iN) + \int_{x+iN}^{x+iy}\gamma^*(\mathrm{d}\theta) = \theta\circ\gamma(x+iN) + \int_{x+iN}^{x+iy}\gamma^*\,\mathrm{Re}(\omega_g).$$

Appealing again to the calculation following Lemma 1.5.7.2 and the exponential decay of $|(\widetilde{g}|_{k+2}\gamma)|$ near $i\infty$, this integral is bounded independently of y for $a \le x \le a+h$. $\square$

By Lemma 1.7.2.1 this completes the proof of the injectivity of $\mathrm{Re}(\alpha)$.

1.7.7. Computation of cohomology groups. The final step in the proof of Theorem 1.7.1.1 is carried out in the next lemma.

LEMMA 1.7.7.1. $2\dim_{\mathbf{C}} S_{k+2}(\Gamma,\mathbf{C}) = \dim_{\mathbf{R}} \widetilde{\mathrm{H}}^1(X_\Gamma, \mathcal{U}_\Gamma^k)$.

PROOF. Since Theorem 1.7.1.1 for $k=0$ is a consequence of the Hodge decomposition for $\overline{X}_\Gamma$, we may assume that $k>0$. Using the Kodaira-Spencer isomorphism between $\overline{\omega}_\Gamma^{\otimes 2}$ and $\Omega^1_{\overline{X}_\Gamma}(\text{cusps})$, we find that the line bundle $\overline{\omega}_\Gamma$ on the smooth projective curve $\overline{X}_\Gamma$ has degree $g-1+\frac{1}{2}\#\text{cusps}$ where g is the genus of $\overline{X}_\Gamma$. In particular, we note that this implies that $\overline{\omega}_\Gamma$ has degree at least -1 and that there are an even number of cusps.

Since

$$S_{k+2}(\Gamma,\mathbf{C}) = \mathrm{H}^0(\overline{X}_\Gamma, \overline{\omega}_\Gamma^{\otimes k}\otimes\Omega^1_{\overline{X}_\Gamma}),$$

we are most interested in the line bundle $\overline{\omega}_\Gamma^{\otimes k}\otimes\Omega^1_{\overline{X}_\Gamma}$, which has degree

$$\deg\left(\overline{\omega}_\Gamma^{\otimes k}\otimes\Omega^1_{\overline{X}_\Gamma}\right) = k\left(g-1+\tfrac{1}{2}\#\text{cusps}\right)+2g-2. \tag{1.7.7.1}$$

Since we are assuming that $k>0$, the Riemann-Roch theorem and (1.7.7.1) imply that

$$\dim_{\mathbf{C}} S_{k+2}(\Gamma,\mathbf{C}) = (k+1)(g-1)+\tfrac{k}{2}\#\text{cusps} \tag{1.7.7.2}$$

except possibly when $\deg\overline{\omega}_\Gamma$ is -1 or 0.

We claim that these last two possibilities cannot occur. When $\deg\overline{\omega}_\Gamma = -1$, then $g=0$ and there are no cusps, so $X_\Gamma = \overline{X}_\Gamma \simeq \mathbf{CP}^1$. This space is simply connected, yet X_Γ has universal cover $X \simeq \mathfrak{h}$, which is a contradiction. The same

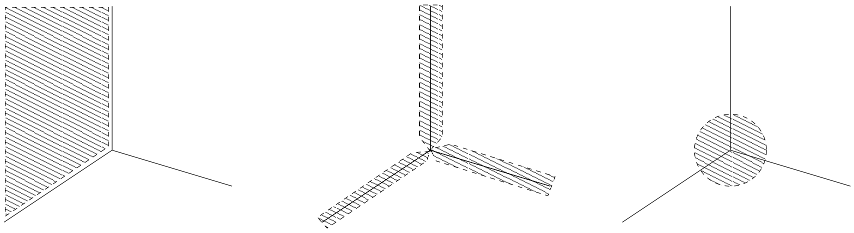


FIGURE 10. A good covering of X_Γ

argument shows that if $\deg \overline{\omega}_\Gamma = 0$ and $g = 0$ then there are exactly 2 cusps, so $\overline{X}_\Gamma \simeq \mathbf{CP}^1$ with $X_\Gamma = \overline{X}_\Gamma - \{0, \infty\}$. Then $\pi_1(X_\Gamma) \simeq \mathbf{Z}$ and therefore the stabilizer of a cusp is a finite index subgroup of Γ. By conjugating Γ in $\mathrm{SL}_2(\mathbf{R})$, it follows that Γ is commensurable with $\left(\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right)^{\mathbf{Z}}$ for some $h > 0$. However, such a subgroup does not have finite covolume in $\mathrm{SL}_2(\mathbf{R})$, giving a contradiction. This leaves the case where $\deg \overline{\omega}_\Gamma = 0$ and $g = 1$. In this case we again have no cusps, so $X_\Gamma = \overline{X}_\Gamma$. But this is impossible since the universal cover of an elliptic curve is $\mathbf{C}$, and $X \simeq \mathfrak{h}$ is not biholomorphic to $\mathbf{C}$.

We conclude that $\deg \overline{\omega}_\Gamma > 0$, so in particular

$$M_{-k}(\Gamma, \mathbf{C}) = \mathrm{H}^0(\overline{X}_\Gamma, \overline{\omega}_\Gamma^{\otimes -k}) = 0$$

for $k > 0$. That is, there are no modular forms for Γ of negative weight. This will be used later.

Using the long exact sequence extending (1.7.4.1) to the right, by (1.7.7.2) it suffices to show

- $\dim_{\mathbf{R}} \mathrm{H}^1(X_\Gamma, \mathcal{U}_\Gamma^k) = (k+1)(2g - 2 + \#\mathrm{cusps})$;
- $\dim_{\mathbf{R}} \mathrm{H}^1(\Delta_x^*, \mathcal{U}_\Gamma^k) = 1$ for each cusp $x \in \overline{X}_\Gamma - X_\Gamma$;
- The map $\mathrm{H}^2_c(X_\Gamma, \mathcal{U}_\Gamma^k) \to \mathrm{H}^2(X_\Gamma, \mathcal{U}_\Gamma^k)$ is injective.

We can triangulate $\overline{X}_\Gamma$ in such a way that all cusps are included among the vertices. Then it is easy to find a *finite* ordered cover $\mathfrak{U} = \{U_i\}$ of X_Γ by opens such that all U_i, $U_i \cap U_j$ and $U_i \cap U_j \cap U_k$ are contractible, while all 4-fold overlaps are empty. Indeed, we can simply take the cover to consist of the interiors of the triangles, small neighborhoods of the "open" edges (i.e., without the vertices) which are pairwise disjoint, and small disks around each non-cuspidal vertex. (See Figure 10.) Note that the first type of neighborhood in Figure 10 is really the *entire* interior of a triangle.

Clearly $\mathfrak{U}$ is an acyclic covering for local systems, so Čech cohomology with respect to $\mathfrak{U}$ may be used to compute to cohomology of *any* local system $\mathcal{F}$ on X_Γ. The Čech complex $\mathrm{C}^\bullet(\mathfrak{U}, \mathcal{F})$ only has terms in degrees 0, 1 and 2. If $\mathcal{F}$ is a local system of d-dimensional $\mathbf{R}$-vector spaces, then each $\mathrm{C}^i(\mathfrak{U}, \mathcal{F})$ is an $\mathbf{R}$-vector space with (finite) dimension depending *only* on d and the combinatorics of the overlaps. In particular, it follows that

$$\chi(\mathcal{F}) = \sum_{i=0}^{2} (-1)^i \dim_{\mathbf{R}} \mathrm{H}^i(X_\Gamma, \mathcal{F}) = \sum_{i=0}^{2} (-1)^i \dim_{\mathbf{R}} \mathrm{C}^i(\mathfrak{U}, \mathcal{F})$$

depends only on d, and not on $\mathcal{F}$. Taking $\mathcal{F} = \mathbf{R}^d$ now shows that the Euler characteristic $\chi(\mathcal{F})$ is nothing other than $d\chi(X_\Gamma)$, where by $\chi(X_\Gamma)$ we mean $\chi(\underline{\mathbf{R}})$.

It follows that

$$\dim_{\mathbf{R}} \mathrm{H}^1(X_\Gamma, \mathcal{U}_\Gamma^k) = \dim_{\mathbf{R}} \mathrm{H}^0(X_\Gamma, \mathcal{U}_\Gamma^k) + \dim_{\mathbf{R}} \mathrm{H}^2(X_\Gamma, \mathcal{U}_\Gamma^k) - (k+1)\chi(X_\Gamma).$$

By the Mayer-Vietoris sequence, removing points from $\overline{X}_\Gamma$ causes Euler characteristics to drop by 1 for each point removed, so $\chi(X_\Gamma) = 2 - 2g - \#\text{cusps}$. Since Δ_x^* has Euler characteristic 0, we can use an argument as above to conclude that $\mathcal{U}_\Gamma^k$ has vanishing Euler characteristic on Δ_x^*. Thus,

$$\dim_{\mathbf{R}} \mathrm{H}^1(\Delta_x^*, \mathcal{U}_\Gamma^k) = \dim_{\mathbf{R}} \mathrm{H}^0(\Delta_x^*, \mathcal{U}_\Gamma^k).$$

From this, we see that it now suffices to prove:

- $\mathrm{H}^0(X_\Gamma, \mathcal{U}_\Gamma^k)$, $\mathrm{H}^2(X_\Gamma, \mathcal{U}_\Gamma^k)$ and $\mathrm{H}^2_{\mathrm{c}}(X_\Gamma, \mathcal{U}_\Gamma^k)$ all vanish.
- $\dim_{\mathbf{R}} \mathrm{H}^0(\Delta_x^*, \mathcal{U}_\Gamma^k) = 1$.

The second of these assertions says that the action of $\left(\begin{smallmatrix} 1 & h \\ 0 & 1 \end{smallmatrix}\right)$ on $\mathbf{R}^2$ has a 1-dimensional invariant subspace for the induced action on $\mathrm{Sym}^k_{\mathbf{R}}(\mathbf{R}^2)$. This is obvious.

Meanwhile, using our Čech covering and a Mayer-Vietoris argument as in the proof of Poincaré duality, one can show that for *any* local system $\mathcal{F}$ of finite-dimensional $\mathbf{R}$-vector spaces on X_Γ, the cup product pairing

$$\mathrm{H}^i(X_\Gamma, \mathcal{F}) \otimes \mathrm{H}^{2-i}_{\mathrm{c}}(X_\Gamma, \mathcal{F}^\vee) \to \mathrm{H}^2_{\mathrm{c}}(X_\Gamma, \underline{\mathbf{R}}) \simeq \mathbf{R}$$

is a perfect pairing for any i. (Over opens in X_Γ where $\mathcal{F}$ is constant, this assertion is just the classical Poincaré duality.) Since $\mathcal{U}_\Gamma^k$ is self-dual via the pairing T_Γ in (1.7.6.2), we conclude that

$$\begin{aligned} \mathrm{H}^2(X_\Gamma, \mathcal{U}_\Gamma^k) &\simeq \mathrm{H}^0_{\mathrm{c}}(X_\Gamma, \mathcal{U}_\Gamma^k) \subseteq \mathrm{H}^0(X_\Gamma, \mathcal{U}_\Gamma^k) \\ \mathrm{H}^2_{\mathrm{c}}(X_\Gamma, \mathcal{U}_\Gamma^k) &\simeq \mathrm{H}^0(X_\Gamma, \mathcal{U}_\Gamma^k). \end{aligned}$$

Thus, we are left with proving that $\mathrm{H}^0(X_\Gamma, \mathcal{U}_\Gamma^k) = 0$ for $k > 0$ (this is of course false if $k = 0$, but the Riemann-Roch application at the beginning of the proof required $k > 0$, so there is no inconsistency). What we need to prove is that the standard Γ-action on $V_k = \mathrm{Sym}^2_{\mathbf{R}} \mathbf{R}^2$ has no invariants if $k > 0$. To see this, choose $v \in V_k^\Gamma$ and use the perfect $\mathrm{SL}_2(\mathbf{R})$-equivariant pairing

$$T : V_k \otimes_{\mathbf{R}} V_k \to \mathbf{R}$$

defined by the kth symmetric power of (1.7.6.1). Following Shimura, define

$$p(z) = T\big(v \otimes (ze_1 + e_2)^k\big)$$

for $z \in \mathbf{C}$, where $e_1 = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, $e_2 = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$. Clearly p is a polynomial, hence p is holomorphic on $\mathfrak{h}$. If $p = 0$, then looking at its coefficients implies that

$$T\big(v \otimes (e_1^i e_2^{k-i})\big) = 0$$

for all $0 \le i \le k$, so by perfectness of T we must have $v = 0$. It therefore suffices to show that $p = 0$.

For this, we will show that $p \in M_{-k}(\Gamma, \mathbf{C})$, which we already saw is $\{0\}$. The transformation formula is easy: for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, we have $v = \gamma v$ so

$$\begin{aligned} p\left(\frac{az+b}{cz+d}\right) &= T\left(\gamma v \otimes \left(\frac{az+b}{cz+d}e_1 + e_2\right)^k\right) \\ &= T\left(v \otimes \left(\frac{az+b}{cz+d}\gamma^{-1}e_1 + \gamma^{-1}e_2\right)^k\right) \\ &= (cz+d)^{-k}T\big(v \otimes (ze_1 + e_2)^k\big), \end{aligned}$$

since $\det \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = 1$. Thus,

$$p(z) = (cz+d)^k p\left(\frac{az+b}{cz+d}\right).$$

To show that $p \in M_{-k}(\Gamma, \mathbf{C})$, it remains to prove a suitable boundedness near the cusps. If there are no cusps, then there is nothing to check. If there is some cusp $\overline{x} \in \overline{X}_\Gamma - X_\Gamma$, then we pick $x \in \mathrm{cusps}_X(\Gamma)$ lifting $\overline{x}$ and as usual choose $g \in \mathrm{SL}_2(\mathbf{R})$ "moving" x to ∞. We must show that

$$(p|_{-k}g)(z) \stackrel{\text{def}}{=} (cz+d)^k p\left(\frac{az+b}{cz+d}\right) \tag{1.7.7.3}$$

is bounded near $i\infty$. But by the above *definition* of p, it is clear that (1.7.7.3) is a polynomial in z, yet it is *invariant* under a translation $z \mapsto z + h$ for some $h > 0$ (as $\mathrm{Stab}_{g^{-1}\Gamma g}(\infty) = g^{-1}\mathrm{Stab}_\Gamma(x)g$ contains non-scalars). Thus, $p|_{-k}g$ must be constant, so it is trivially bounded near $i\infty$. This completes the proof. $\square$

In the special case where Γ is a finite index subgroup of $\mathrm{SL}_2(\mathbf{Z})$, we have $\mathcal{U}_\Gamma^k \simeq \mathrm{R}^1 f_{\Gamma *}\underline{\mathbf{R}}$ for the universal elliptic curve $f_\Gamma : E_\Gamma \to X_\Gamma$ and using the next lemma, Theorem 1.7.1.1 takes the form of an $\mathbf{R}$-linear isomorphism

$$S_{k+2}(\Gamma, \mathbf{C}) \oplus \overline{S_{k+2}(\Gamma, \mathbf{C})} \xrightarrow{\simeq} \widetilde{\mathrm{H}}^1(X_\Gamma, \mathrm{Sym}^k_{\underline{\mathbf{Z}}}\, \mathrm{R}^1 f_{\Gamma *}\underline{\mathbf{Z}}) \otimes_{\mathbf{Z}} \mathbf{C}. \tag{1.7.7.4}$$

LEMMA 1.7.7.2. *If $\mathcal{F}$ is a local system of $\mathbf{Z}$-modules on X_Γ, then for any flat $\mathbf{Z}$-module R, the natural maps*

$$\widetilde{\mathrm{H}}^1(X_\Gamma, \mathcal{F}) \otimes_{\mathbf{Z}} R \to \widetilde{\mathrm{H}}^1(X_\Gamma, \mathcal{F} \otimes_{\underline{\mathbf{Z}}} \underline{R}) \tag{1.7.7.5}$$

$$\mathrm{R}^1 f_{\Gamma *}\underline{\mathbf{Z}} \otimes_{\underline{\mathbf{Z}}} \underline{R} \to \mathrm{R}^1 f_{\Gamma *}\underline{R} \tag{1.7.7.6}$$

are isomorphisms.

PROOF. By proper base change, the fact that (1.7.7.6) is an isomorphism amounts to the map

$$\mathrm{H}^1(E, \underline{\mathbf{Z}}) \otimes_{\mathbf{Z}} R \to \mathrm{H}^1(E, \underline{R}) \tag{1.7.7.7}$$

being an isomorphism for any flat $\mathbf{Z}$-module R (with E a classical elliptic curve). Formation of compactly supported cohomology on a locally compact Hausdorff space commutes with direct limits [**33**, Chapter III, 2.2]. Thus, writing R as the direct limit of finite free $\mathbf{Z}$-modules (for which (1.7.7.7) is obviously an isomorphism) shows that (1.7.7.7) is indeed an isomorphism for any $\mathbf{Z}$-flat R.

By a similar argument over X_Γ, the map (1.7.7.5) is surjective. For injectivity, it suffices to show that

$$\mathrm{H}^1(X_\Gamma, \mathcal{F}) \otimes_{\mathbf{Z}} R \to \mathrm{H}^1(X_\Gamma, \mathcal{F} \otimes_{\mathbf{Z}} \underline{R})$$

is injective. Arguing as in the proof of Lemma 1.7.7.1 via a *finite* Čech covering, this map is an isomorphism since $(\cdot)\otimes_{\mathbf{Z}} R$ commutes with *finite* products and formation of cohomology of complexes of $\mathbf{Z}$-modules. $\square$

Let us now explain how the Eichler-Shimura isomorphism naturally motivates the étale cohomology groups to be considered later. Since the elliptic curve $E_\Gamma \to X_\Gamma$ is algebraic, it makes sense to consider the image $\widetilde{\mathrm{H}}^1_{\text{ét}}(X_\Gamma, \mathrm{Sym}^k_{\mathbf{Z}_l} \mathrm{R}^1 f_{\Gamma *}\mathbf{Z}_l)$ of the étale cohomology group $\mathrm{H}^1_{\mathrm{c},\text{ét}}(X_\Gamma, \mathrm{Sym}^k_{\mathbf{Z}_l} \mathrm{R}^1 f_{\Gamma *}\mathbf{Z}_l)$ in $\mathrm{H}^1_{\text{ét}}(X_\Gamma, \mathrm{Sym}^k_{\mathbf{Z}_l} \mathrm{R}^1 f_{\Gamma *}\mathbf{Z}_l)$. The comparison theorems of étale cohomology yield (using Lemma 1.7.7.2 with $R = \mathbf{Z}_l$) a canonical isomorphism

$$\begin{aligned}\widetilde{\mathrm{H}}^1_{\text{ét}}(X_\Gamma, \mathrm{Sym}^k_{\mathbf{Z}_l} \mathrm{R}^1 f_{\Gamma *}\mathbf{Z}_l) &\simeq \widetilde{\mathrm{H}}^1(X_\Gamma, \mathrm{Sym}^k_{\underline{\mathbf{Z}}_l} \mathrm{R}^1 f_{\Gamma *}\underline{\mathbf{Z}}_l) \\ &\simeq \widetilde{\mathrm{H}}^1(X_\Gamma, \mathrm{Sym}^k_{\underline{\mathbf{Z}}} \mathrm{R}^1 f_{\Gamma *}\underline{\mathbf{Z}}) \otimes_{\mathbf{Z}} \mathbf{Z}_l.\end{aligned}$$

It is for this reason that the isomorphism (1.7.7.4) provides the link between classical modular forms and étale cohomology. The appearance of "two" copies of $S_{k+2}(\Gamma, \mathbf{C})$ in the left side of (1.7.7.4) is ultimately where the 2-dimensionality of our representation spaces will come from. In order to exploit (1.7.7.4) we will need to give a *direct* purely cohomological definition of Hecke operators on $\widetilde{\mathrm{H}}^1$ in a manner compatible with the classical definitions on the left hand side of (1.7.7.4).

When $k = 0$ (i.e., the weight 2 case) we have

$$S_2(\Gamma, \mathbf{C}) \oplus \overline{S_2(\Gamma, \mathbf{C})} \simeq \widetilde{\mathrm{H}}^1(X_\Gamma, \underline{\mathbf{Z}}) \otimes_{\mathbf{Z}} \mathbf{C} = \mathrm{H}^1(\overline{X}_\Gamma, \underline{\mathbf{C}})$$

and $\mathrm{H}^1(\overline{X}_\Gamma, \underline{\mathbf{Z}}_l)$ can be studied using the l-adic Tate module of the Jacobian of $\overline{X}_\Gamma$, *without* requiring the general theory of étale cohomology. This is why the weight 2 picture is much simpler: the subtle local system $\mathrm{Sym}^k_{\underline{\mathbf{Z}}} \mathrm{R}^1 f_{\Gamma *}\underline{\mathbf{Z}}$ is just $\underline{\mathbf{Z}}$ when $k = 0$.

CHAPTER 2

Level Structures and Hecke Operators

Throughout this chapter, Γ denotes a subgroup of $\mathrm{SL}_2(\mathbf{Z})$ of finite index acting freely on $\mathfrak{h}$. We initially make no regularity assumption on the cusps of Γ. We want to understand the universal properties of $E_\Gamma \to X_\Gamma$ and $\overline{E}_\Gamma \to \overline{X}_\Gamma$ for appropriate subgroups Γ. It is important to keep in mind that results should be stated in their natural intrinsic form, valid for an arbitrary choice of $E_\Gamma \to X_\Gamma$ (or $E_X \to X$) and not depending in an essential way on quirky feature of the classical upper half-plane model $\mathcal{E}^{\mathrm{an}} \to \mathfrak{h}$. For example, although one can classically identify the eigenvalues of Hecke operators with q-expansion coefficients at a particular cusp (and this is certainly an important fact in applications), it clears the mind to think in terms of eigenvalues rather than q-expansions.

We will prove that the classical upper half-plane formulas for Hecke operators agree (via the Eichler-Shimura isomorphism) with abstract cohomological definitions of Hecke operators, but one must bear in mind that the *explicit* form of such formulas in the classical context is sometimes an artifact of the particular choice of the classical models as quotients of $\mathfrak{h}$. The cohomological definitions are what will generalize to the context of étale cohomology, paving the way for the connection between Hecke eigenvalues on classical spaces of modular forms and Frobenius eigenvalues in étale cohomology (to which the Weil conjectures apply).

2.1. Moduli spaces for elliptic curves

2.1.1. Universal properties. We begin by studying $E_\Gamma \to X_\Gamma$ without worrying about generalized elliptic curves. By Theorem 1.5.1.1 and Theorem 1.5.2.2, the pair

$$(E_X \to X,\ \alpha_X^\vee : \underline{\mathbf{Z}}^2 \simeq \mathrm{R}^1 f_{X*}(\underline{\mathbf{Z}})^\vee)$$

is a universal object for the moduli problem of isomorphism classes of pairs $(f : E \to S, \alpha^\vee)$ where $f : E \to S$ is an elliptic curve and $\alpha^\vee : \underline{\mathbf{Z}}^2 \simeq \mathrm{R}^1 f_*(\underline{\mathbf{Z}})^\vee$ is dual to an i-permissible trivialization. Furthermore, by Theorems 1.5.1.1 and 1.5.2.2 again, the left $\mathrm{SL}_2(\mathbf{Z})$-action defined on this moduli problem by

$$(f : E \to S,\ \alpha^\vee) \mapsto (f : E \to S,\ \alpha^\vee \circ \gamma^t)$$

for $\gamma \in \mathrm{SL}_2(\mathbf{Z})$ corresponds to the *usual* action of $\mathrm{SL}_2(\mathbf{Z})$ on $\mathfrak{h}$ via the isomorphism $\psi_{\binom{1}{0},\binom{0}{1}} : X \xrightarrow{\simeq} \mathfrak{h}$. Since this is related to the $[\cdot]$ operation through *inversion* on Γ, the quotient of $E_X \to X$ by this action is *still* $f_\Gamma : E_\Gamma \to X_\Gamma$.

The element $\alpha_X^\vee$ is easily seen to induce a well-defined global section

$$\alpha_\Gamma^\vee \in \mathrm{H}^0(X_\Gamma, \Gamma\backslash \underline{\mathrm{Isom}}^+(\underline{\mathbf{Z}}^2, \mathrm{R}^1 f_{\Gamma*}(\underline{\mathbf{Z}})^\vee))$$

where $\underline{\mathrm{Isom}}^+(\underline{\mathbf{Z}}^2, \mathrm{R}^1 f_{\Gamma*}(\underline{\mathbf{Z}})^\vee)$ is the sheaf on X_Γ of "dual to i-permissible" isomorphisms and $\Gamma \subseteq \mathrm{SL}_2(\mathbf{Z})$ acts through the transpose action on $\mathbf{Z}^2$. To see this, let $\Delta_\Gamma \subseteq X_\Gamma$ be a small disc with $\Delta \subseteq X$ one of the discs over Δ_Γ. Then the isomorphism $\underline{\mathbf{Z}}^2|_{\Delta_\Gamma} \simeq \mathrm{R}^1 f_{\Gamma *}(\underline{\mathbf{Z}})^\vee|_{\Delta_\Gamma}$ induced by the cartesian diagram

$$\begin{tikzcd} E_X \arrow[r, "\simeq"] \arrow[d, "f_X"] & E_\Gamma \arrow[d, "f_\Gamma"] \\ \Delta \arrow[r, "\simeq"'] & \Delta_\Gamma \end{tikzcd}$$

is well-defined (independent of the choice of Δ) up to Γ-action, since the Γ-action on X acts transitively on fibers over X_Γ. This translates into the resulting element in $\mathrm{H}^0(\Delta_\Gamma, \underline{\mathrm{Isom}}^+(\underline{\mathbf{Z}}^2, \mathrm{R}^1 f_{\Gamma *}(\underline{\mathbf{Z}})^\vee))$ being well-defined up to Γ-action. It follows that $\alpha_X^\vee$ yields a global section of the quotient sheaf of $\underline{\mathrm{Isom}}^+(\underline{\mathbf{Z}}^2, \mathrm{R}^1 f_{\Gamma *}(\underline{\mathbf{Z}})^\vee)$ by the action of Γ. Note, however, that because of the construction of the quotient sheaf, α_Γ need not correspond to a Γ-orbit of global isomorphisms $\underline{\mathbf{Z}}^2 \simeq \mathrm{R}^1 f_{\Gamma *}(\underline{\mathbf{Z}})^\vee$.

LEMMA 2.1.1.1. *The pair $(f_\Gamma : E_\Gamma \to X_\Gamma,\ \alpha_\Gamma^\vee)$ is universal for isomorphism classes of pairs $(f : E \to S,\ \alpha^\vee)$ with $f : E \to S$ an elliptic curve and $\alpha^\vee$ a global section of the sheaf $\Gamma\backslash\underline{\mathrm{Isom}}^+(\underline{\mathbf{Z}}^2, \mathrm{R}^1 f_*(\underline{\mathbf{Z}})^\vee)$ on S.*

PROOF. We may cover S by opens S_j such that $\alpha^\vee|_{S_j}$ is induced by some map $\alpha_j^\vee : \underline{\mathbf{Z}}^2|_{S_j} \simeq \mathrm{R}^1 f_*(\underline{\mathbf{Z}})^\vee|_{S_j}$. By universality of $(E_X \to X,\ \alpha_X^\vee)$, there is a unique map $S_j \to X$ pulling $(E_X \to X,\ \alpha_X^\vee)$ back to $(E|_{S_j} \to S_j,\ \alpha_j^\vee)$. By *definition*, changing $\alpha_j^\vee$ by the action of Γ amounts to composing $S_j \to X$ with the action of Γ on X. Thus, the map $S_j \to X_\Gamma$ is intrinsic to the pair $(E \to S,\ \alpha^\vee)$. It follows that we may glue to get a well-defined map $S \to X_\Gamma$ which pulls $(E_\Gamma \to X_\Gamma,\ \alpha_\Gamma^\vee)$ back to a pair on S which is *locally* isomorphic to the given $(E \to S,\ \alpha)$. Since $X \to X_\Gamma$ is a *covering space*, it follows that the map $S \to X_\Gamma$ is uniquely determined by this local property.

It remains to show that pairs $(E \to S,\ \alpha^\vee)$ with

$$\alpha^\vee \in \mathrm{H}^0(S, \Gamma\backslash\underline{\mathrm{Isom}}^+(\underline{\mathbf{Z}}^2, \mathrm{R}^1 f_*(\underline{\mathbf{Z}})^\vee))$$

admit no non-trivial automorphisms, so the above local isomorphisms globalize. Since elliptic curves admits no non-trivial infinitesimal automorphisms, it suffices to consider the case of $S = \mathrm{Sp}\,\mathbf{C}$. We need to show that if E is an elliptic curve with $\alpha : \mathrm{H}_1(E, \mathbf{Z}) \simeq \mathbf{Z}^2$ an i-permissible trivialization, and if $\varphi : E \simeq E$ is an automorphism fitting into a commutative diagram

$$\begin{tikzcd} \mathrm{H}^1(E, \underline{\mathbf{Z}}) \arrow[r, "\alpha"] \arrow[d, "\mathrm{H}^1(\varphi)"'] & \mathbf{Z}^2 \arrow[d, "\gamma"] \\ \mathrm{H}^1(E, \underline{\mathbf{Z}}) \arrow[r, "\alpha"] & \mathbf{Z}^2 \end{tikzcd}$$

for some $\gamma \in \Gamma$, then $\varphi = 1$. The existence of such a diagram implies that the point in X corresponding to $(E,\ \alpha)$ is fixed by γ. This implies that $\gamma = 1$, which in turn implies that $\varphi = 1$. $\square$

2.1.2. Full level N-structures. For any elliptic curve $E \to S$, the natural map

$$\mathrm{R}^1 f_* \underline{\mathbf{Z}} \otimes_{\underline{\mathbf{Z}}} \underline{\mathbf{Z}/N} \to \mathrm{R}^1 f_* \underline{\mathbf{Z}/N} \tag{2.1.2.1}$$

is an isomorphism. Indeed, by proper base change we can assume $S = \mathrm{Sp}\,\mathbf{C}$, in which case it is clear. Let $E[N]$ denote the "scheme-theoretic" kernel of the multiplication-by-N map on E; here the fibre product is taken with respect to $N : E \to E$ and the identity section $S \to E$. The map $N : E \to E$ is finite étale of degree N^2 since these properties can be checked on fibers (where they are well known). Thus, we may view $E[N]$ as a local system of rank 2 free $\mathbf{Z}/N$-modules on S. We claim that there is a unique isomorphism

$$\mathrm{R}^1 f_*(\underline{\mathbf{Z}/N})^\vee \simeq E[N] \tag{2.1.2.2}$$

inducing the canonical isomorphism

$$\mathrm{H}_1(E_s, \mathbf{Z}/N) \simeq E_s[N]$$

arising from the uniformization $E_s \simeq \mathrm{H}^0(E_s, \Omega^1_{E_s})^\vee / \mathrm{H}_1(E_s, \mathbf{Z})$ on fibers. Uniqueness is clear, and for existence we note that the classical uniformization was relativized in Theorem 1.3.4.2 to an isomorphism

$$E \simeq f_*(\Omega^1_{E/S})^\vee / \mathrm{R}^1 f_*(\underline{\mathbf{Z}})^\vee,$$

with $\mathrm{R}^1 f_*(\underline{\mathbf{Z}})^\vee \hookrightarrow f_*(\Omega^1_{E/S})^\vee$ inducing a cocompact lattice on fibers. This gives an isomorphism

$$E[N] \simeq \mathrm{R}^1 f_*(\underline{\mathbf{Z}})^\vee \otimes_{\underline{\mathbf{Z}}} \underline{\mathbf{Z}/N} \simeq \mathrm{R}^1 f_*(\underline{\mathbf{Z}/N})^\vee$$

having the desired properties.

Combining these considerations with Lemma 2.1.1.1 leads us to the next theorem. For a classical elliptic curve E, let

$$\langle \cdot, \cdot \rangle_N : E[N] \otimes E[N] \to \mu_N$$

be the Weil pairing (according to the sign conventions in §0.1). Fix $i = \sqrt{-1} \in \mathbf{C}$ and define $\underline{\mathrm{Isom}}^i((\underline{\mathbf{Z}/N})^2, E[N])$ to be the sheaf of sets on S consisting of isomorphisms $\alpha^\vee : (\underline{\mathbf{Z}/N})^2 \simeq E[N]$ which satisfy

$$\langle \alpha^\vee(1,0), \alpha^\vee(0,1) \rangle_N = e^{-\frac{2\pi i}{N}}$$

on fibers. Note the minus sign (this is related to the anti-commutativity condition in the definition of i-permissibility).

Theorem 2.1.2.1. *Fix $N \geq 3$ and let*

$$\Gamma = \Gamma(N) = \ker\bigl(\mathrm{SL}_2(\mathbf{Z}) \to \mathrm{SL}_2(\mathbf{Z}/N)\bigr).$$

Then for any $f : E \to S$, the isomorphism $\mathrm{R}^1 f_*(\underline{\mathbf{Z}/N})^\vee \simeq E[N]$ induces an identification of sheaves

$$\Gamma \backslash \underline{\mathrm{Isom}}^+(\underline{\mathbf{Z}}^2, \mathrm{R}^1 f_*(\underline{\mathbf{Z}})^\vee) \simeq \underline{\mathrm{Isom}}^i(\underline{\mathbf{Z}/N}^2, E[N])$$

compatible with base change on S.

In particular, $(E_\Gamma \to X_\Gamma,\ \alpha^\vee_\Gamma)$ is the universal elliptic curve equipped with a global trivialization of its N-torsion where the basis has Weil pairing $e^{-2\pi i/N}$. This universal object is identified via $\psi_{\binom{1}{0},\binom{0}{1}} : X \xrightarrow{\simeq} \mathfrak{h}$ with $\Gamma\backslash\mathcal{E}^{\mathrm{an}} \to \Gamma\backslash\mathfrak{h}$, having fiber $\mathbf{C}/[1,z]$ over $z \in \Gamma\backslash\mathfrak{h}$ with ordered N-torsion basis $\{z/N, 1/N\}$.

Remark 2.1.2.2. If we flip the order of the basis in our family over $\Gamma(N)\backslash\mathfrak{h}$, we arrive at the "classical" basis $\{1/N, z/N\}$ with the Weil pairing $e^{\frac{2\pi i}{N}}$.

PROOF. Since $N \geq 3$, the freeness of the $\Gamma(N)$-action on $\mathfrak{h}$ is clear. Thus, the preceding theory applies. It is clear from the definition of Γ that

$$\Gamma\backslash\underline{\mathrm{Isom}}(\underline{\mathbf{Z}}^2, \mathrm{R}^1 f_*(\underline{\mathbf{Z}})^\vee) \simeq \underline{\mathrm{Isom}}(\underline{\mathbf{Z}/N}^2, E[N]).$$

The only point we need to check is that if E is a classical elliptic curve and $\alpha^\vee : \mathbf{Z}^2 \simeq \mathrm{H}_1(E, \mathbf{Z})$ is an isomorphism, then the dual α is i-permissible if and only if $\langle\alpha^\vee(1,0), \alpha^\vee(0,1)\rangle_N = e^{-\frac{2\pi i}{N}}$. This property is unaffected by the $\mathrm{SL}_2(\mathbf{Z})$-action on $\alpha^\vee$, so it clearly suffices to choose *some* i-permissible α and to check that $\langle\alpha^\vee(1,0), \alpha^\vee(0,1)\rangle_N = e^{-\frac{2\pi i}{N}}$ for this α. Passing to the universal case and looking in its fibers, we may assume that $E = \mathbf{C}/[1, z]$ with basis $\{z, 1\}$ for $\mathrm{H}_1(E, \mathbf{Z})$. We need $\langle 1/N, z/N\rangle_N = e^{\frac{2\pi i}{N}}$, but we noted in §0.1 that this holds for our Weil pairing sign convention.

$\square$

By descent, the left $\mathrm{SL}_2(\mathbf{Z})$-action on $E_X \to X$ from the universality of $(E_X \to X,\ \alpha_X : \mathrm{R}^1 f_X(\underline{\mathbf{Z}})^\vee \simeq \underline{\mathbf{Z}}^2)$ induces a left $\mathrm{SL}_2(\mathbf{Z})/\Gamma(N) \simeq \mathrm{SL}_2(\mathbf{Z}/N)$-action on $E_{\Gamma(N)} \to X_{\Gamma(N)}$. This is *exactly* the action induced by the canonical left 'transpose' action of $\mathrm{SL}_2(\mathbf{Z}/N)$ on "full level N structures" $(\underline{\mathbf{Z}/N})^2 \simeq E[N]$. This action is transitive on cusps of $\overline{X}_{\Gamma(N)}$, so all cusps have ramification degree exactly N (as this is clear at ∞_X after using $\psi_{\binom{1}{0},\binom{0}{1}} : X \simeq \mathfrak{h}$, though it can also be seen more conceptually once we identify this compactification with a moduli space for generalized elliptic curves over analytic spaces).

For any elliptic curve $f : E \to S$, we define the relative Weil pairing

$$\langle\cdot,\cdot\rangle_N : E[N] \times E[N] \simeq \mathrm{R}^1 f_*(\underline{\mathbf{Z}/N})^\vee \times \mathrm{R}^1 f_*(\underline{\mathbf{Z}/N})^\vee \to \underline{\mathbf{Z}/N}$$

to be induced by applying $\underline{\mathbf{Z}/N} \otimes_{\underline{\mathbf{Z}}} (\cdot)$ to the perfect pairing

$$\mathrm{R}^1 f_*\underline{\mathbf{Z}} \otimes \mathrm{R}^1 f_*\underline{\mathbf{Z}} \xrightarrow{\cup} \mathrm{R}^2 f_*\underline{\mathbf{Z}} \simeq \underline{\mathbf{Z}}(-1) \xrightarrow{2\pi i} \underline{\mathbf{Z}}.$$

This is compatible with base change and on fibers induces exactly the Weil pairing as previously defined. Thus, $X_{\Gamma(N)}$ is the base of the universal family of triples $(E \to S; Q_1, Q_2)$ with $E \to S$ an elliptic curve and $Q_1, Q_2 \in E[N](S)$ satisfying $\langle Q_1, Q_2\rangle_N = e^{-\frac{2\pi i}{N}}$. For a primitive Nth root of unity $\zeta_N \in \mu_N(\mathbf{C})$, we define $Y(N)^{\zeta_N-\mathrm{can}}$ to be the base of a universal family for the moduli problem of triples $(E \to S; P, P')$ with $\langle P, P'\rangle_N = \zeta_N \in \mu_N(\mathbf{C})$. For example, if $\zeta_N = e^{2\pi i c/N}$ with $c \in (\mathbf{Z}/N)^\times$, then $(E_{\Gamma(N)} \to X_{\Gamma(N)}; Q_1, -cQ_2)$ is such a family, as is $(E_{\Gamma(N)} \to X_{\Gamma(N)}; cQ_2, Q_1)$. The resulting map

$$\mathfrak{h} \simeq X \twoheadrightarrow X_{\Gamma(N)} = Y(N)^{\zeta_N-\mathrm{can}}$$

pulls back the universal family on $Y(N)^{\zeta_N-\mathrm{can}}$ to $\mathcal{E}^{\mathrm{an}} \to \mathfrak{h}$ with N-torsion basis $\{-z/N, c/N\}$ over $z \in \mathfrak{h}$ in the first example, and $\{c/N, z/N\}$ in the second example.

2.1.3. $\Gamma_1(N)$-structures. The action of $\mathrm{SL}_2(\mathbf{Z})$ on the above moduli problem is through the transpose action of $\mathrm{SL}_2(\mathbf{Z}/N)$ on isomorphisms $\mathbf{Z}/N \times \mathbf{Z}/N \simeq E[N]$; it translates into the action

$$\big(E; (P_1, P_2)\big) \mapsto \big(E, (P_1, P_2)g^t\big)$$

since $(P_1, P_2)g^t = (\cdot, P_2)$ if and only if $g \in \Gamma_1(N)$. and $\Gamma_1(N)$ acts freely on $\mathfrak{h}$ when $N \geq 4$. By Lemma 2.1.1.1 and Theorem 2.1.2.1, if we define

$$Y_1(N) \stackrel{\text{def}}{=} \Gamma_1(N) \backslash X$$

for $N \geq 4$ then we get a universal object $(E_1(N) \to Y_1(N), \mathcal{P})$ for the moduli problem of structures $(E \to S; \mathcal{P})$ where $\mathcal{P} : \underline{\mathbf{Z}/N} \hookrightarrow E[N]$ is a "point of exact order N". Moreover, this universal analytic family over $Y_1(N)$ has pullback $\mathcal{E}^{\text{an}} \to \mathfrak{h}$ under

$$\mathfrak{h} \xleftarrow{\psi_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)}} X \to \Gamma_1(N) \backslash X$$

and so has fiber

$$\left(\mathbf{C}/[1, z]; \tfrac{1}{N}\right) \simeq \left(\mathbf{C}^\times / q^{\mathbf{Z}}, \zeta_N\right)$$

over $z \in \mathfrak{h}$; as always, $q = e^{2\pi i z}$. Note that the cusp ∞ on $Y_1(N)$ has ramification degree 1, while the cusp 0 has ramification degree N. This is "why" this standardization of analytic models is better suited for arithmetic considerations with points as we've labelled them. For $N \geq 4$, the rule $(E; (P_1, P_2)) \rightsquigarrow (E; P_2)$ from $e^{-2\pi i/N}$-canonical full level N structures to $\Gamma_1(N)$-structures corresponds to the natural projection $\Gamma(N) \backslash X \to \Gamma_1(N) \backslash X$ in terms of the above models. Thus, by comparison with $\overline{X}_{\Gamma(N)}$ we see that all cusps on $X_1(N) \stackrel{\text{def}}{=} \overline{X}_{\Gamma_1(N)}$ have ramification degree dividing N.

2.2. Moduli spaces for generalized elliptic curves

2.2.1. $\Gamma_1(N)$-structures on generalized elliptic curves. It is classical that $\Gamma(N)$ has regular cusps for $N \geq 3$ and that $\Gamma_1(N)$ has regular cusps for $N \geq 5$. We will be exclusively interested in the case of $\Gamma_1(N)$ later on, so we consider only this case here.

DEFINITION 2.2.1.1. A $\Gamma_1(N)$-*structure* on a generalized elliptic curve $E \to S$ is a closed immersion $\underline{\mathbf{Z}/N} \hookrightarrow E^{\text{sm}}[N]$ of group objects which meets all irreducible components of all fibers. If $P \in E^{\text{sm}}(S)$ is the image of 1, we also denote this data by (E, P) or $(E, \langle P \rangle)$.

On a generalized elliptic curve with a $\Gamma_1(N)$-structure, the closed immersion must meet all irreducible components of the fibers. Thus, it is clear that these fibers must all be either smooth or else d-gon singular fibers for various d dividing N.

LEMMA 2.2.1.2. *$\Gamma_1(N)$-structures on generalized elliptic curves do not admit non-trivial automorphisms for $N \geq 5$.*

PROOF. It suffices to work on infinitesimal fibers. By GAGA and the algebraicity of 1-dimensional proper analytic spaces, we may use the algebraic result [**14**, Ch I, 1.10] to see that (analytic) generalized elliptic curves do not admit non-trivial infinitesimal automorphisms. Thus, it suffices to work over $\operatorname{Sp} \mathbf{C}$. For classical elliptic curves the lemma is classical, even with $N \geq 4$. For the case of generalized elliptic curves, we must consider a standard d-gon E (for d dividing N) and a $P \in E^{\text{sm}}[N]$ whose image in the order d cyclic component group $\pi_0(E^{\text{sm}})$ is a generator and with $dP \in (E^{\text{sm}})^0 \simeq \mathbf{G}_m$ having exact order $\frac{N}{d}$. This is exactly the meaning of a $\Gamma_1(N)$-structure on a standard d-gon. We will show that non-trivial elements in $\operatorname{Aut}(E)$ do not fix P if $N > 2$ and $N \neq 4$. By [**14**, Chapter I, 1.10] or a direct analysis, we see that $\mathrm{Aut}(E) = \mathbf{Z}/2 \ltimes \mu_d(\mathbf{C})$ where $\mathbf{Z}/2$ is generated by the "inversion" sending (x, j) to $(x^{-1}, -j)$ and $\zeta \in \mu_d(\mathbf{C})$ acts by $(x, j) \mapsto (x \cdot \zeta^j, j)$; here (x, j) denotes the point on the jth $\mathbf{CP}^1$ with coordinate x. (These actions are visibly compatible with gluing along 0 and ∞.)

We have $P = (x, j)$ for some $j \in (\mathbf{Z}/d)^\times$ and $x \in \mu_N$ with exact order N. We must consider the question of whether or not an element of $\mathrm{Aut}(E)$ fixes this (x, j). If $(x, j) = (x \cdot \zeta^j, j)$, then $\zeta^j = 1$, yet $j \in (\mathbf{Z}/d)^\times$; thus $\zeta = 1$. If on the other hand $(x, j) = (x^{-1}\zeta^{-j}, -j)$, then $2j \equiv 0 \pmod{d}$, so that d divides 2 and $x^2 = \zeta^{-j}$. If $d = 1$, then $\zeta = 1$ and therefore $x^2 = \pm 1$, which contradicts the assumption that $N > 2$. Thus $d = 2$ and N is even, yet $x^2 = \zeta^{-j} = \pm 1$ and x^2 is a primitive $(N/2)$th root of unity. Thus, $N > 2$ and we arrive at the two possibilities $N = 4$, $d = 2$, $j = 1$, $x^2 = 1$. This is ruled out by the condition $N \neq 4$. $\square$

Now fix $N \geq 5$ and let $X_1(N) = \overline{X}_{\Gamma_1(N)}$, so over $X_1(N)$ there is a unique generalized elliptic curve $\overline{E}_1(N) \to X_1(N)$ extending the universal elliptic curve $E_1(N) \to Y_1(N)$, subject to the conditions of Corollary 1.6.5.2, and this data is uniquely algebraic. It is *not* yet clear that the "point of order N" on $E_1(N)$ is algebraic. We must prove

LEMMA 2.2.1.3. *The $\Gamma_1(N)$-structure on $E_1(N) \to Y_1(N)$ uniquely extends to a section of $\overline{E}_1(N) \to X_1(N)$ and as such defines a $\Gamma_1(N)$-structure on $\overline{E}_1(N) \to X_1(N)$.*

REMARK 2.2.1.4. By GAGA, this structure over $X_1(N)$ is "algebraic", hence so is the original universal analytic $\Gamma_1(N)$-structure over $Y_1(N)$.

PROOF. The uniqueness is clear and existence is therefore a local question near each cusp. Let $\overline{x} \in X_1(N)$ be a cusp with ramification degree d (which divides N). Choose an oriented basis $\{e_1, e_2\}$ of $\mathbf{Z}^2$ for which the isomorphism $\psi_{e_1,e_2} : X \xrightarrow{\simeq} \mathfrak{h}$ sends a representative $x \in \overline{\pi}_\Gamma^{-1}(\overline{x})$ to ∞. We use this isomorphism to identify a punctured neighborhood of $\overline{x} \in X_1(N)$ with

$$\{z \in \mathfrak{h} \mid \mathrm{Im}(z) \gg 0\}/(z \sim z + d),$$

over which $E_1(N)$ is identified with $\underline{\mathrm{Tate}}_d^* \to \Delta^*$ and the fibers of $(E_1(N); P)$ must have the form $(\mathbf{C}/[1, z]; \frac{a+bz}{N})$ where a, b are independent of z and $\gcd(a, b)$ is relatively prime to N.

Clearly d is the smallest positive integer such that

$$\left(\mathbf{C}/[1, z]; \tfrac{a+bz}{N}\right) \simeq \left(\mathbf{C}/[1, z + d]; \tfrac{a+b(d+z)}{N}\right)$$

whenever $\mathrm{Im}(z) \gg 0$ (for our fixed a, b). The existence of such isomorphisms says exactly that d is the smallest positive integer such that $a + bd \equiv a \pmod{N}$, or equivalently that $bd \equiv 0 \pmod{N}$. Since d divides N, the condition that d is the exact ramification degree of $\overline{x}$ says that $b = b' \cdot \frac{N}{d}$ with $\gcd(b', d) = 1$. Thus, with respect to the exponential isomorphism

$$\left(\mathbf{C}/[1, z], \tfrac{a+bz}{N}\right) \simeq \left(\mathbf{C}^\times/q_d^{d\mathbf{Z}}, \zeta_N^a \cdot q_d^{b'}\right),$$

where $q_d = e^{\frac{2\pi i z}{d}}$ is a uniformizer near $\overline{x}$, we see that $\underline{\mathrm{Tate}}_d \to \Delta$ and the section $\zeta_N^a \cdot q^{b'}$ over Δ (cf. Theorem 1.6.3.1) yields the desired extension of the $\Gamma_1(N)$-structure around $\overline{x}$ (since $b' \in (\mathbf{Z}/d)^\times$). $\square$

2.2.2. Moduli-theoretic interpretation of the cusps. The interpretation of the cusps in terms of moduli is given by the next theorem.

THEOREM 2.2.2.1. *For $N \geq 5$, the $\Gamma_1(N)$-structure on $\overline{E}_1(N) \to X_1(N)$ is universal in the category of analytic spaces. The associated algebraic family is universal in the category of* $\mathbf{C}$*-schemes.*

The definitions of (generalized) elliptic curves and $\Gamma_1(N)$-structures carry over verbatim to the setting of $\mathbf{Z}[1/N]$-schemes, provided we replace 'fiber' by 'geometric fiber'. The theory of such objects is developed in [**14**]. This theory makes essential use of algebraic stacks to construct moduli spaces which *a fortiori* are shown to be schemes in favorable situations. We will exploit this algebraic theory in order to settle problems on the analytic side.

Before proving Theorem 2.2.2.1, we will need some lemmas. The basic idea is to prove that $\Gamma_1(N)$-structures are "locally algebraic" so that we can use the pullback from the already established algebraic theory, using artinian bases (which are both analytic *and* algebraic) to ensure the uniqueness results needed for globalization.

LEMMA 2.2.2.2. *Let $E \to S$ be a generalized elliptic curve and let $s_0 \in S$ be such that E_{s_0} is a d-gon. Then each fiber of $E \to S$ near s_0 is either smooth or a d-gon.*

PROOF. Consider the map $d : E^{\mathrm{sm}} \to E^{\mathrm{sm}}$. This is quasi-finite and étale, as these properties can both be checked on fibers. Thus, $E^{\mathrm{sm}}[d] \to S$ is separated, quasi-finite, and étale with fibers obviously of order at most d^2 (by Corollary 1.6.2.5). Since étale maps are local isomorphisms in the analytic setting, it is clear that the locus U of $s \in S$ with $\#E_s^{\mathrm{sm}}[d] = d^2$ is open. Upon replacing S by U we can without loss of generality assume that $\#E_s^{\mathrm{sm}}[d] = d^2$ for all $s \in S$. Thus, all singular fibers of $E \to S$ must be nd-gons for various n, and $E^{\mathrm{sm}}[d] \to S$ is visibly *finite* étale of degree d^2.

Let $u : E \to \overline{E}$ be the contraction (in the sense of Lemma 1.6.4.5) of all components *not* meeting the Cartier divisor $E^{\mathrm{sm}}[d]$. Thus, $\overline{E}$ has all fibers either smooth or d-gons. We want to prove the openness of the set of $s \in S$ for which the proper map $u_s : E_s \to \overline{E}_s$ is an isomorphism. Since E and $\overline{E}$ are S-flat, this forces u to be *flat* at all points of E_s. Thus, by the geometric nature of contractions we see that u_s is an isomorphism if and only if u is étale at all points of E_s, or merely is quasi-finite at all points of E_s. But if $\Omega^1_{E/\overline{E}}$ vanishes at a point of E then u is automatically quasi-finite at such a point, so if $\Omega^1_{E/\overline{E}}$ vanishes along E_s then u_s is an isomorphism. The support of the coherent sheaf $\Omega^1_{E/\overline{E}}$ an analytic set $Z \subseteq E$. The image Z' of Z in S is an analytic set since $E \to S$ is proper, so $S - Z'$ gives the desired neighborhood of s_0. □

PROOF OF THEOREM 2.2.2.1. In [**14**, Chapter IV, Section 2], for $N \geq 3$ a universal moduli scheme, projective and smooth of pure relative dimension 1 over $\mathbf{Z}[1/N]$, is constructed for generalized elliptic curves $E \to S$ (over variable $\mathbf{Z}[1/N]$-schemes S) equipped with isomorphisms $(\mathbf{Z}/N)^2 \simeq E^{\mathrm{sm}}[N]$ meeting all irreducible components of all geometric fibers. (In particular all of these fibers are N-gons or are smooth.) Moreover, the locus of singular fibers of this family is a Cartier divisor in the base. The same technique applies in the case of $\Gamma_1(N)$-structures when $N \geq 5$ (thanks to the algebraic analogue of Lemma 2.2.1.2). Thus, by base change there is a universal algebraic family of $\Gamma_1(N)$-structures $E_1(N)_{DR} \to X_1(N)_{DR}$ over

$\operatorname{Spec}\mathbf{C}$, with $X_1(N)_{DR} \to \operatorname{Spec}\mathbf{C}$ projective smooth with pure dimension 1, and the locus of singular fibers of $E_1(N)_{DR} \to X_1(N)_{DR}$ is a Cartier divisor (here, 'DR' is shorthand for 'Deligne-Rappaport').

Since our $\overline{E}_1(N) \to X_1(N)$ with its $\Gamma_1(N)$ structure is *algebraic*, there exists a canonical map $X_1(N) \to X_1(N)_{DR}$ of smooth projective algebraic curves over $\operatorname{Spec}\mathbf{C}$ which we need to prove is an isomorphism. (We do *not* need to know that $X_1(N)_{DR}$ is a priori connected.) For this, it suffices to check bijectivity outside of a finite set of points. Since the support of a Cartier divisor in $X_1(N)_{DR}$ is a *finite* set, it suffices to study the map $Y_1(N) \to Y_1(N)_{DR}$ between the loci of smooth fibers. But this is bijective on closed points by GAGA and the fact that $E_1(N) \to Y_1(N)$ is *universal* in the category of analytic spaces (or alternately because its fibers are known explicitly). This establishes that the "algebraic" model of $\overline{E}_1(N) \to X_1(N)$ is universal in the category of $\mathbf{C}$-schemes. In particular, any $\Gamma_1(N)$-structure on $E \to S$ with an artinian analytic base S is the pullback by a *unique* map $S \to X_1(N)$, since such S are always algebraic. This "infinitesimal" uniqueness reduces the universality question for $\overline{E}_1(N) \to X_1(N)$ in the analytic world to a local question, since we can always reduce questions of uniqueness to infinitesimal fibers.

By Lemma 2.2.2.2, it suffices to study a $\Gamma_1(N)$-structure on an $f : E \to S$ whose singular fibers are d-gons for a fixed divisor d of N. We want to realize this as a pullback of $\overline{E}_1(N) \to X_1(N)$, since this has a universal property in the algebraic world. The idea is to use the analytic theory of Proj in §1.6.4 to locally realize our analytic $\Gamma_1(N)$-structure as a base change of the analytification of an algebraic $\Gamma_1(N)$-structure. Since $E^{\mathrm{sm}}[d] \to S$ is finite étale of degree d^2, $D = E^{\mathrm{sm}}[d]$ is a Cartier divisor in E and $\mathcal{L} = \mathcal{O}(D)^{\otimes 3}$ is ample on fibers. In fact,

- $f_*\mathcal{L}$ is locally free of rank $3d^2$, of formation compatible with arbitrary base change;
- $f^*f_*\mathcal{L} \to \mathcal{L}$ is surjective;
- $\oplus_{n\geq 0} f_*\mathcal{L}^{\otimes n}$ is generated by its degree 1 term over $f_*\mathcal{O}_E = \mathcal{O}_S$ in *universal* sufficiently large degrees (i.e., independent of S).

To prove these facts, note that by the analytic theory of cohomology and base change and the algebraic theory of Grothendieck duality on fibers to see that for all $n \geq 1$, the divisor $3nD$ is a very ample divisor on fibers with higher cohomology vanishing and global sections of dimension $3d^2n$. This gives the first two properties above.

By base change, the third property in universal sufficiently high degrees can be checked on fibers, where it follows from quasi-compactness of the Hilbert scheme $\underline{\mathrm{Hilb}}^{3d^2t}_{\mathbf{P}^{3d^2-1}_{\mathbf{C}}}$ (or more generally, the quasi-compactness of Hilbert schemes $\underline{\mathrm{Hilb}}^{Q}_{\mathbf{P}^r_{\mathbf{Z}}}$ for all $r \geq 1$ and $Q \in \mathbf{Q}[t]$). Indeed, if H is any noetherian scheme, $f : Z \to H$ is any proper flat map, and $\mathcal{L}$ is an H-ample invertible sheaf on Z whose restriction to fibers over H is very ample with vanishing higher cohomology, then we claim that

$$\bigoplus_{n\geq 0} f_*\mathcal{L}^{\otimes n} \tag{2.2.2.1}$$

is generated in sufficiently high degrees by the degree 1 term over $\mathcal{O}_H$. To see this, note that by cohomology and base change, the sheaf $f_*\mathcal{L}$ is locally free of formation compatible with base change. By the very ampleness condition, it therefore follows that the canonical *proper* H-map

$$Z \to \mathbf{P}(f_*\mathcal{L})$$

is a closed immersion on fibers over H and therefore is a closed immersion. Thus, by [**27**, 4.4.4] we conclude that $\mathcal{L}$ is very ample relative to f, so the noetherian condition and [**27**, 2.3.4.1] imply the third property above.

It now follows from Lemma 1.6.4.3 that we can construct a diagram

$$\begin{tikzcd} E \arrow[r, "\simeq"] \arrow[dr] & \operatorname{Proj} \bigoplus_{n\geq 0} f_*\mathcal{L}^{\otimes n} \arrow[r, hook] \arrow[d] & \mathbf{P}(f_*\mathcal{L}) \\ & S & \end{tikzcd}$$

where $\mathbf{P}(\mathcal{E}) = \operatorname{Proj}(\operatorname{Sym}(\mathcal{E}))$ for a locally free coherent sheaf $\mathcal{E}$ on S and the closed immersion is induced by the canonical map (which is surjective in high degrees)

$$\operatorname{Sym}(f_*\mathcal{L}) \twoheadrightarrow \bigoplus_{n\geq 0} f_*\mathcal{L}^{\otimes n}.$$

Working locally on S we may assume that $f_*\mathcal{L} \simeq \mathcal{O}_S^{\oplus 3d^2}$, so we have a closed S-immersion $E \hookrightarrow \mathbf{CP}^{3d^2-1} \times S$ with Hilbert polynomial $3d^2t$ on fibers over S. By Lemma 2.2.3.1 below, for any $n \geq 1$ and any $Q \in \mathbf{Q}[t]$, the analytification of the finite-type $\mathbf{C}$-scheme $\underline{\mathrm{Hilb}}^Q_{\mathbf{P}^n_{\mathbf{C}}/\mathbf{C}}$ and the universal closed subscheme of $\mathbf{P}^n_{\mathbf{C}} \times \underline{\mathrm{Hilb}}^Q_{\mathbf{P}^n_{\mathbf{C}}/\mathbf{C}}$ are universal for *analytic* flat families $Z \hookrightarrow \mathbf{CP}^n$ with Hilbert polynomial Q on fibers. Since $\underline{\mathrm{Hom}}$ and $\underline{\mathrm{Isom}}$ schemes are constructed in terms of Quot schemes in a completely formal manner which carries over to the analytic setting (see Theorem 2.2.3.2), we conclude that $E \hookrightarrow \mathbf{CP}^{3d^2-1} \times S$ with its structure of $\underline{\mathbf{Z}/N} \hookrightarrow E$ and "translation by $1 \in \mathbf{Z}/N$" map $\Phi : E \xrightarrow{\simeq} E$ satisfying $\Phi^N = 1$ is the "base change" from an algebraic situation over a locally finite-type $\mathbf{C}$-scheme T, or more precisely is the pullback of a T-flat closed subscheme $C \hookrightarrow \mathbf{P}^{3d^2-1}_{\mathbf{C}} \times_{\mathbf{C}} T$ equipped with a closed T-immersion $\underline{\mathbf{Z}/N} \hookrightarrow C$ and a T-automorphism $\Phi : C \xrightarrow{\simeq} C$ with $\Phi^N = 1$.

By [**14**, Ch I, §1.5] and [**26**, 4.1.7], the combined conditions that $C \to T$ be a *DM semistable* curve of genus 1 and $\underline{\mathbf{Z}/N} \hookrightarrow C$ factor through C^{sm} as an ample relative Cartier divisor is is represented by an open subscheme of T. Of course, $S \to T$ must factor through this open subscheme. By the *proof* of [**35**, Ch I, Cor 1.3.5], the condition "$\Phi(0) = 1$" is represented by a closed subscheme of T and this is universal in the analytic world as well. We may therefore suppose that $C \to T$ is a DM semistable genus 1 curve, $\underline{\mathbf{Z}/N} \hookrightarrow C$ is an ample relative Cartier divisor, and $\Phi(0) = 1$.

Now by [**14**, Ch II, §3], there exists a unique structure of generalized elliptic curve on $C \to T$ with identity section 0 such that the $\mathbf{Z}/N$-action of Φ is translation by $1 \in \underline{\mathbf{Z}/N} \hookrightarrow C^{\mathrm{sm}}$. The ampleness forces $\underline{\mathbf{Z}/N} \hookrightarrow C$ to be a $\Gamma_1(N)$-structure. Moreover, by the *uniqueness* of such structures on infinitesimal fibers over T as well as GAGA over artinian bases, we conclude that the isomorphism $E \simeq C^{\mathrm{an}} \times_{T^{\mathrm{an}}} S$ *must* respect the generalized elliptic curve structure on E. Thus $E \to S$ with its $\Gamma_1(N)$-structure is obtained from the universal algebraic object $\overline{E}_1(N) \to X_1(N)$ by composing $S \to T^{\mathrm{an}}$ with the analytification of some $T \to X_1(N)$. As we noted earlier, infinitesimal fibers ensure the uniqueness of $S \to X_1(N)$, so we finally have analytic universality. $\square$

2.2.3. Analytic Hilbert and Quot spaces. A key fact used in the above proof (as well as in the proof of Lemma 1.6.4.3(4)) is the following.

LEMMA 2.2.3.1. *Fix $n \geq 1$ and $Q \in \mathbf{Q}[t]$. Let $H = \underline{\mathrm{Hilb}}^{Q}_{\mathbf{P}^n_{\mathbf{C}}/\mathbf{C}}$ be the Hilbert scheme with $Z \hookrightarrow H \times_{\mathbf{C}} \mathbf{P}^n_{\mathbf{C}}$ the universal closed subscheme, flat of finite presentation over H, with Hilbert polynomial Q on fibers (Grothendieck proved H exists and is projective over $\mathbf{C}$). Then $Z^{\mathrm{an}} \hookrightarrow H^{\mathrm{an}} \times \mathbf{CP}^n$ has the analogous universal property in the category of analytic spaces.*

Rather than prove this result directly, it is more convenient (and in fact necessary, in view of arguments with $\underline{\mathcal{H}om}$-functors in the preceding section) to prove a more general result. Let $f : X \to S$ be a proper morphism and define the Hilbert functor $\underline{\mathrm{Hilb}}_{X/S}$ to classify closed immersions $Z \hookrightarrow X \times_S T$ which are T-flat, for variable analytic spaces T over S. If $\mathcal{L}$ is an invertible sheaf on X which is ample on the fibers, then the Hilbert polynomials of the fibers of $Z \to T$ with respect to the T-ample $\mathcal{L}|_Z$ naturally break up $\underline{\mathrm{Hilb}}_{X/S}$ into a disjoint union of Zariski clopen subfunctors $\underline{\mathrm{Hilb}}^{Q}_{X/S}$ for $Q \in \mathbf{Q}[t]$ (by the first part of Lemma 1.6.4.4, whose proof didn't make use of Hilbert functors). Thus, representability of $\underline{\mathrm{Hilb}}_{X/S}$ is equivalent to that of $\underline{\mathrm{Hilb}}^{Q}_{X/S}$ for all $Q \in \mathbf{Q}[t]$. Even more generally, for a coherent sheaf $\mathcal{E}$ on X, one can define the Quot-functor $\underline{\mathrm{Quot}}_{\mathcal{E}/X/S}$ which classifies T-flat coherent quotients of $\mathcal{E}_T$ on X_T, for variable analytic spaces $T \to S$. If we fix an S-ample invertible sheaf $\mathcal{L}$ on X, then this Quot-functor naturally breaks up into a disjoint union of Zariski clopen subfunctors $\underline{\mathrm{Quot}}^{Q}_{\mathcal{E}/X/S}$. The representability of these functors in the complex analytic category was proven by Grothendieck in [**21**], based on earlier complex analytic results of Grauert-Remmert. Here is Grothendieck's theorem:

THEOREM 2.2.3.2. *Let $f : X \to S$ be proper and assume there is an invertible sheaf $\mathcal{L}$ on X which is ample on the fibers. Fix $Q \in \mathbf{Q}[t]$ and a coherent sheaf $\mathcal{E}$ on X. Then $\underline{\mathrm{Quot}}^{Q}_{\mathcal{E}/X/S}$ is represented by a proper analytic space over S which is "projective" locally on S and is of formation compatible with analytification.*

In particular, $\underline{\mathrm{Hilb}}_{X/S}$ exists as a separated analytic space over S and is compatible with analytification. The same holds for $\underline{\mathrm{Hom}}(X, Y)$ and $\underline{\mathrm{Isom}}(X, Y)$ for proper X and Y which admit invertible sheaves that are ample on fibers.

We note in passing that it is possible to use the theory of formal schemes to give a rather different proof of the analytic input of Grauert-Remmert which is used in the proof of Theorem 2.2.3.2. This alternate approach seems necessary if one wants to prove Theorem 2.2.3.2 in the context of rigid analytic spaces, as is done in [**6**]; the the methods in [**6**, §3.4–3.5] are written to also apply verbatim in the complex analytic case.

2.2.4. Hecke operators and diamond operators. With a satisfactory analytic theory of cusps on $X_1(N)$ now constructed, we can make the following intrinsic definition of the cusp at ∞. We remind the reader that $N \geq 5$.

DEFINITION 2.2.4.1. The *cusp* ∞ on $X_1(N)$ corresponds to the $\Gamma_1(N)$-structure given by $\zeta_N = e^{\frac{2\pi i}{N}}$ on the 1-gon $(\underline{\mathrm{Tate}}_1)_0$.

Consider triples (E, P, ω) consisting of a generalized elliptic curve $E \to S$, a $\Gamma_1(N)$-structure $P : \underline{\mathbf{Z}/N} \hookrightarrow E$, and a generator ω of $\varepsilon^*\Omega^1_{E^{\mathrm{sm}}/S}$. Due to the existence of a universal analytic family over $X_1(N)$ and Theorem 1.6.5.3, we can (following Katz) view a modular form

$$f \in M_k(\Gamma_1(N), \mathbf{C}) = \mathrm{H}^0(X_1(N), \overline{\omega}^{\otimes k})$$

as a "rule" *compatible with base change* that assigns to any such triple an element $f(E, P, \omega) \in \mathrm{H}^0(S, \mathcal{O}_S)$ in such a way that

$$f(E, P, \lambda\omega) = \lambda^{-k} f(E, P, \omega)$$

for all $\lambda \in \mathrm{H}^0(S, \mathcal{O}_S^\times)$. The dictionary is that such a "rule" f gives a well-defined section $f(E, P, \omega) \cdot \omega^{\otimes k}$ of $(\varepsilon^*\Omega^1_{E^{\mathrm{sm}}/S})^{\otimes k}$, independent of ω. Conversely, any global section $f \in M_k(\Gamma_1(N), \mathbf{C})$ canonically pulls back to a section of $\varepsilon^*\Omega^1_{E^{\mathrm{sm}}/S}$ which can be uniquely written

$$f(E, P, \omega) \cdot \omega^{\otimes k}.$$

We will also sometimes identify f with the "1-form valued rule" given by $f(E, P) = f(E, P, \omega) \cdot \omega^{\otimes k}$, which as we saw above is well-defined. Using only smooth elliptic curves in the above considerations gives an element of $\mathrm{H}^0(Y_1(N), \omega^{\otimes k})$.

For example, with $0 < |q| < 1$, we can define

$$f_\infty(q) = f\left(\mathbf{C}^\times/q^{\mathbf{Z}}, \zeta_N, \mathrm{d}t/t\right)$$

and by Theorem 1.6.6.2 this is *exactly* the evaluation at $q \in \Delta^*$ of the q-expansion of f at ∞. Of course, we can identify the cusps of $X_1(N)$ with the set of $\Gamma_1(N)$-structures on standard d-gons for all d dividing N, and $f \in M_k(\Gamma_1(N), \mathbf{C})$ lies in $S_k(\Gamma_1(N), \mathbf{C})$ if and only if evaluation on each of these finitely many structures is 0.

We can use this point of view to define *Hecke operators* somewhat geometrically, and we will check compatibility with the classical Hecke operators via the inclusion

$$\mathrm{H}^0(X_1(N), \overline{\omega}^{\otimes k}) \hookrightarrow \mathrm{H}^0(\mathfrak{h}, \mathcal{O}_{\mathfrak{h}})$$

from Lemma 1.5.7.2. Later on we will give another, more cohomological, definition in weight at least 2, and we will identify that definition with the geometric definition which we now give.

Viewing $f \in M_k(\Gamma_1(N), \mathbf{C})$ as a "rule" as above, for n relatively prime to N define the "rule"

$$\langle n \rangle f(E, P, \omega) = f(E, n \cdot P, \omega).$$

There seems to be no risk of confusion with the notation $\langle P \rangle$ for a $\Gamma_1(N)$-structure $\underline{\mathbf{Z}/N} \hookrightarrow E^{\mathrm{sm}}$. For any prime p and $\Gamma_1(N)$-structure (E, P) on a smooth elliptic curve $E \to S$ over a (path) connected base for which there is a global isomorphism $E[p] \simeq \underline{\mathbf{Z}/p} \times \underline{\mathbf{Z}/p}$, define

$$T_p f(E, P, \omega) = p^{k-1} \sum_{\substack{C \subseteq E[p] \\ C \cap \langle P \rangle = \{0\} \\ \#C = p}} f(E/C, P \bmod C, \breve{\pi}_C^* \omega)$$

where $\pi_C : E \to E/C$ has dual $\breve{\pi}_C$ and the sum is taken over all $p+1$ distinct degree p subgroups $C \hookrightarrow E[p]$ having trivial intersection with the subgroup $\langle P \rangle = \underline{\mathbf{Z}/N} \hookrightarrow E$ generated by P. Since this latter expression is clearly compatible with localization over S and analytic spaces are locally (path) connected, it globalizes to define $T_p f(E, P, \omega)$ for any $\Gamma_1(N)$-structure (E, P) on a smooth elliptic curve $E \to S$. The operator T_p therefore acts on $\mathrm{H}^0(Y_1(N), \omega^{\otimes k})$. In order to check if this operation preserves the subspace $\mathrm{H}^0(X_1(N), \overline{\omega}^{\otimes k})$, it suffices to analyze what happens near the cusps. In fact, we have:

THEOREM 2.2.4.2. *The operators defined above are preserve* $\mathrm{H}^0(X_1(N), \overline{\omega}^{\otimes k})$ *as well as the subspace of cusp forms, and coincide with the classical operators.*

PROOF. Since the definition of $\langle n \rangle$ makes sense on generalized elliptic curves with $\Gamma_1(N)$-structures, it automatically takes $M_k(\Gamma_1(N), \mathbf{C})$ back to itself. Also, if f is actually a cusp form, which is to say that $f(E, P, \omega) = 0$ for all Néron polygons E, then clearly the same is true for $\langle n \rangle (f)$. Thus $\langle n \rangle$ preserves the spaces of modular forms and cusp forms.

In order to study T_p and to make the connection with the classical theory, we note first of all that the universal family over $Y_1(N)$ is identified near each cusp with a $\Gamma_1(N)$-structure on some $\underline{\mathrm{Tate}}_d^*$ for d dividing N. Thus, by evaluating on such structures, with ω trivialized by $\mathrm{d}t/t$, we can study what happens as q goes to 0 in order to detect membership in $M_k(\Gamma_1(N), \mathbf{C})$ or $S_k(\Gamma_1(N), \mathbf{C})$. Also, recall that the isomorphism $\psi_{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)} : X \xrightarrow{\simeq} \mathfrak{h}$ identifies $\mathcal{E}^{\mathrm{an}} \to \mathfrak{h}$ with $E_X \to X$ in such a way that the section $\frac{1}{N} \hookrightarrow \mathcal{E}^{\mathrm{an}}$ corresponds *exactly* to the construction $E_1(N) \to Y_1(N) = \Gamma_1(N)\backslash\mathfrak{h}$, and $-q\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)^\vee$ corresponds to $2\pi i\mathrm{d}\tau$ in $\mathcal{E}^{\mathrm{an}}$. Thus, the map

$$z \mapsto f\left(\mathbf{C}/[1, z], 1/N, 2\pi i\mathrm{d}\tau\right)$$

is *exactly* the element in $\mathrm{H}^0(\mathfrak{h}, \mathcal{O}_{\mathfrak{h}})$ associated to $f \in \mathrm{H}^0(X_1(N), \overline{\omega}^{\otimes k})$. In particular, since

$$(\mathbf{C}/[1, z], 1/N, 2\pi i\mathrm{d}\tau) \simeq \left(\mathbf{C}^\times/q^{\mathbf{Z}}, \zeta_N, \frac{dt}{t}\right)$$

(with $q = e^{2\pi i z}$), evaluation of f on triples $(\mathbf{C}/[1, z], 1/N, 2\pi i\mathrm{d}\tau)$ suffices to describe the structure near ∞. Since an element in $\mathrm{H}^0(Y_1(N), \omega^{\otimes k})$ is uniquely determined by knowing it in a punctured neighborhood of ∞, we see that the comparison with the classical operators can be determined by looking near ∞.

For example, choosing $g = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z})$ with $g \equiv \left(\begin{smallmatrix}n^{-1} & *\\ 0 & n\end{smallmatrix}\right) \bmod N$, we find that

$$\begin{aligned}
\langle n \rangle f\left(\mathbf{C}/[1, z], 1/N, 2\pi i\mathrm{d}\tau\right) &= f\left(\mathbf{C}/[1, z], n/N, 2\pi i\mathrm{d}\tau\right)\\
&= f\left(\mathbf{C}/[az + b, cz + d], (cz + d)/N, 2\pi i\mathrm{d}\tau\right)\\
&= f\left(\mathbf{C}/[1, gz], 1/N, (cz + d) \cdot 2\pi i\mathrm{d}\tau\right)\\
&= (cz + d)^{-k} f(g(z))\\
&= (f|_k g)(z).
\end{aligned}$$

But this is just the classical definition of a diamond operator, as desired.

To study the T_p, we note that $f(E, P, (1/p)\omega) = p^k f(E, P, \omega)$ and $\pi_C : E \to E/C$ has degree p, so we easily compute

$$T_p f\left(\mathbf{C}/[1, z], 1/N, 2\pi i\mathrm{d}\tau\right) = \frac{1}{p} \sum_{\substack{C \subseteq [\frac{1}{p}, \frac{z}{p}]/[1, z] \\ \frac{1}{p} \notin C \\ \#C = p}} f\left(\mathbf{C}/[1, z, C], 1/N, 2\pi i\mathrm{d}\tau\right).$$

If p divides N, then this specializes to

$$T_p f(z) = \frac{1}{p}\sum_{j=0}^{p-1} f\left(\mathbf{C}/\left[1, \tfrac{z+j}{p}\right], 1/N, 2\pi i \mathrm{d}\tau\right) = \frac{1}{p}\sum_{j=0}^{p-1} f\left(\frac{z+j}{p}\right).$$

Again, this is just the classical definition. When p does not divide N, then

$$\begin{aligned} T_p f(z) &= \frac{1}{p}\sum_{j=0}^{p-1} f\left(\frac{z+j}{p}\right) + \frac{1}{p} f\left(\mathbf{C}/\left[1/p, z\right], 1/N, 2\pi i \mathrm{d}\tau\right) \\ &= \frac{1}{p}\sum_{j=0}^{p-1} f\left(\frac{z+j}{p}\right) + \frac{1}{p} f\left(\mathbf{C}/[1, pz], p/N, (2\pi i/p)\mathrm{d}\tau\right) \\ &= \frac{1}{p}\sum_{j=0}^{p-1} f\left(\frac{z+j}{p}\right) + p^{k-1}\langle p\rangle f(pz); \end{aligned}$$

this is just the classical definition as desired.

It is now unnecessary to prove directly that $T_p f$ is a modular form (resp. cusp form) when f is, since we can appeal to the classical theory. We give a direct proof anyway. By the discussion in the proof of Lemma 2.2.1.3, the structures of interest near the cusps are $\underline{\mathrm{Tate}}_d^* \to \Delta^*$ with point $\zeta^a q^b$ of order N; here $b \in (\mathbf{Z}/d)^\times$, $a \in \mathbf{Z}/N$ with $\gcd(a, N/d) = 1$, and $\zeta = e^{\frac{2\pi i}{N}}$. Let

$$f_{(d;a,b)} = f\left(\underline{\mathrm{Tate}}_d^*, \zeta^a q^b, \mathrm{d}t/t\right)$$

denote the q-expansion at this cusp. Recalling how $\overline{\omega}_\Gamma$ is *constructed* using $\mathrm{d}t/t$ near cusps, it is enough to study

$$T_p f\left(\mathbf{C}^\times/q^{d\mathbf{Z}}, \zeta^a q^b, \mathrm{d}t/t\right)$$

for $q \in \Delta^*$ and q going to 0.

First suppose that p does not divide N. The order p subgroups of $\mathbf{C}^\times/q^{d\mathbf{Z}}$ are those generated by $\zeta_p = e^{2\pi i/p}$ and by $\zeta_p^j q^{d/p}$ for $0 \le j \le p-1$. Since the degree p isogeny $\mathbf{C}^\times/q^{\mathbf{Z}} \to \mathbf{C}^\times/q^{\mathbf{Z}}$ (via pth powers on $\mathbf{C}^\times$) pulls $\mathrm{d}t/t$ back to $p\,\mathrm{d}t/t$, we calculate that

$$\begin{aligned} & (T_p f)\left(\mathbf{C}^\times/q^{d\mathbf{Z}}, \zeta^a q^b, \mathrm{d}t/t\right) \\ = \;& p^{k-1} f\left(\mathbf{C}^\times/(q^{d\mathbf{Z}}, \zeta_p), \zeta^a q^b, p\,\mathrm{d}t/t\right) + p^{k-1}\textstyle\sum_{j=0}^{p-1} f\left(\mathbf{C}^\times/(\zeta_p^j q^{d/p}), \zeta^a q^b, p\,\mathrm{d}t/t\right) \\ = \;& p^{k-1} f\left(\mathbf{C}^\times/(q^{dp\mathbf{Z}}), \zeta^{ap} q^{bp}, \mathrm{d}t/t\right) + \tfrac{1}{p}\textstyle\sum_{j=0}^{p-1} f\left(\mathbf{C}^\times/(\zeta_p^j q^{\frac{d}{p}}), \zeta^a q^b, \mathrm{d}t/t\right) \\ = \;& p^{k-1} f_{(d;ap,b)}(q^p) + \begin{cases} \frac{1}{p}\sum_{j=0}^{p-1} f_{(d;a,bp)}(\zeta_p^{j/d} q^{1/p}) & p \nmid d \\ \frac{1}{p}\sum_{j=0}^{p-1} f_{(d/p,a-jbN/d,b)}(\zeta_d^j q) & p \mid d \end{cases} \\ = \;& p^{k-1} f_{(d;ap,b)}(q^p) + \begin{cases} \frac{1}{p}\sum_{q'^p=q} f_{(d;a,bp)}(q') & p \nmid d \\ \frac{1}{p}\sum_{j=0}^{p-1} f_{(d/p,a-jbN/d,b)}(\zeta_d^j q) & p \mid d \end{cases} \end{aligned}$$

This obviously has the required analytic properties near cusps $(d; a, b)$. In the case that p divides N,the same analysis works after dropping one of the terms in the above sums (depending solely on d, a, b).

$\square$

It follows from Theorem 2.2.4.2 that the operators $\langle n\rangle$ and T_p commute with each other. Motivated by the classical theory, we define T_n for all $n \geq 1$ via the formulas

$$\begin{aligned} T_1 &= 1 \\ T_{p^{r+1}} &= T_p T_{p^r} - p^{k-1}\langle p\rangle T_{p^{r-1}} \\ T_{nm} &= T_n T_m \qquad \gcd(m,n) = 1. \end{aligned}$$

(Here we are taking $\langle p\rangle = 0$ if p divides N.) If we write

$$f_\infty = \sum_{n=1}^{\infty} a_n(f) e^{2\pi i n z}$$

for the q-expansion at ∞ of $f \in M_k(\Gamma_1(N), \mathbf{C})$, we have

$$a_1(T_n f) = a_n(f)$$

by the *classical theory.* We let

$$\mathbf{T}_1(N) \subseteq \mathrm{End}_{\mathbf{C}}(S_k(\Gamma_1(N), \mathbf{C})$$

denote the $\mathbf{Z}$-module generated by the T_p and $\langle n\rangle$ for all primes p and $n \in (\mathbf{Z}/N)^\times$; we suppress k from the notation, as it is always understood.

2.3. Hecke operators and cohomology

2.3.1. Geometric definition of Hecke operators. Fix $N \geq 5$ and let

$$f_N : E_1(N) \to Y_1(N)$$

be the universal elliptic curve. We want to define $\mathbf{Z}$-linear maps

$$T \in \mathrm{End}_{\mathbf{Z}}\big(\widetilde{\mathrm{H}}^1(Y_1(N), \mathrm{Sym}^k_{\underline{\mathbf{Z}}}\, \mathrm{R}^1 f_{N*}\underline{\mathbf{Z}})\big)$$

for $T = T_p$ or $T = \langle n\rangle$. We wish to do this in such a way that after tensoring with $\mathbf{C}$ these operators yield the corresponding classical linear operators $T \oplus \overline{T}$ on

$$S_{k+2}(\Gamma_1(N), \mathbf{C}) \oplus \overline{S_{k+2}(\Gamma_1(N), \mathbf{C})}$$

via the Eichler-Shimura isomorphism (1.7.7.4).

In order to identify our cohomological construction with the classical operators on weight $k+2$ cusp forms, we shall make use of the moduli-theoretic description of Hecke operators in Theorem 2.2.4.2. For $n \in (\mathbf{Z}/N)^\times$, define an automorphism

$$I_n : X_1(N) \xrightarrow{\simeq} X_1(N)$$

via the automorphism $(E, P) \mapsto (E, n \cdot P)$ on $\Gamma_1(N)$-structures on generalized elliptic curves. This restricts to the analogous automorphism of $Y_1(N)$ and is given by a cartesian square

(2.3.1.1)
$$\begin{tikzcd} \overline{E}_1(N) \arrow[r, "\simeq"] \arrow[d, "f_N"'] & \overline{E}_1(N) \arrow[d, "f_N"] \\ X_1(N) \arrow[r, "I_n", "\simeq"'] & X_1(N) \end{tikzcd}$$

where the isomorphism $\overline{E}_1(N) \simeq I_n^* \overline{E}_1(N)$ identifies $n \cdot \mathcal{P}$ with $I_n^*(\mathcal{P})$. That is, I_n is induced by the $\Gamma_1(N)$-structure $(\overline{E}_1(N), n \cdot \mathcal{P})$ via universality of $(\overline{E}_1(N), \mathcal{P})$. There is an analogous setup over $Y_1(N)$.

Using proper base change, we have an isomorphism $I_n^* \mathrm{R}^1 f_{N*}\underline{\mathbf{Z}} \simeq \mathrm{R}^1 f_{N*}\underline{\mathbf{Z}}$. We can therefore define operations via pullback along I_n on

$$\mathrm{H}^1_{\mathrm{c}}(Y_1(N), \mathrm{Sym}^k \mathrm{R}^1 f_{N*}\underline{\mathbf{Z}}) \quad \textit{and} \quad \mathrm{H}^1(Y_1(N), \mathrm{Sym}^k \mathrm{R}^1 f_{N*}\underline{\mathbf{Z}})$$

compatibly with the inclusion of compact cohomology into ordinary cohomology. We therefore get an induced map denoted

$$\langle n \rangle \in \mathrm{End}_{\mathbf{Z}}\big(\widetilde{\mathrm{H}}^1(Y_1(N), \mathrm{Sym}^k \mathrm{R}^1 f_{N*}\underline{\mathbf{Z}})\big).$$

To define a T_p operator here for any prime p, we must first define the $\Gamma_1(N,p)$ moduli problem. A $\Gamma_1(N,p)$*-structure* is a triple (E,P,C) where $E \to S$ is a generalized elliptic curve, $P \in E^{\mathrm{sm}}(S)$ has exact order N, and $C \hookrightarrow E^{\mathrm{sm}}$ is a degree p relative Cartier divisor which is a closed subgroup object. We further require that the subgroup generated by P and C meets all irreducible components of fibers, and if p divides N we insist that C and the subgroup generated by P have trivial intersection. This final condition for p dividing N says that

$$(\langle P \rangle [p]) \times C \to E^{\mathrm{sm}}[p]$$

is an isomorphism of étale group objects. If p does not divide N, then the $\Gamma_1(N,p)$ moduli problem is the quotient of the $\Gamma_1(Np)$ problem by the action of $(\mathbf{Z}/p)^\times$ on $\mathbf{Z}/Np \simeq \mathbf{Z}/N \times \mathbf{Z}/p$. The methods used to study $\Gamma_1(N)$ structures carry over essentially verbatim to $\Gamma_1(N,p)$ structures, and the same algebraicity and universal properties carry over; here the universal family

$$\overline{E}_1(N,p) \to X_1(N,p)$$

has a proper smooth *connected* curve as base, and is obtained by "adding" degenerate Tate curves to the universal elliptic curve family $E_1(N,p) \to Y_1(N,p)$ over the complement of the cusps.

One can replace p by any integer n with distinct prime factors to get a similar universal family. Integers n with repeated prime factors require an additional "cyclicity" condition, but we will not need to consider such cases.

Define the degeneracy maps

$$\pi_1, \pi_2 : Y_1(N,p) \to Y_1(N)$$

by

$$\begin{aligned}(E,P,C) &\overset{\pi_1}{\mapsto} (E,P)\\ (E,P,C) &\overset{\pi_2}{\mapsto} (E/C, P \bmod C)\end{aligned}$$

respectively.

The fibers of the π_j over $Y_1(N)$ have *constant* degree $p+1$ (resp. p) if p does not divide N (resp. p does divide N). An easy argument with discs then shows that the π_j are covering maps (i.e., they are finite étale maps). These maps are encoded in the diagram

(2.3.1.2)
$$\begin{tikzcd}
 & & E_1(N,p) \arrow[rr, "\varphi"] \arrow[dl] \arrow[dr, "u"] & & E_1(N,p)/\mathcal{C} \arrow[dl, "v"'] \arrow[dr] & & \\
 & E_1(N) \arrow[dr, "f_N"'] & & Y_1(N,p) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & & E_1(N) \arrow[dl, "f_N"] & \\
 & & Y_1(N) & & Y_1(N) & &
\end{tikzcd}$$

where the squares are *cartesian*, φ is the "universal" p-isogeny over $Y_1(N,p)$, and $E_1(N,p)/\mathcal{C}$ has $\Gamma_1(N)$-structure $\mathcal{P} \bmod \mathcal{C}$, where

$$(E_1(N,p) \to X_1(N,p), \mathcal{P}, \mathcal{C})$$

is the universal $\Gamma_1(N,p)$-structure. We have a map $\varphi^* : \mathrm{R}^1 v_* \underline{\mathbf{Z}} \to \mathrm{R}^1 u_* \underline{\mathbf{Z}}$ by the functoriality of pullback, and isomorphisms

$$\pi_2^* \mathrm{R}^1 f_{N*} \underline{\mathbf{Z}} \simeq \mathrm{R}^1 v_* \underline{\mathbf{Z}}$$
$$\pi_1^* \mathrm{R}^1 f_{N*} \underline{\mathbf{Z}} \simeq \mathrm{R}^1 u_* \underline{\mathbf{Z}}$$

by proper base change. Since π_2 is *proper* (even finite), it defines δ-functorial compatible pullback maps on compact cohomology as well as ordinary cohomology. (This compatibility can be checked in degree 0 by a universal δ-functor argument.) We therefore get an induced pullback map on the image $\widetilde{\mathrm{H}}^*$ of $\mathrm{H}^*_{\mathrm{c}}$ in H^*. Note that in *fixed* degree, $\widetilde{\mathrm{H}}^*$ is *not* a δ-functor.

We can also define trace maps, thanks to the next lemma.

Lemma 2.3.1.1. *Let $\pi : X \to Y$ be a finite étale map of analytic spaces. Then π_* is an exact functor from $\mathcal{O}_X$-modules to $\mathcal{O}_Y$-modules, taking injectives to injectives. Thus, there exist canonical isomorphisms*

$$\mathrm{H}^*(X, \cdot) \simeq \mathrm{H}^*(Y, \pi_*(\cdot))$$
$$\mathrm{H}^*_{\mathrm{c}}(X, \cdot) \simeq \mathrm{H}^*_{\mathrm{c}}(Y, \pi_*(\cdot))$$

of δ-functors, induced by the obvious equalities in degree 0, and these are compatible with the map $\mathrm{H}^*_{\mathrm{c}} \to \mathrm{H}^*$. Thus, there is an induced natural isomorphism of functors

$$\widetilde{\mathrm{H}}^*(X, \cdot) \simeq \widetilde{\mathrm{H}}^*(Y, \pi_*(\cdot))$$

in each fixed degree.

Proof. Since π_* has *exact* adjoint π^* and π_* is exact by the local structure of analytic finite étale maps, everything is clear. ☐

In the setting of Lemma 2.3.1.1 we can define obvious trace maps

$$\pi_* \pi^* \to \mathrm{id}$$

on $\mathcal{O}_Y$-modules, so we may define "trace maps"

$$\mathrm{tr}_\pi : \mathrm{H}^*(X, \pi^*(\cdot)) \simeq \mathrm{H}^*(Y, \pi_* \pi^*(\cdot)) \to \mathrm{H}^*(Y, \cdot)$$

and similarly for $\mathrm{H}^*_{\mathrm{c}}$ and $\widetilde{\mathrm{H}}^*$. This construction is obviously δ-functorial for H^* and $\mathrm{H}^*_{\mathrm{c}}$.

Returning to the situation of interest, we have trace maps

$$\mathrm{tr}_{\pi_1} : \mathrm{H}^*(Y_1(N,p), \pi_1^*(\cdot)) \to \mathrm{H}^*(Y_1(N), \cdot)$$

and likewise for $\mathrm{H}^*_{\mathrm{c}}$ and $\widetilde{\mathrm{H}}^*$. We can also use the same argument to get the same results with abelian sheaves in Lemma 2.3.1.1 using topological pullbacks π^*. Note that when the map $\pi : X \to Y$ is finite *étale* , topological pullbacks on $\mathcal{O}_Y$-modules *coincide* with module pullbacks. Thus, the entire preceding discussion surrounding Lemma 2.3.1.1 is a genuine *special case* of the more general analogue for abelian sheaves (e.g. local systems of $\mathbf{C}$-modules).

We can now define a composite map

$$\begin{tikzcd}
\mathrm{H}^1_{\mathrm{c}}(Y_1(N), \mathrm{Sym}^k \mathrm{R}^1 f_{N*}\underline{\mathbf{Z}}) \arrow[r, "\pi_2^*"] & \mathrm{H}^1_{\mathrm{c}}(Y_1(N,p), \mathrm{Sym}^k \mathrm{R}^1 v_*\underline{\mathbf{Z}}) \arrow[d, "\varphi^*"] \\
& \mathrm{H}^1_{\mathrm{c}}(Y_1(N,p), \mathrm{Sym}^k \mathrm{R}^1 u_*\underline{\mathbf{Z}}) \arrow[d, equal] \\
\mathrm{H}^1_{\mathrm{c}}(Y_1(N), \mathrm{Sym}^k \mathrm{R}^1 f_{N*}\underline{\mathbf{Z}}) & \mathrm{H}^1_{\mathrm{c}}(Y_1(N,p), \pi_1^* \mathrm{Sym}^k \mathrm{R}^1 f_{N*}\underline{\mathbf{Z}}) \arrow[l, "\mathrm{tr}_{\pi_1}"']
\end{tikzcd}$$

and analogously with H^1 instead of $\mathrm{H}^1_{\mathrm{c}}$. By standard universal δ-functor argument, this composite is compatible with the map $\mathrm{H}^1_{\mathrm{c}} \to \mathrm{H}^1$, so we finally have a composite map

$$T_p = \mathrm{tr}_{\pi_1} \circ \varphi^* \circ \pi_2^* \in \mathrm{End}_{\mathbf{Z}}\big(\widetilde{\mathrm{H}}^1(Y_1(N), \mathrm{Sym}^k \mathrm{R}^1 f_{N*}\underline{\mathbf{Z}})\big).$$

2.3.2. Compatibility with classical definition. The main result is the following.

THEOREM 2.3.2.1. *Let $\Gamma = \Gamma_1(N)$ with $N \geq 5$ and let $T = T_p$ for some p, or $T = \langle n \rangle$ for some n relatively prime to N. Via the Eichler-Shimura isomorphism*

$$S_{k+2}(\Gamma, \mathbf{C}) \oplus \overline{S_{k+2}(\Gamma, \mathbf{C})} \xrightarrow{\simeq} \widetilde{\mathrm{H}}^1(Y_1(N), \mathrm{Sym}^k \mathrm{R}^1 f_{N*}\underline{\mathbf{Z}}) \otimes_{\mathbf{Z}} \mathbf{C} \tag{2.3.2.1}$$

the $\mathbf{C}$-linear action of $T \otimes 1$ on the right side coincides with the direct sum of the classical T and its conjugate on the left side.

REMARK 2.3.2.2. We will later compare the Petersson product on the left side of (2.3.2.1) with a sheaf cohomology cup product on the right side. Such a relation was hinted at in the proof of Lemma 1.7.6.1.

PROOF. We may compare these operators after composing with the injective map $\widetilde{\mathrm{H}}^1 \hookrightarrow \mathrm{H}^1$. By the *definition* of the Eichler-Shimura isomorphism, we then need to prove (for $\mathcal{U}^k = \mathrm{Sym}^k \mathrm{R}^1 f_{N*}\underline{\mathbf{C}}$, $\Omega^1 = \Omega^1_{X_1(N)}$ and $\omega = f_{N*}\Omega^1_{\overline{E}_1(N)/X_1(N)}$)

that the diagram

$$\begin{tikzcd}
\mathrm{H}^0(X_1(N), \overline{\omega}^{\otimes k} \otimes \Omega^1) \arrow[r, "\langle n \rangle"] \arrow[d, hook] & \mathrm{H}^0(X_1(N), \overline{\omega}^{\otimes k} \otimes \Omega^1) \arrow[d, hook] \\
\mathrm{H}^0(Y_1(N), \omega^{\otimes k} \otimes \Omega^1) \arrow[d] & \mathrm{H}^0(Y_1(N), \omega^{\otimes k} \otimes \Omega^1) \arrow[d] \\
\mathrm{H}^0(Y_1(N), \mathcal{U}^k \otimes \Omega^1) \arrow[d, "\partial"'] & \mathrm{H}^0(Y_1(N), \mathcal{U}^k \otimes \Omega^1) \arrow[d, "\partial"] \\
\mathrm{H}^1(Y_1(N), \mathcal{U}^k) \arrow[r, "I_n^*"] & \mathrm{H}^1(Y_1(N), \mathcal{U}^k)
\end{tikzcd}$$

commutes for all n relatively prime to N (with the top map the classical map $\langle n \rangle$ as "geometrized" via Theorem 2.2.4.2 and the Kodaira-Spencer isomorphism) and that

$$\begin{tikzcd}
 & \mathrm{H}^0(Y_1(N), \omega^{\otimes k} \otimes \Omega^1) \arrow[d] \\
\mathrm{H}^0(X_1(N), \overline{\omega}^{\otimes k} \otimes \Omega^1) \arrow[ur, hook] \arrow[dd, "T_p"] & \mathrm{H}^0(Y_1(N), \mathcal{U}^k \otimes \Omega^1) \arrow[d, "\partial"] \\
 & \mathrm{H}^1(Y_1(N), \mathcal{U}^k) \arrow[d, "\pi_2^*"] \\
\mathrm{H}^0(X_1(N), \overline{\omega}^{\otimes k} \otimes \Omega^1) \arrow[dd, hook] & \mathrm{H}^1(Y_1(N,p), \mathrm{Sym}^k \mathrm{R}^1 v_* \underline{\mathbf{C}}) \arrow[d, "\varphi^*"] \\
 & \mathrm{H}^1(Y_1(N,p), \mathrm{Sym}^k \mathrm{R}^1 u_* \underline{\mathbf{C}}) \arrow[d, "\simeq"] \\
\mathrm{H}^0(Y_1(N), \omega^{\otimes k} \otimes \Omega^1) \arrow[d] & \mathrm{H}^1(Y_1(N,p), \pi_1^* \mathcal{U}^k) \arrow[d, "\mathrm{tr}_{\pi_1}"] \\
\mathrm{H}^0(Y_1(N), \mathcal{U}^k \otimes \Omega^1) \arrow[r, "\partial"] & \mathrm{H}^1(Y_1(N), \mathcal{U}^k)
\end{tikzcd}$$

commutes for all p, where the T_p on the left side is again the classical map.

Since the Kodaira-Spencer isomorphism is compatible with base change, it follows that base change by $I_n : Y_1(N) \xrightarrow{\simeq} Y_1(N)$ reduces the case of $\langle n \rangle$ to checking that

$$I_n^* \in \mathrm{End}\big(\mathrm{H}^0(Y_1(N), \omega^{\otimes (k+2)})\big)$$

and

$$\langle n \rangle \in \mathrm{End}\big(\mathrm{H}^0(X_1(N), \overline{\omega}^{\otimes (k+2)})\big)$$

are compatible in the obvious sense. This requires a little care to justify without being obscure. The point is that by *definition*, I_n sends a geometric point $x =$

$(E_0, P_0) \in Y_1(N)$ to $I_n(x) = (E_0, n\cdot P_0) \in Y_1(N)$, and the restriction of the universal cartesian diagram (2.3.1.1) over these two points corresponds to an automorphism $E_0 \simeq E_0$ pulling the "universal point" $n\cdot P_0$ over $I_n(x)$ back to n times the "universal point" P_0 over x. That is, the automorphism $E_0 \simeq E_0$ *fixes* the point nP_0 of exact order $N \geq 5$, so it must be the *identity*.

We conclude that for any global section f of $\omega^{\otimes(k+2)}$ over $Y_1(N)$, the image of $I_n^*(f)$ in the fiber at $x = (E_0, P_0)$ is an element of $\omega_{x_0}^{\otimes(k+2)} \simeq \mathrm{H}^0(E_0, \Omega^1)^{\otimes(k+2)}$ which is *equal* to the image of f in the fiber at $I_n(x) = (E_0, n \cdot P_0)$. This in turn is an element of $\omega_{I_n(x)}^{\otimes k} = \mathrm{H}^0(E_0, \Omega^1)^{\otimes(k+2)}$. Thus, if $f \in \mathrm{H}^0(X_1(N), \overline{\omega}^{\otimes(k+2)})$ then viewing f as a function on moduli gives

$$\begin{aligned}(I_n^* f)(x) &= (I_n^* f)(E_0, P_0)\\ &= f(E_0, nP_0)\\ &= (\langle n\rangle f)(E_0, P_0)\\ &= (\langle n\rangle f)(x)\end{aligned}$$

by Theorem 2.2.4.2. Thus, as we vary x we conclude that $I_n^* f$ and $\langle n\rangle f$ coincide as elements of $\mathrm{H}^0(Y_1(N), \omega^{\otimes(k+2)})$, which is what we wanted.

We turn now to T_p. The first step in the analysis of the relevant diagram is to use δ-functor considerations to eliminate the appearance of H^1's. More precisely, since π_1 is finite étale , the natural map of invertible sheaves

$$\pi_1^* \Omega^1_{Y_1(N)} \to \Omega^1_{Y_1(N,p)}$$

is an isomorphism. In fact, if we interpret π_1^* as *topological* pullback (on the level of underlying abelian sheaves), then the short exact sequence

$$0 \to \underline{\mathbf{C}} \to \mathcal{O}_{Y_1(N,p)} \xrightarrow{\mathrm{d}} \Omega^1_{Y_1(N,p)} \to 0$$

is *exactly* π_1^* applied to the sequence

$$0 \to \underline{\mathbf{C}} \to \mathcal{O}_{Y_1(N)} \xrightarrow{\mathrm{d}} \Omega^1_{Y_1(N)} \to 0.$$

Thus, we can form the following diagram in which all squares *commute* by δ-functoriality of pullback and trace maps (for *abelian* sheaves):

$$\begin{array}{ccc}
\mathrm{H}^0\big(Y_1(N), \mathcal{U}^k \otimes_{\underline{\mathbf{C}}} \Omega^1_{Y_1(N)}\big) & \xrightarrow{\partial} & \mathrm{H}^1\big(Y_1(N), \mathcal{U}^k\big)\\
\big\downarrow{\scriptstyle \pi_2^*} & & \big\downarrow{\scriptstyle \pi_2^*}\\
\mathrm{H}^0\big(Y_1(N,p), \mathrm{Sym}^k\, \mathrm{R}^1 v_* \underline{\mathbf{C}} \otimes_{\underline{\mathbf{C}}} \Omega^1_{Y_1(N,p)}\big) & \xrightarrow{\partial} & \mathrm{H}^1\big(Y_1(N,p), \mathrm{Sym}^k\, \mathrm{R}^1 v_* \underline{\mathbf{C}}\big)\\
\big\downarrow{\scriptstyle \varphi^*} & & \big\downarrow{\scriptstyle \varphi^*}\\
\mathrm{H}^0\big(Y_1(N,p), \mathrm{Sym}^k\, \mathrm{R}^1 u_* \underline{\mathbf{C}} \otimes_{\underline{\mathbf{C}}} \Omega^1_{Y_1(N,p)}\big) & \xrightarrow{\partial} & \mathrm{H}^1\big(Y_1(N,p), \mathrm{Sym}^k\, \mathrm{R}^1 u_* \underline{\mathbf{C}}\big)\\
\big\downarrow{\scriptstyle \simeq} & & \big\downarrow{\scriptstyle \simeq}\\
\mathrm{H}^0\big(Y_1(N,p), \pi_1^*(\mathcal{U}^k \otimes_{\underline{\mathbf{C}}} \Omega^1_{Y_1(N)})\big) & \xrightarrow{\partial} & \mathrm{H}^1\big(Y_1(N,p), \pi_1^* \mathcal{U}^k\big)\\
\big\downarrow{\scriptstyle \mathrm{tr}_{\pi_1}} & & \big\downarrow{\scriptstyle \mathrm{tr}_{\pi_1}}\\
\mathrm{H}^0\big(Y_1(N), \mathcal{U}^k \otimes_{\underline{\mathbf{C}}} \Omega^1_{Y_1(N)}\big) & \xrightarrow{\partial} & \mathrm{H}^1\big(Y_1(N), \mathcal{U}^k\big)
\end{array}$$

The left column fits into another commutative diagram

$$\begin{tikzcd}
\mathrm{H}^0\big(Y_1(N), \omega^{\otimes k} \otimes_{\mathcal{O}} \Omega^1_{Y_1(N)}\big) \arrow[r] \arrow[d, "\pi_2^*"'] & \mathrm{H}^0\big(Y_1(N), \mathcal{U}^k \otimes_{\underline{\mathbf{C}}} \Omega^1_{Y_1(N)}\big) \arrow[d, "\pi_2^*"] \\
\mathrm{H}^0\big(Y_1(N,p), \omega^{\otimes k}_{\pi_2^* E_1(N)} \otimes_{\mathcal{O}} \Omega^1_{Y_1(N,p)}\big) \arrow[r] \arrow[d, "\varphi^*"'] & \mathrm{H}^0\big(Y_1(N,p), \operatorname{Sym}^k \mathrm{R}^1 v_* \underline{\mathbf{C}} \otimes_{\underline{\mathbf{C}}} \Omega^1_{Y_1(N,p)}\big) \arrow[d, "\varphi^*"] \\
\mathrm{H}^0\big(Y_1(N,p), \omega^{\otimes k}_{\pi_1^* E_1(N)} \otimes_{\mathcal{O}} \Omega^1_{Y_1(N,p)}\big) \arrow[r] \arrow[d, "\simeq"'] & \mathrm{H}^0\big(Y_1(N,p), \pi_1^*(\operatorname{Sym}^k \mathrm{R}^1 u_* \underline{\mathbf{C}} \otimes_{\underline{\mathbf{C}}} \Omega^1_{Y_1(N,p)})\big) \arrow[d, "\simeq"] \\
\mathrm{H}^0\big(Y_1(N,p), \pi_1^*(\omega^{\otimes k} \otimes \Omega^1_{Y_1(N)})\big) \arrow[r] \arrow[d, "\mathrm{tr}_{\pi_1}"'] & \mathrm{H}^0\big(Y_1(N,p), \pi_1^*(\mathcal{U}^k \otimes \Omega^1_{Y_1(N)})\big) \arrow[d, "\mathrm{tr}_{\pi_1}"] \\
\mathrm{H}^0(Y_1(N), \omega^{\otimes k} \otimes \Omega^1) \arrow[r] & \mathrm{H}^0(Y_1(N), \mathcal{U}^k \otimes \Omega^1)
\end{tikzcd}$$

The commutativity of the second above diagram amounts to the fact that for all elliptic curve $f : E \to S$, the left side of the Hodge filtration

$$0 \to \omega_{E/S} \to \mathrm{R}^1 f_* \underline{\mathbf{C}} \otimes_{\underline{\mathbf{C}}} \mathcal{O}_S \to \omega_{E/S}^{-1} \to 0$$

is compatible with base change on S *and* with maps $\varphi : E \to E'$ of elliptic curves over S. This last compatibility means that the diagram

(2.3.2.2)
$$\begin{tikzcd}
\omega_{E/S} \arrow[r, hook] & \mathrm{R}^1 f_* \underline{\mathbf{C}} \otimes_{\underline{\mathbf{C}}} \mathcal{O}_S \\
\omega_{E'/S} \arrow[r, hook] \arrow[u] & \mathrm{R}^1 f'_* \underline{\mathbf{C}} \otimes_{\underline{\mathbf{C}}} \mathcal{O}_S \arrow[u]
\end{tikzcd}$$

commutes. By the *very definition* of the rows in (2.3.2.2) as coboundaries of the sheaf cohomology sequence of the exact sequences

$$0 \to f^{-1}\mathcal{O}_S \to \mathcal{O}_E \xrightarrow{\mathrm{d}} \Omega^1_{E/S} \to 0$$

$$0 \to f^{-1}\mathcal{O}_S \to \mathcal{O}_{E'} \xrightarrow{\mathrm{d}} \Omega^1_{E'/S} \to 0$$

the commutativity of (2.3.2.2) follows immediately from the *commutative* diagram

$$\begin{tikzcd}
0 \arrow[r] & f^{-1}\mathcal{O}_S \arrow[r] & \mathcal{O}_E \arrow[r, "\mathrm{d}"] & \Omega^1_{E/S} \arrow[r] & 0 \\
0 \arrow[r] & \varphi^{-1} f'^{-1} \mathcal{O}_S \arrow[r] \arrow[u, equal] & \varphi^{-1}\mathcal{O}_{E'} \arrow[r, "\varphi^{-1}(\mathrm{d})"] \arrow[u] & \varphi^{-1}\Omega^1_{E'/S} \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

and the δ-functoriality of pullback in sheaf cohomology.

Putting this all together, we now see that we need to show that the diagram

$$\begin{tikzcd}
\mathrm{H}^0\big(X_1(N),\overline{\omega}^{\otimes(k+2)}(-\mathrm{cusps})\big) \arrow[r, hook] \arrow[d, "T_p"'] & \mathrm{H}^0\big(Y_1(N),\omega^{\otimes(k+2)}\big) \arrow[d, "\underline{\mathrm{KS}}"] \\
\mathrm{H}^0\big(X_1(N),\overline{\omega}^{\otimes(k+2)}(-\mathrm{cusps})\big) \arrow[d, hook] & \mathrm{H}^0\big(Y_1(N),\omega^{\otimes k}\otimes\Omega^1_{Y_1(N)}\big) \arrow[d, "\pi_2^*"] \\
\mathrm{H}^0\big(Y_1(N),\omega^{\otimes(k+2)}\big) \arrow[dd, "\underline{\mathrm{KS}}"'] & \mathrm{H}^0\big(Y_1(N,p),\omega^{\otimes k}_{\pi_2^*E_1(N)}\otimes\Omega^1_{Y_1(N,p)}\big) \arrow[d, "\varphi^*"] \\
& \mathrm{H}^0\big(Y_1(N,p),\omega^{\otimes k}_{\pi_1^*E_1(N)}\otimes\Omega^1_{Y_1(N,p)}\big) \arrow[d, equal] \\
\mathrm{H}^0\big(Y_1(N),\omega^{\otimes k}\otimes\Omega^1_{Y_1(N)}\big) & \mathrm{H}^0\big(Y_1(N,p),\pi_1^*(\omega^{\otimes k}\otimes\Omega^1_{Y_1(N)})\big) \arrow[l, "\mathrm{tr}_{\pi_1}"']
\end{tikzcd}$$

commutes.

Since the Kodaira-Spencer map is compatible with base change, we can insert the commutative squares

$$\begin{tikzcd}
\mathrm{H}^0\big(Y_1(N),\omega^{\otimes(k+2)}\big) \arrow[r, "\underline{\mathrm{KS}}"] \arrow[d, "\pi_2^*"'] & \mathrm{H}^0\big(Y_1(N),\omega^{\otimes k}\otimes\Omega^1_{Y_1(N)}\big) \arrow[d, "\pi_2^*"] \\
\mathrm{H}^0\big(Y_1(N,p),\omega^{\otimes(k+2)}_{\pi_2^*E_1(N)}\big) \arrow[r, "\underline{\mathrm{KS}}"] & \mathrm{H}^0\big(Y_1(N,p),\omega^{\otimes k}_{\pi_2^*E_1(N)}\otimes\Omega^1_{Y_1(N,p)}\big)
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{H}^0\big(Y_1(N,p),\omega^{\otimes(k+2)}_{\pi_1^*E_1(N)}\big) \arrow[r, "\underline{\mathrm{KS}}"] \arrow[d, "\simeq"'] & \mathrm{H}^0\big(Y_1(N,p),\omega^{\otimes k}_{\pi_1^*E_1(N)}\otimes\Omega^1_{Y_1(N,p)}\big) \arrow[d, "\simeq"] \\
\mathrm{H}^0\big(Y_1(N),\pi_{1*}\pi_1^*\omega^{\otimes(k+2)}\big) \arrow[r, "\pi_{1*}\pi_1^*\underline{\mathrm{KS}}"] \arrow[d, "\mathrm{tr}_{\pi_1}"'] & \mathrm{H}^0\big(Y_1(N),\pi_{1*}\pi_1^*(\omega^{\otimes k}\otimes\Omega^1_{Y_1(N)})\big) \arrow[d, "\mathrm{tr}_{\pi_1}"] \\
\mathrm{H}^0\big(Y_1(N),\omega^{\otimes(k+2)}\big) \arrow[r, "\underline{\mathrm{KS}}"] & \mathrm{H}^0\big(Y_1(N),\omega^{\otimes k}\otimes\Omega^1\big)
\end{tikzcd}$$

to arrive at the diagram

$$\begin{tikzcd}
\mathrm{H}^0\big(X_1(N),\overline{\omega}^{\otimes(k+2)}(-\mathrm{cusps})\big) \arrow[r, hook] \arrow[d, "T_p"'] & \mathrm{H}^0\big(Y_1(N),\omega^{\otimes(k+2)}\big) \arrow[d, "\pi_2^*"] & \\
\mathrm{H}^0\big(X_1(N),\overline{\omega}^{\otimes(k+2)}(-\mathrm{cusps})\big) \arrow[d, hook] & \mathrm{H}^0\big(Y_1(N,p),\omega^{\otimes(k+2)}_{\pi_2^*E_1(N)}\big) \arrow[r, "\underline{\mathrm{KS}}"] \arrow[d, "\frac{1}{p}\varphi^*"] & \mathrm{H}^0\big(Y_1(N,p),\omega^{\otimes k}_{\pi_2^*E_1(N)}\otimes\Omega^1\big) \arrow[d, "\varphi^*"] \\
\mathrm{H}^0\big(Y_1(N),\omega^{\otimes(k+2)}\big) & \mathrm{H}^0\big(Y_1(N,p),\omega^{\otimes(k+2)}_{\pi_1^*E_1(N)}\big) \arrow[l, "\mathrm{tr}_{\pi_1}"] \arrow[r, "\underline{\mathrm{KS}}"'] & \mathrm{H}^0\big(Y_1(N,p),\omega^{\otimes k}_{\pi_1^*E_1(N)}\otimes\Omega^1\big)
\end{tikzcd}$$

and we want the outside edge to commute. We prove this by showing that both subdiagrams commute.

The smaller diagram follows from the more precise claim that the diagram of sheaves on $Y_1(N,p)$

(2.3.2.3)
$$\begin{tikzcd} \omega^{\otimes 2}_{\pi_2^* E_1(N)} \arrow[rr, "(1/p)\varphi^*"] \arrow[dr, "\underline{\mathrm{KS}}"'] & & \omega^{\otimes 2}_{\pi_1^* E_1(N)} \arrow[dl, "\underline{\mathrm{KS}}"] \\ & \Omega^1_{Y_1(N,p)} & \end{tikzcd}$$

commutes, where $\varphi : \pi_1^* E_1(N) \to \pi_2^* E_1(N)$ is the "universal p-isogeny" $E \to E/C$ over a "point" $(E, P, C) \in Y_1(N,p)$. Since $Y_1(N,p)$ is connected, it suffices to check this commutativity after pullback by the open immersions $\Delta^* \hookrightarrow Y_1(N,p)$ corresponding to the triple $(\underline{\mathrm{Tate}}_p^*, \zeta_N, \langle q \rangle)$ with fiber $(\mathbf{C}^\times/q^{p\mathbf{Z}}, \zeta_N, \langle q \rangle)$ over $q \in \Delta^*$. Since $\underline{\mathrm{KS}}$ commutes with base change, this local problem comes down to studying the canonical p-isogeny

$$\begin{tikzcd} \underline{\mathrm{Tate}}_p^* \arrow[rr, "\varphi"] \arrow[dr] & & \underline{\mathrm{Tate}}_1^* \arrow[dl] \\ & \Delta^* & \end{tikzcd}$$

formed by quotienting $\underline{\mathrm{Tate}}_p^*$ by "$q^{\mathbf{Z}}$" (see Lemma 2.3.2.3 below for details on an explication of φ, inducing $\mathbf{C}^\times/q^{p\mathbf{Z}} \to \mathbf{C}^\times/q^{\mathbf{Z}}$ on fibers). We want to show that

$$\underline{\mathrm{KS}}_{\underline{\mathrm{Tate}}_1^*/\Delta^*}\left(\frac{\mathrm{d}t}{t}\right)^{\otimes 2} \stackrel{?}{=} \underline{\mathrm{KS}}_{\underline{\mathrm{Tate}}_p^*/\Delta^*}\left((1/p)\varphi^*\left(\frac{\mathrm{d}t}{t}\right)^{\otimes 2}\right).$$

The left side is just $\mathrm{d}q/q$. Also, as can easily be checked on fibers, $\varphi^*(\mathrm{d}t/t) = \mathrm{d}t/t$ (see Lemma 2.3.2.3). Thus we must show

$$\underline{\mathrm{KS}}_{\underline{\mathrm{Tate}}_p^*/\Delta^*}\left(\frac{\mathrm{d}t}{t}\right)^{\otimes 2} \stackrel{?}{=} p \cdot \frac{\mathrm{d}q}{q}.$$

More generally, as is shown in Lemma 2.3.2.3 below, $\underline{\mathrm{Tate}}_n^* \to \Delta^*$ is the *base change* of $\underline{\mathrm{Tate}}_1^* \to \Delta^*$ by the degree n covering $\Delta^* \to \Delta^*$ given by $q \mapsto q^n$ and $\mathrm{d}t/t$ is the base change of $\mathrm{d}t/t$. By the base change compatibility of the Kodaira-Spencer map, we therefore have that

$$\underline{\mathrm{KS}}_{\underline{\mathrm{Tate}}_n^*/\Delta^*}\left(\frac{\mathrm{d}t}{t}\right)^{\otimes 2}$$

is the base change by $q \mapsto q^n$ of

$$\underline{\mathrm{KS}}_{\underline{\mathrm{Tate}}_1^*/\Delta^*}(\left(\frac{\mathrm{d}t}{t}\right)^{\otimes 2} = \frac{\mathrm{d}q}{q}.$$

This in turn is just $\mathrm{d}q^n/q^n = n\mathrm{d}q/q$. Thus, (2.3.2.3) commutes as desired.

We are left with verifying that

$$\begin{tikzcd}
\mathrm{H}^0\big(X_1(N), \overline{\omega}^{\otimes(k+2)}(-\mathrm{cusps})\big) \arrow[r, hook] \arrow[d, "T_p"'] & \mathrm{H}^0\big(Y_1(N), \omega^{\otimes(k+2)}\big) \arrow[d, "\pi_2^*"] \\
\mathrm{H}^0\big(X_1(N), \overline{\omega}^{\otimes(k+2)}(-\mathrm{cusps})\big) \arrow[d, hook] & \mathrm{H}^0\big(Y_1(N,p), \omega^{\otimes(k+2)}_{\pi_2^* E_1(N)}\big) \arrow[d, "\frac{1}{p}\varphi^*"] \\
\mathrm{H}^0\big(Y_1(N), \omega^{\otimes(k+2)}\big) & \mathrm{H}^0\big(Y_1(N,p), \omega^{\otimes(k+2)}_{\pi_1^* E_1(N)}\big) \arrow[l, "\mathrm{tr}_{\pi_1}"]
\end{tikzcd}$$

commutes. We have now eliminated the Kodaira-Spencer maps and what we will do works in weight 1, as well as weights 2 and higher.

We can work with the operator

$$T_p \in \mathrm{End}\big(\mathrm{H}^0(Y_1(N), \omega^{\otimes k})\big)$$

defined via the "rule" as in Theorem 2.2.4.2, using the universal family of (smooth) elliptic curves with $\Gamma_1(N)$-structure over $Y_1(N)$. Choose $f \in \mathrm{H}^0(Y_1(N), \omega^{\otimes k})$. In the setup of Theorem 2.2.4.2, we have

$$\begin{aligned}
T_p f(E,P) &= T_p f(E,P,\omega) \cdot \omega^{\otimes k} \\
&= p^{k-1} \sum_C f\big(E/C, P \bmod C, \breve{\pi}_C^*(\omega)\big) \cdot \omega^{\otimes k} \\
&= \frac{1}{p} \sum_C f\big(E/C, P \bmod C, \breve{\pi}_C^*(\omega)\big) (p\omega)^{\otimes k}.
\end{aligned}$$

On the other hand, $p\,\omega = \pi_C^* \breve{\pi}_C^*(\omega)$, so

$$\begin{aligned}
T_p f(E,P) &= \frac{1}{p} \sum_C \pi_C^*\big(f(E/C, P \bmod C, \breve{\pi}_C^* \omega)\big) \cdot (\breve{\pi}_C^* \omega)^{\otimes k} \\
&= \frac{1}{p} \sum_C \pi_C^*\big(f(E/C, P \bmod C)\big).
\end{aligned}$$

This gives a formula for f viewed as a "1-form valued rule" rather than a "rule" with values in $\mathrm{H}^0(S, \mathcal{O}_S)$. Likewise, we have an alternate description $\langle n \rangle f(E,P) = f(E, n \cdot P)$.

Thus, we want to prove that for any $\Gamma_1(N)$-structure (E,P) over a connected base for which there is a global trivialization

$$E[p] \simeq \underline{\mathbf{Z}/p} \times \underline{\mathbf{Z}/p},$$

we have

$$\mathrm{tr}_{\pi_1} \circ \varphi^* \circ \pi_2^*(f)(E,P) \stackrel{?}{=} \sum_C \pi_C^*\big(f(E/C, P \bmod C)\big) \tag{2.3.2.4}$$

as 1-forms on E.

Now by the *definition* of π_1, we have for $g \in \mathrm{H}^0\big(Y_1(N,p), \omega^{\otimes k}_{\pi_1^* E_1(N)}\big)$ that

$$\mathrm{tr}_{\pi_1}(g)(E,P) = \sum_C g(E,P,C)$$

as 1-forms on E. By the *definition* of φ and π_2, we have

$$(\varphi^* \pi_2^* f)(E,P,C) = \pi_C^*(f(E/C, P \bmod C))$$

since φ is the "universal degree p-isogeny" on the universal elliptic curve over $Y_1(N,p)$. This verifies (2.3.2.4) and thus completes the proof. $\square$

A key lemma in the above proof was the following.

LEMMA 2.3.2.3. *For $n \geq 1$ there is a unique cartesian diagram*

$$\begin{tikzcd} \underline{\mathrm{Tate}}_n^* \arrow[r] \arrow[d] & \underline{\mathrm{Tate}}_1^* \arrow[d] \\ \Delta^* \arrow[r, "q \mapsto q^n"'] & \Delta^* \end{tikzcd}$$

inducing the identity $\mathbf{C}^\times/q^{n\mathbf{Z}} = \mathbf{C}^\times/q^{n\mathbf{Z}}$ on the fiber over $q \in \Delta^*$. The pullback of $\mathrm{d}t/t$ under this map is $\mathrm{d}t/t$.

There is also a unique étale map of elliptic curves over Δ^*

$$\begin{tikzcd} \underline{\mathrm{Tate}}_n^* \arrow[rr] \arrow[dr] & & \underline{\mathrm{Tate}}_1^* \arrow[dl] \\ & \Delta^* & \end{tikzcd}$$

realizing $\underline{\mathrm{Tate}}_1^*$ as the quotient of $\underline{\mathrm{Tate}}_n^*$ by $\underline{\mathbf{Z}/n} = \langle q \rangle \hookrightarrow \underline{\mathrm{Tate}}_n^*$ and inducing the canonical quotient map $\mathbf{C}^\times/q^{n\mathbf{Z}} \to \mathbf{C}^\times/q^{\mathbf{Z}}$ on the fiber over $q \in \Delta^*$. The pullback of $\mathrm{d}t/t$ under this map is $\mathrm{d}t/t$.

PROOF. The uniqueness is clear in each case, as is the computation of the pullback of $\mathrm{d}t/t$ (as this can be checked on fibers). Existence is an easy exercise using the presentation of $\underline{\mathrm{Tate}}_d^*$ in terms of glued annuli; however one must be careful with $d = 1$. $\square$

The cohomological Hecke operators on $\widetilde{\mathrm{H}}^1(Y_1(N), \mathrm{Sym}^k \mathrm{R}^1 f_{N*}\underline{\mathbf{Z}})$ induce analogous operators on $\widetilde{\mathrm{H}}^1(Y_1(N), \mathrm{Sym}^k \mathrm{R}^1 f_{N*}\underline{\mathbf{Q}})$ by tensoring with $\mathbf{Q}$ over $\mathbf{Z}$ (cf. Lemma 1.7.7.2) and likewise with $\mathbf{C}$ in place of $\mathbf{Q}$. By the Mayer-Vietoris sequence, the group

$$\mathrm{H}^1(Y_1(N), \mathrm{Sym}^k \mathrm{R}^1 f_{N*}\underline{\mathbf{Z}})$$

is finitely generated, so $\widetilde{\mathrm{H}}^1(Y_1(N), \mathrm{Sym}^k \mathrm{R}^1 f_{N*}\underline{\mathbf{Z}})$ is finitely generated as well. In fact, we have the following more precise result.

LEMMA 2.3.2.4. *The group $\widetilde{\mathrm{H}}^1(Y_1(N), \mathrm{Sym}^k \mathrm{R}^1 f_{N*}\underline{\mathbf{Z}})$ is torsion-free and the group $\mathrm{H}^1(Y_1(N), \mathrm{Sym}^k \mathrm{R}^1 f_{N*}\underline{\mathbf{Z}})$ has no p-torsion for p not dividing N. When p divides N, this group has no p-torsion if $k = 0$, and for $k > 0$ the p-torsion subgroup is 1-dimensional, isomorphic to*

$$\mathrm{H}^0\big(Y_1(N), \mathrm{Sym}^k \mathrm{R}^1 f_{N*}\underline{\mathbf{Z}/p}\big) \simeq \mathbf{F}_p[e_1, e_2]_k^{\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)} = \mathbf{F}_p e_1^k.$$

REMARK 2.3.2.5. The method of proof of Theorem 1.2.1.6 shows that the sheaf $\mathrm{R}^1 f_{N*}\underline{\mathbf{Z}/p}$ is a local system, although this is clear anyway since f_N is topologically a locally trivial fibration.

PROOF. We claim that the complex of local systems

$$0 \to \mathrm{Sym}^k \mathrm{R}^1 f_{N*}\underline{\mathbf{Z}} \xrightarrow{p} \mathrm{Sym}^k \mathrm{R}^1 f_{N*}\underline{\mathbf{Z}} \to \mathrm{Sym}^k \mathrm{R}^1 f_*\underline{\mathbf{Z}/p} \to 0 \tag{2.3.2.5}$$

is a short exact sequence. By proper base change, it suffices to check this on fibers, where it follows from the standard fact that the natural map

$$\mathrm{H}^1(X, \underline{\mathbf{Z}}) \otimes_{\mathbf{Z}} \mathbf{Z}/p \to \mathrm{H}^1(X, \underline{\mathbf{Z}/p})$$

is an isomorphism for any compact Riemann surface X.

The cohomology sequence of (2.3.2.5) identifies the p-torsion in the cohomology group $\mathrm{H}^1(Y_1(N), \mathrm{Sym}^k \mathrm{R}^1 f_{N*}\underline{\mathbf{Z}})$ with the cokernel of

$$\mathrm{H}^0\big(Y_1(N), \mathrm{Sym}^k \mathrm{R}^1 f_{N*}\underline{\mathbf{Z}}\big) \to \mathrm{H}^0\big(Y_1(N), \mathrm{Sym}^k \mathrm{R}^1 f_{N*}\underline{\mathbf{Z}/p}\big).$$

This latter map is identified with the natural map

$$\mathbf{Z}[e_1, e_2]_k^{\Gamma_1(N)} \to (\mathbf{Z}/p)[e_1, e_2]_k^{\Gamma_1(N)} \tag{2.3.2.6}$$

where $\Gamma_1(N) \subseteq \mathrm{SL}_2(\mathbf{Z})$ acts in the standard way on $e_1 = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$, $e_2 = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$; for a ring R, we write $R[x_1, \ldots, x_n]_k$ for the R-submodule of homogeneous forms of degree k. When $k = 0$, (2.3.2.6) is surjective, completing the proof in that case.

We now assume that $k > 0$. For any field F, the invariants in $F[e_1, e_2]_k$ under $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ is the space $F \cdot e_1^k$. Indeed, if $k = 1$ (or 0) this is clear. In general, we induct on k as follows. Suppose

$$f \in F[e_1, e_2]_k^{\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)},$$

so $f(e_1, e_1 + e_2) = f(e_1, e_2)$ for a homogeneous f of degree $k > 1$. Let ∂_i denote the partial derivative in the i^{th} variable. Then for our f we have $(\partial_2 f)(e_1, e_1 + e_2) = (\partial_2 f)(e_1, e_2)$, so by induction $(\partial_2 f)(e_1, e_2) \in F \cdot e_1^{k-1}$. This settles the case of F with characteristic 0. If F has positive characteristic p, then we can only say that $f = h(e_1, e_2^p)$, so $h(e_1, e_2^p) = h(e_1, e_1^p + e_2^p)$. Thus,

$$(\partial_1 h)(e_1, e_2^p) = (\partial_1 h)(e_1, e_1^p + e_2^p).$$

That is, $(\partial_1 f)(e_1, e_2) = (\partial_1 f)(e_1, e_1 + e_2)$, so by induction we have $\partial_1 f \in F \cdot e_1^{k-1}$. Thus, $f = ae_1^k + g(e_1^p, e_2^p)$ and $g(e_1, e_2)$ satisfies the same properties as f. The polynomial g is homogeneous of degree $\frac{k}{p}$, so induction now concludes the argument.

It follows that

$$\mathbf{Z}[e_1, e_2]_k^{\Gamma_1(N)} \subseteq \mathbf{Q}[e_1, e_2]_k^{\Gamma_1(N)} \subseteq \mathbf{Q} \cdot e_1^k,$$

yet $\left(\begin{smallmatrix}1&0\\N&1\end{smallmatrix}\right) \in \Gamma_1(N)$ does *not* fix e_1^k for $k > 0$. Thus, $\mathbf{Z}[e_1, e_2]_k^{\Gamma_1(N)} = 0$, so the p-torsion in $\mathrm{H}^1(Y_1(N), \mathrm{Sym}^k \mathrm{R}^1 f_{N*}\underline{\mathbf{Z}})$ is identified with $\mathbf{F}_p[e_1, e_2]_k^{\Gamma_1(N)}$ for all $k > 0$. Furthermore, $\mathbf{F}_p[e_1, e_2]_k^{\Gamma_1(N)} \subseteq \mathbf{F}_p \cdot e_1^k$. If p does not divide N, then $\Gamma_1(N) \to \mathrm{SL}_2(\mathbf{F}_p)$ is surjective and $\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$ does *not* fix e_1^k, so $\mathbf{F}_p[e_1, e_2]_k^{\Gamma_1(N)} = 0$ in this case. When p divides N, we obviously have $\mathbf{F}_p[e_1, e_2]_k^{\Gamma_1(N)} = \mathbf{F}_p e_1^k$. This settles the assertions about ordinary cohomology.

It remains to check that when $k > 0$ and $p|N$, the injection

$$\mathbf{F}_p e_1^k = \mathrm{H}^0\big(Y_1(N), \mathrm{Sym}^k \mathrm{R}^1 f_{N*}\underline{\mathbf{Z}/p}\big) \hookrightarrow \mathrm{H}^1\big(Y_1(N), \mathrm{Sym}^k \mathrm{R}^1 f_{N*}\underline{\mathbf{Z}}\big)$$

meets $\widetilde{\mathrm{H}}^1(Y_1(N), \mathrm{Sym}^k \mathrm{R}^1 f_{N*}\underline{\mathbf{Z}})$ precisely at $\{0\}$. In other words, we want to find a cusp $x \in X_1(N)$ such that the image of e_1^k in $\mathrm{H}^1(\Delta_x^*, \mathrm{Sym}^k \mathrm{R}^1 f_{N*}\underline{\mathbf{Z}})$ is not zero. Said differently, we claim that

$$\mathrm{H}^0(\Delta_x^*, \mathrm{Sym}^k \mathrm{R}^1_{N*}\underline{\mathbf{Z}}) \to \mathrm{H}^0(\Delta_x^*, \mathrm{Sym}^k \mathrm{R}^1 f_{N*}\underline{\mathbf{Z}/p}) \tag{2.3.2.7}$$

can not hit e_1^k for all cusps x.

Concretely, (2.3.2.7) is just

$$\mathbf{Z}[e_1, e_2]_k^{\mathrm{Stab}_{\Gamma_1(N)}(x)} \to \mathbf{F}_p[e_1, e_2]_k^{\mathrm{Stab}_{\Gamma_1(N)}(x)}. \tag{2.3.2.8}$$

Consider the cusp $x = 0$ with stabilizer $\left(\begin{smallmatrix} 1 & 0 \\ N & 1 \end{smallmatrix}\right)^{\mathbf{Z}}$. Clearly $\mathbf{Z}[e_1, e_2]_k^{\mathrm{Stab}_{\Gamma_1(N)}(x)}$ consists of all homogeneous f of degree k with $f(e_1 + Ne_2, e_2) = f(e_1, e_2)$. A variation of our earlier argument shows that such f lie in $\mathbf{Z} \cdot e_2^k$, so (2.3.2.8) cannot hit e_1^k in characteristic p. $\square$

We arrive now at the following interesting fact.

THEOREM 2.3.2.6. *For all primes p not dividing N, the complex*

$$0 \to \widetilde{\mathrm{H}}^1\big(Y_1(N), \mathrm{Sym}^k \mathrm{R}^1 f_{N*}\underline{\mathbf{Z}}\big) \xrightarrow{p} \widetilde{\mathrm{H}}^1\big(Y_1(N), \mathrm{Sym}^k \mathrm{R}^1 f_{N*}\underline{\mathbf{Z}}\big) \longrightarrow \widetilde{\mathrm{H}}^1\big(Y_1(N), \mathrm{Sym}^k \mathrm{R}^1 f_{N*}\underline{\mathbf{Z}/p}\big) \to 0 \tag{2.3.2.9}$$

is a short exact sequence.

PROOF. By Lemma 2.3.2.4, we have the injectivity on the left. To verify the surjectivity on the right, we will check that the analogous complex in compact cohomology is short exact. To begin, since H^0_c vanishes on local systems on the non-compact $Y_1(N)$, we obtain a left exact sequence

$$0 \to \mathrm{H}^1_c(Y_1(N), \mathrm{Sym}^k \mathrm{R}^1 f_{N*}\underline{\mathbf{Z}}) \xrightarrow{p} \mathrm{H}^1_c(Y_1(N), \mathrm{Sym}^k \mathrm{R}^1 f_{N*}\underline{\mathbf{Z}}) \to \mathrm{H}^1_c(Y_1(N), \mathrm{Sym}^k \mathrm{R}^1 f_{N*}\underline{\mathbf{Z}/p}). \tag{2.3.2.10}$$

To check exactness on the right, it is enough to check that the $\mathbf{Z}$-rank of the finite free $\mathbf{Z}$-module $\mathrm{H}^1_c(Y_1(N), \mathrm{Sym}^k \mathrm{R}^1 f_{N*}\underline{\mathbf{Z}})$ equals the $(\mathbf{Z}/p)$-rank of the $(\mathbf{Z}/p)$-vector space $\mathrm{H}^1_c(Y_1(N), \mathrm{Sym}^k \mathrm{R}^1 f_{N*}\underline{\mathbf{Z}/p})$. By Lemma 1.7.7.2, this $\mathbf{Z}$-rank is just the $\mathbf{Q}$-dimension of $\mathrm{H}^1_c(Y_1(N), \mathrm{Sym}^k \mathrm{R}^1 f_{N*}\underline{\mathbf{Q}})$. We now use Poincaré duality for local systems on $Y_1(N)$ consisting of finite dimensional vector spaces over a *field*. Since the Euler characteristic of $Y_1(N)$ is independent of the coefficient field, the argument of §1.7.7 and the fact that H^0_c vanishes on local systems reduces us to proving equality of the ranks of

$$\mathrm{H}^0(Y_1(N), \mathrm{Sym}^k \mathrm{R}^1 f_{N*}\underline{\mathbf{Q}})$$

and

$$\mathrm{H}^0(Y_1(N), \mathrm{Sym}^k \mathrm{R}^1 f_{N*}\underline{\mathbf{Z}/p}).$$

This comparison follows by the argument of Lemma 2.3.2.4, since p does not divide N. This completes the proof that (2.3.2.10) is short exact, and thus proves surjectivity on the right side of (2.3.2.9).

To verify the exactness in the middle of (2.3.2.9), we have to check that

$$\dim_{\mathbf{Q}} \widetilde{\mathrm{H}}^1\big(Y_1(N), \mathrm{Sym}^k \mathrm{R}^1 f_{N*}\underline{\mathbf{Q}}\big) = \dim_{\mathbf{F}_p} \widetilde{\mathrm{H}}^1\big(Y_1(N), \mathrm{Sym}^k \mathrm{R}^1 f_{N*}\underline{\mathbf{Z}/p}\big).$$

Following the method of calculation of $\dim \widetilde{\mathrm{H}}^1$ in the proof of the Eichler-Shimura isomorphism, this will follow from (1.7.4.1) provided that we have surjectivity there on the right. This surjectivity amounts to $\mathrm{H}^2_c(Y_1(N), \mathcal{F}_k) \to \mathrm{H}^2(Y_1(N), \mathcal{F}_k)$ being injective for $\mathcal{F}_k = \mathrm{Sym}^k \mathrm{R}^1 f_{N*}\underline{F}$ with $F = \mathbf{Q}$ or $F = \mathbf{F}_p$. But by Poincaré duality for local systems we have

$$\mathrm{H}^2_c(Y_1(N), \mathcal{F}_k) = \mathrm{H}^0(Y_1(N), \mathcal{F}_k) = 0$$

since p does not divide N. $\square$

2.3.3. Products on cohomology. The final classical structure that we want to make cohomological is the Petersson product, at least up to some universal non-zero constant in $\mathbf{C}^\times$ depending only on k and Γ. (This constant comes from viewing the "true" Petersson product as normalized by $\mathrm{vol}(\Gamma\backslash\mathfrak{h})^{-1}$.) This part has nothing to do with Hecke operators, so it can be established in the same general setting as Theorem 1.7.1.1.

Let $\Gamma \subseteq \mathrm{SL}_2(\mathbf{R})$ be as in Theorem 1.7.1.1. Let $\mathcal{F}, \mathcal{G}, \mathcal{H}$ be abelian sheaves on X_Γ and let $\mathcal{F} \otimes_{\underline{\mathbf{Z}}} \mathcal{G} \to \mathcal{H}$ be a bilinear pairing. We have a "commutative diagram" of cup products

$$\begin{array}{ccccc}
\mathrm{H}^i_{\mathrm{c}}(X_\Gamma, \mathcal{F}) & \times & \mathrm{H}^j(X_\Gamma, \mathcal{G}) & \longrightarrow & \mathrm{H}^{i+j}_{\mathrm{c}}(X_\Gamma, \mathcal{H}) \\
\downarrow & & \uparrow & & \| \\
\mathrm{H}^i(X_\Gamma, \mathcal{F}) & \times & \mathrm{H}^j_{\mathrm{c}}(X_\Gamma, \mathcal{G}) & \longrightarrow & \mathrm{H}^{i+j}_{\mathrm{c}}(X_\Gamma, \mathcal{H}) \\
\uparrow & & \uparrow & & \| \\
\mathrm{H}^i_{\mathrm{c}}(X_\Gamma, \mathcal{F}) & \times & \mathrm{H}^j_{\mathrm{c}}(X_\Gamma, \mathcal{G}) & \longrightarrow & \mathrm{H}^{i+j}_{\mathrm{c}}(X_\Gamma, \mathcal{H})
\end{array}$$

It follows that there is a *well-defined* pairing

$$\widetilde{\mathrm{H}}^i(X_\Gamma, \mathcal{F}) \times \widetilde{\mathrm{H}}^j(X_\Gamma, \mathcal{G}) \to \mathrm{H}^{i+j}_{\mathrm{c}}(X_\Gamma, \mathcal{H})$$

compatible with all three rows above. We will be especially interested in this pairing when $\mathcal{F}$ is a local system of finite dimensional $\mathbf{Q}$-modules, $\mathcal{G} = \mathcal{F}^\vee = \mathcal{H}om(\mathcal{F}, \underline{\mathbf{Q}}(1))$ and $\mathcal{H} = \underline{\mathbf{Q}}(1)$ with the canonical pairing $\mathcal{F} \otimes \mathcal{G} \to \mathcal{H}$. In this case, Poincaré duality for local systems ensures that the pairings

$$\mathrm{H}^i_{\mathrm{c}}(X_\Gamma, \mathcal{F}) \times \mathrm{H}^{2-i}(X_\Gamma, \mathcal{F}^\vee) \to \mathrm{H}^2_{\mathrm{c}}(X_\Gamma, \underline{\mathbf{Q}}(1)) \simeq \mathbf{Q}$$

and

$$\mathrm{H}^i(X_\Gamma, \mathcal{F}) \times \mathrm{H}^{2-i}_{\mathrm{c}}(X_\Gamma, \mathcal{F}^\vee) \to \mathrm{H}^2_{\mathrm{c}}(X_\Gamma, \underline{\mathbf{Q}}(1)) \simeq \mathbf{Q}$$

are perfect pairings of finite dimensional $\mathbf{Q}$-vector spaces. Since (as we have already noted) these pairings are compatible with the natural maps from compactly support cohomology to ordinary cohomology, it follows that the induced pairing

$$\widetilde{\mathrm{H}}^i(X_\Gamma, \mathcal{F}) \times \widetilde{\mathrm{H}}^{2-i}(X_\Gamma, \mathcal{F}) \to \mathrm{H}^2_{\mathrm{c}}(X_\Gamma, \underline{\mathbf{Q}}(1)) \simeq \mathbf{Q}$$

is a perfect pairing and likewise with $\mathbf{Q}$ replaced by *any* field.

An interesting example for us is the local system $\mathcal{U}^k_\Gamma$ of $\underline{\mathbf{R}}$-modules. This has a natural self-duality induced from one on $\mathcal{U}_\Gamma$, via descent as in the proof of Lemma 1.7.6.1. More precisely, using the "relative integration" map as at the end of Theorem 1.2.2.2, the pairing

$$V_1 \otimes V_1 \to \mathbf{R} \tag{2.3.3.1}$$

from (1.7.6.1) is the *negative* of the cup product pairing

$$\mathrm{R}^1 f_{X*}\underline{\mathbf{R}} \otimes_{\underline{\mathbf{R}}} \mathrm{R}^1 f_{X*}\underline{\mathbf{R}} \to \mathrm{R}^2 f_{X*}\underline{\mathbf{R}} \overset{\int}{\simeq} \underline{\mathbf{R}},$$

as can be checked on fibers over X (where it just says that $\gamma_1^\vee \cup \gamma_2^\vee = -1$; this is forced by i-permissibility). In order to make things independent of $i = \sqrt{-1} \in \mathbf{C}$ and hence more "algebraic", we should divide the integral by $2\pi i$. That is, instead of (2.3.3.1) we should consider the scaled pairing

$$V_1 \otimes V_1 \to \underline{\mathbf{R}}(-1).$$

Carrying out descent and passing to kth tensor powers yields from this a perfect pairing

$$\mathcal{U}_\Gamma^k \otimes_{\underline{\mathbf{R}}} \mathcal{U}_\Gamma^k \to \underline{\mathbf{R}}(-k) \tag{2.3.3.2}$$

on X_Γ. Thus, there is a perfect pairing

$$\textstyle\frac{k}{\Gamma}\langle\cdot,\cdot\rangle : \widetilde{\mathrm{H}}^1(X_\Gamma, \mathcal{U}_\Gamma^k) \otimes_{\mathbf{R}} \widetilde{\mathrm{H}}^1(X_\Gamma, \mathcal{U}_\Gamma^k) \xrightarrow{\cup} \mathrm{H}^2_c(X_\Gamma, \underline{\mathbf{R}}(-k)) \simeq \mathbf{R}(-k-1). \tag{2.3.3.3}$$

When Γ is a subgroup of finite index in $\mathrm{SL}_2(\mathbf{Z})$, (2.3.3.2) is just the pairing induced on kth symmetric powers of the extension of scalars from $\underline{\mathbf{Z}}$ to $\underline{\mathbf{R}}$ of the pairing

$$\mathrm{R}^1 f_{\Gamma *}\underline{\mathbf{Z}} \otimes_{\underline{\mathbf{Z}}} \mathrm{R}^1 f_{\Gamma *}\underline{\mathbf{Z}} \xrightarrow{\cup} \mathrm{R}^2 f_{\Gamma *}\underline{\mathbf{Z}} \simeq \underline{\mathbf{Z}}(-1).$$

This can be used to define a pairing

$$\textstyle\frac{k}{\Gamma}\langle\cdot,\cdot\rangle : \widetilde{\mathrm{H}}^1(X_\Gamma, \mathrm{Sym}^k \mathrm{R}^1 f_{\Gamma *}\underline{\mathbf{Z}}) \otimes_{\mathbf{Z}} \widetilde{\mathrm{H}}^1(X_\Gamma, \mathrm{Sym}^k \mathrm{R}^1 f_{\Gamma *}\underline{\mathbf{Z}}) \to \mathbf{Z}(-k-1) \tag{2.3.3.4}$$

which recovers (2.3.3.3).

We want to show that the Eichler-Shimura isomorphism essentially identifies (2.3.3.3) with the (unnormalized) Petersson product on $S_{k+2}(\Gamma, \mathbf{C})$; recall that the Petersson pairing is defined on modular forms by

$$\langle f, g\rangle_\Gamma = \int_{X_\Gamma} f(z)\overline{g}(z) y^{k+2} \frac{\mathrm{d}x\,\mathrm{d}y}{y^2}.$$

This was observed in a crude form in the proof of Lemma 1.7.6.1.

Note that the above pairings $\frac{k}{\Gamma}\langle\cdot,\cdot\rangle$ (over $\mathbf{R}$ or over $\mathbf{Z}$) are symmetric when k is odd and are alternating when k is even. Indeed, the cup product is alternating and the pairing on $\mathcal{U}_\Gamma$ (or on $\mathrm{R}^1 f_{\Gamma *}\underline{\mathbf{Z}}$) is alternating (as it comes from a cup product), so that the total sign of the pairing is $(-1)^{k+1}$.

THEOREM 2.3.3.1. *For Γ as above, consider the $\mathbf{C}$-linear Eichler-Shimura isomorphism*

$$\mathrm{sh}_\Gamma : S_{k+2}(\Gamma, \mathbf{C}) \oplus \overline{S_{k+2}(\Gamma, \mathbf{C})} \xrightarrow{\simeq} \widetilde{\mathrm{H}}^1(X_\Gamma, \mathcal{U}_\Gamma^k) \otimes_{\mathbf{R}} \mathbf{C} \simeq \widetilde{\mathrm{H}}^1(X_\Gamma, \mathcal{U}_\Gamma^k(\mathbf{C})).$$

The $\mathbf{C}$-bilinear pairing $\frac{k}{\Gamma}\langle\cdot,\cdot\rangle$ on the right side makes each of the subspaces

$$S_{k+2}(\Gamma, \mathbf{C}), \quad \overline{S_{k+2}(\Gamma, \mathbf{C})}$$

annihilate itself and in general

$$\left\langle \mathrm{sh}_\Gamma(f_1 + \overline{g}_1), \mathrm{sh}_\Gamma\, {\textstyle\frac{k}{\Gamma}}(f_2 + \overline{g}_2)\right\rangle = \frac{c_k}{(2\pi i)^{k+1}}\left(\langle f_1, g_2\rangle_\Gamma + (-1)^{k+1}\langle f_2, g_1\rangle_\Gamma\right)$$

where $\langle\cdot,\cdot\rangle_\Gamma$ is the Petersson product and $c_k = (-1)^k(4\pi)^{k+1}(2\pi i)^{k+1}$.

In the above formula, we adopt the notation $\overline{v} = 1 \otimes v \in \mathbf{C} \otimes_{\mathbf{C}} V$ for an element v in a $\mathbf{C}$-vector space V, where the base change is by complex conjugation. In particular, when V is a space of holomorphic functions, $\overline{v}$ should not be confused with the complex conjugate function (which is almost never holomorphic). The context should make it clear whether the notation $\overline{f}$ denotes a conjugate function or an element $1 \otimes f$ in a conjugate vector space.

PROOF. By the definition of sh_Γ, we have $\mathrm{sh}_\Gamma(\overline{g}) = \overline{\mathrm{sh}_\Gamma(g)}$. Thus, the alternating and symmetric properties of $\frac{k}{\Gamma}\langle\cdot,\cdot\rangle$ reduce us to showing that

- each of $S_{k+2}(\Gamma, \mathbf{C})$ and $\overline{S_{k+2}(\Gamma, \mathbf{C})}$ annihilates itself;
- $\frac{k}{\Gamma}\langle \mathrm{sh}_\Gamma(f), \mathrm{sh}_\Gamma(g)\rangle = \frac{c_k}{(2\pi i)^{k+1}}\langle f, g\rangle_\Gamma$.

Note that

$$\begin{aligned}(2.3.3.5)\qquad \mathrm{sh}_\Gamma(f+\overline{g}) &= \mathrm{sh}_\Gamma(f)+\overline{\mathrm{sh}_\Gamma(g)}\\ &= \mathrm{Re}\big(\mathrm{sh}_\Gamma(f+g)\big)+i\,\mathrm{Re}\big(\mathrm{sh}_\Gamma(-i(g-f))\big).\end{aligned}$$

Define

$${}_k(f,g) = {}^k_\Gamma\,\langle \mathrm{Re}(\mathrm{sh}_\Gamma(f)), \mathrm{Re}(\mathrm{sh}_\Gamma(g))\rangle$$

so by (2.3.3.5) we have

(2.3.3.6)

$$\begin{aligned}{}^k_\Gamma\,\langle \mathrm{sh}_\Gamma(f_1+\overline{g}_1), \mathrm{sh}_\Gamma(f_2+\overline{g}_2)\rangle &= {}_k\big(f_1+g_1, f_2+g_2\big) - {}_k\big(i(g_1-f_1), i(g_2-f_2)\big) +\\ &\quad {}_k\big(i(g_1-f_1), f_2+g_2\big)i + {}_k\big(f_1+g_1, i(g_2-f_2)\big)i.\end{aligned}$$

We claim that

$$(2.3.3.7)\qquad {}_k(if,g) = {}_k(f,-ig);$$

this simplifies (2.3.3.6) to

$$2\big({}_k(f_1,g_2)+{}_k(g_1,f_2)\big)+2i\big({}_k(ig_1,f_2)+{}_k(f_1,ig_2)\big),$$

which shows that each of $S_{k+2}(\Gamma,\mathbf{C})$ and $\overline{S_{k+2}(\Gamma,\mathbf{C})}$ annihilates itself (i.e., we get 0 if $g_1=g_2=0$ or $f_1=f_2=0$). We also claim that

$$(2.3.3.8)\qquad {}^k_\Gamma\,\langle \mathrm{sh}_\Gamma(f), \mathrm{sh}_\Gamma(\overline{g})\rangle = 2\cdot{}_k(f,g)+2\cdot{}_k(f,ig)i.$$

If we could show that

$$(2.3.3.9)\qquad {}_k(f,g) = \begin{cases}\frac{c_k}{2(2\pi i)^k}\,\mathrm{Re}\,{}^k_\Gamma\,\langle f,g\rangle & k \text{ odd}\\ \frac{c_k}{2(2\pi i)^k}\,i\,\mathrm{Im}\,{}^k_\Gamma\,\langle f,g\rangle & k \text{ even}\end{cases}$$

then we would immediately obtain (2.3.3.7) and one checks by an easy calculation that it also yields (2.3.3.8).

Using Lemma 1.7.2.1 and the calculations of Lemma 1.7.6.1, for

$$\omega_f = (2\pi i)^{k+1}(ze_1+e_2)^k f(z)\mathrm{d}z$$

we compute

$$\begin{aligned}{}_k(f,g) &= \frac{1}{(2\pi i)^{k+1}}\int_{X_\Gamma} T_\Gamma\big(\mathrm{Re}(\omega_f), \mathrm{Re}(\omega_\Gamma)\big)\\ &= \frac{1}{(2\pi i)^{k+1}}\int_{X_\Gamma}\frac{(2\pi i)^{2(k+1)}}{4}\big(\det\left(\begin{smallmatrix} z & \overline{z}\\ 1 & 1\end{smallmatrix}\right) f\overline{g}\,\mathrm{d}z\mathrm{d}\overline{z} + \det\left(\begin{smallmatrix} \overline{z} & z\\ 1 & 1\end{smallmatrix}\right)\overline{f}g\,\mathrm{d}\overline{z}\mathrm{d}z\big)\\ &= \frac{-(2\pi i)^{k+1}(2i)^{k+1}}{2}\int_{X_\Gamma}\frac{f\overline{g}+(-1)^{k+1}\overline{f}g}{2}y^k\,\mathrm{d}x\mathrm{d}y.\end{aligned}$$

Since

$$\frac{f\overline{g}+(-1)^{k+1}\overline{f}g}{2} = \begin{cases}\mathrm{Re}(f\overline{g}) & k \text{ odd};\\ i\,\mathrm{Im}(f\overline{g}) & k \text{ even};\end{cases}$$

and $-(2\pi i)^{k+1}(2i)^{k+1} = (-1)^k(4\pi)^{k+1}$, (2.3.3.9) is proved. $\square$

2.3.4. Commutativity of the Hecke algebra. In the classical theory, it is shown that the operators T_p and $\langle n\rangle$ all commute and that their adjoints with respect to the Petersson product are $\langle p\rangle T_p$ (if p does not divide N) and $\langle n^{-1}\rangle$, respectively. We want to recover these results using integral cohomology and Theorems 2.3.2.1 and 2.3.3.1. To this end, for $N \geq 5$ we introduce the important group

$$ {}^k_N W = \widetilde{\mathrm{H}}^1(Y_1(N), \mathrm{Sym}^k\, \mathrm{R}^1 f_{N*}\underline{\mathbf{Z}}) $$

which is a finitely generated *torsion-free* $\mathbf{Z}$-module (by Lemma 2.3.2.4). Let $\mathbf{T}_1(N)$ be the subring of $\mathrm{End}_{\mathbf{Z}}({}^k_N W)$ generated by the T_p (for all p) and the $\langle n\rangle$ (for all $n \in (\mathbf{Z}/N)^\times$); note that $\mathbf{T}_1(N)$ depends on k and is a finite flat $\mathbf{Z}$-algebra (as it lives inside of a finite torsion-free $\mathbf{Z}$-module). We have the pairing

$$ {}^k_N(\cdot,\cdot) : {}^k_N W \otimes_{\mathbf{Z}} {}^k_N W \to \mathbf{Z}(-k-1) \tag{2.3.4.1} $$

from (2.3.3.4). Theorem 1.7.1.1 gives us a $\mathbf{C}$-linear isomorphism

$$ S_{k+2}(\Gamma_1(N),\mathbf{C}) \oplus \overline{S_{k+2}(\Gamma_1(N),\mathbf{C})} \xrightarrow{\simeq} {}^k_N W \otimes_{\mathbf{Z}} \mathbf{C}, $$

arising from an $\mathbf{R}$-linear isomorphism $S_{k+2}(\Gamma_1(N),\mathbf{C}) \simeq {}^k_N W \otimes_{\mathbf{Z}} \mathbf{R}$. By Theorem 2.3.2.1 this isomorphism identifies the cohomologically defined ring $\mathbf{T}_1(N)$ with the classical Hecke ring acting on the modular forms $S_{k+2}(\Gamma_1(N),\mathbf{C})$, since the map

$$ \mathrm{End}_{\mathbf{Z}}({}^k_N W) \to \mathrm{End}_{\mathbf{C}}({}^k_N W \otimes_{\mathbf{Z}} \mathbf{C}) $$

is *injective*. It follows that in order to see that $\mathbf{T}_1(N)$ is commutative, one can appeal to the classical theory. We will give direct cohomological proofs below anyway.

Since $\mathbf{Z}(-k-1) \to \mathbf{C}(-k-1)$ is injective, by Theorem 2.3.3.1 in order to establish adjointness results with respect to the Petersson product over $\mathbf{C}$ it is *equivalent* to show that for $p \nmid N$ and $n \in (\mathbf{Z}/N)^\times$,

$$ {}^k_N(T_p x, y) = {}^k_N(x, \langle p\rangle T_p y), $$
$$ {}^k_N(\langle n\rangle x, y) = {}^k_N(x, \langle n^{-1}\rangle y) $$

in $\mathbf{Z}(-k-1)$. Since $I_{n_1} \circ I_{n_2} = I_{n_1 n_2}$, we must have $\langle n_1\rangle\langle n_2\rangle = \langle n_1 n_2\rangle$, which implies that the $\langle n\rangle$ all commute with each other. Furthermore, since pullback by the automorphism I_n induces the trivial automorphism of $\mathrm{H}^2_c(X_\Gamma, \underline{\mathbf{Z}}) \simeq \mathrm{H}_0(X_\Gamma, \mathbf{Z})$, the compatibility of cup product and pullbacks implies that

$$ {}^k_N(\langle n\rangle x, y) = {}^k_N(x, \langle n\rangle^{-1} y) = {}^k_N(x, \langle n^{-1}\rangle y). $$

The study of the T_p (and of their interaction with the $\langle n\rangle$) is somewhat more delicate. The adjointness for T_p will be addressed in Theorem 2.3.7.3.

THEOREM 2.3.4.1. *The finite flat* $\mathbf{Z}$*-algebra* $\mathbf{T}_1(N) \subseteq \mathrm{End}_{\mathbf{Z}}({}^k_N W)$ *is commutative.*

PROOF. We have already seen that the $\langle n\rangle$ commute. In order to show that T_p and $\langle n\rangle$ also commute, note that $(E,P,C) \to (E, n\cdot P, C)$ defines an automorphism $I_{n,p}$ of $Y_1(N,p)$ compatible with the automorphism I_n of $Y_1(N)$ via both degeneracy maps $\pi_1 : Y_1(N,p) \to Y_1(N)$ and $\pi_2 : Y_1(N,p) \to Y_1(N)$. Since φ is the "universal p-isogeny" $\varphi : E \to E/C$ over $Y_1(N,p)$ and $\varphi(nP) = n\cdot\varphi(P)$, it is straightforward to verify the formal calculations

$$ \varphi^*\pi_2^* I_n^* = \varphi^* I_{n,p}^* \pi_2^* = I_{n,p}^* \varphi^* \pi_2^* $$

via the appropriate commutative diagrams of relative elliptic curves.

Since T_p is defined as $\mathrm{tr}_{\pi_1}\varphi^*\pi_2^*$, to show that T_p and $\langle n\rangle$ commute, it remains to show that

$$\mathrm{tr}_{\pi_1}\circ I_{n,p}^* = I_n^*\circ\mathrm{tr}_{\pi_1}\,.$$

Since $\pi_1\circ I_{n,p} = I_n\circ\pi_1$, this is easily reduced to the claim that for *any* abelian sheaf $\mathcal{F}$ on $Y_1(N)$, the diagram

$$\begin{tikzcd}
\mathrm{H}^1(Y_1(N,p),\pi_1^*\mathcal{F}) \arrow[r, "\mathrm{tr}_{\pi_1}"] \arrow[d] & \mathrm{H}^1(Y_1(N),\mathcal{F}) \arrow[dd] \\
\mathrm{H}^1(Y_1(N,p),I_{n,p}^*\pi_1^*\mathcal{F}) \arrow[d, equal] & \\
\mathrm{H}^1(Y_1(N,p),\pi_1^*I_n^*\mathcal{F}) \arrow[r, "\mathrm{tr}_{\pi_1}"] & \mathrm{H}^1(Y_1(N),I_n^*\mathcal{F})
\end{tikzcd}$$

commutes. Since the commutative diagram

$$\begin{tikzcd}
Y_1(N,p) \arrow[r, "I_{n,p}", "\simeq"'] \arrow[d, "\pi_1"'] & Y_1(N,p) \arrow[d, "\pi_1"] \\
Y_1(N) \arrow[r, "\simeq", "I_n"'] & Y_1(N)
\end{tikzcd}$$

is *cartesian* (as the columns are covering maps of the same degree) it suffices to prove that for any cartesian diagram of topological spaces

$$\begin{tikzcd}
X' \arrow[r, "f"] \arrow[d, "\pi'"'] & X \arrow[d, "\pi"] \\
Y \arrow[r, "g"'] & Y
\end{tikzcd}$$

with π a covering map of degree d, the diagram of δ-functors

$$\begin{tikzcd}
\mathrm{H}^*(X,\pi^*(\cdot)) \arrow[r, "\mathrm{tr}_\pi"] \arrow[d, "f^*"'] & \mathrm{H}^*(Y,\cdot) \arrow[dd, "g^*"] \\
\mathrm{H}^*(X',f^*\pi^*(\cdot)) \arrow[d, equal] & \\
\mathrm{H}^*(X',\pi'^*g^*(\cdot)) \arrow[r, "\mathrm{tr}_{\pi'}"'] & \mathrm{H}^*(Y',g^*(\cdot))
\end{tikzcd}$$

commutes. For H^0 this can be checked locally on Y, so we may reduce to the *obvious* case where π is the trivial degree d covering. Given this, for the general case, it suffices to prove that the δ-functor $\mathrm{H}^*(X,\pi^*\cdot)$ is erasable (and hence universal). Since π_* is exact, by the Leray spectral sequence we have $\mathrm{H}^*(X,\cdot)\simeq\mathrm{H}^*(Y,\pi_*\cdot)$, so it is enough to prove that for a flabby sheaf $\mathcal{F}$ on Y, the sheaf $\pi_*\pi^*\mathcal{F}$ is flabby.

For this, let $\{U_i\}$ be a covering of Y such that π is a trivial covering over U_i. The restriction $\pi_*\pi^*\mathcal{F}|_{U_i}$ is obviously flabby. By Zorn's lemma it is easy to see that a locally flabby sheaf $\mathcal{G}$ on a topological space is flabby (for $s\in\mathcal{G}(U)$, consider pairs (t, V) where V is an open set containing U and $t \in \mathcal{G}(V)$ extends s). This completes the proof that T_p and $\langle n \rangle$ commute.

Finally, we need to show that $T_p T_l = T_l T_p$ for distinct primes p and l. This is a simple generalization of the argument above, by using the diagram

$$\begin{tikzcd}
 & & Y_1(N,pl) \arrow[dl, "q_1"'] \arrow[dr, "q_2"] & & \\
 & Y_1(N,p) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & & Y_1(N,l) \arrow[dl, "\pi_1'"'] \arrow[dr, "\pi_2'"] & \\
Y_1(N) & & Y_1(N) & & Y_1(N)
\end{tikzcd}$$

where

$$q_1(E, P, C_p, C_l) = (E, P, C_p)$$
$$q_2(E, P, C_p, C_l) = (E/C_p, P, C_l);$$

the square $\pi_1' q_2 = \pi_2 q_1$ is *cartesian* by consideration of étaleness and degree. Let (E, P, C_p, C_l) denote the universal family over $Y_1(N, pl)$. Let $\Phi_p : E \to E/C_p$ and $\Phi_l : E/C_p \to E/(C_p, C_l)$ be the natural maps; $\Phi = \Phi_l \circ \Phi_p$ is the "universal degree lp" isogeny over $Y_1(N, pl)$.

We are left then with the formal calculation

$$\begin{aligned}
T_p \circ T_l &= (\mathrm{tr}_{\pi_1} \varphi^* \pi_2^*) \circ (\mathrm{tr}_{\pi_1'} \varphi'^* \pi_2'^*) \\
&= \mathrm{tr}_{\pi_1} \varphi^* \, \mathrm{tr}_{q_1} q_2^* \varphi'^* \pi_2'^* \\
&= \mathrm{tr}_{\pi_1} \mathrm{tr}_{q_1} \Phi_p^* \Phi_l^* q_2^* \pi_2'^* \\
&= \mathrm{tr}_{\pi_1 q_1} \Phi^* (\pi_2' q_2)^*.
\end{aligned}$$

The maps $\pi_1 q_1$ and $\pi_2' q_2 \Phi$ are independent of the choice of ordering of l and p. Thus $T_p T_l = T_l T_p$. We leave it to the reader to write out the diagrams of elliptic curves to justify this formal calculation. □

2.3.5. Modular forms of low level. At low level we do not have fine moduli schemes, but we do have a good theory of modular forms. We now explain how to incorporate these "low level" objects into our cohomological theory, as this is essential in what follows. For an integer M, let $\mathbf{T}_1(N)^{(M)}$ be the subring of $\mathbf{T}_1(N)$ generated by all $\langle n \rangle$ and by the T_p for p not dividing M. To set up a cohomological theory for level $N < 5$, which is *crucial* for our use of newforms later on, we need a lemma.

To motivate things, note that if d divides N and $d \geq 5$, then the natural cartesian diagram

(2.3.5.1)
$$\begin{tikzcd}
E_1(N) \arrow[r] \arrow[d, "f_N"'] & E_1(d) \arrow[d, "f_d"] \\
Y_1(N) \arrow[r, "\pi"'] & Y_1(d)
\end{tikzcd}$$

corresponding to the map $(E, P) \to (E, \frac{N}{d}P)$ induces via pullback a natural map ${}^k_d W \to {}^k_N W$.

LEMMA 2.3.5.1. *The natural map ${}^k_dW \to {}^k_NW$ is injective and is compatible with the actions of $\langle n\rangle$ for all $n \in (\mathbf{Z}/N)^\times$ and with the T_p for p not dividing N and for p dividing d. In particular, if d^∞ denotes the largest divisor of N which is a product of primes dividing d, then there is a well-defined map of rings*

$$\mathbf{T}_1(N)^{(N/d^\infty)} \to \mathbf{T}_1(d)^{(N/d^\infty)}$$

determined by $\langle n\rangle \mapsto \langle n\rangle$ and $T_p \mapsto T_p$ (for p not dividing N/d^∞).

In addition, there is a commutative diagram

(2.3.5.2)

$$\begin{tikzcd}
{}^k_dW \otimes_{\mathbf{Z}} \mathbf{C} \arrow[r] & {}^k_NW \otimes_{\mathbf{Z}} \mathbf{C} \\
S_{k+2}(\Gamma_1(d),\mathbf{C}) \oplus \overline{S_{k+2}(\Gamma_1(d),\mathbf{C})} \arrow[u, "\mathrm{sh}_d"', "\simeq"] \arrow[r] & S_{k+2}(\Gamma_1(N),\mathbf{C}) \oplus \overline{S_{k+2}(\Gamma_1(N),\mathbf{C})} \arrow[u, "\simeq", "\mathrm{sh}_d"']
\end{tikzcd}$$

(*where the bottom map is induced by $f \mapsto f$ in terms of holomorphic functions on $\mathfrak{h}$). In particular, if $N_1, N_2 \geq 1$ are relatively prime and $d \geq 5$, then the intersection*

$$\left({}^{\;\;k}_{dN_1}W \otimes_{\mathbf{Z}} \mathbf{Q}\right) \cap \left({}^{\;\;k}_{dN_2}W \otimes_{\mathbf{Z}} \mathbf{Q}\right) \subseteq {}^{\;\;\;\;k}_{dN_1N_2}W \otimes_{\mathbf{Z}} \mathbf{Q}$$

coincides with ${}^k_dW \otimes_{\mathbf{Z}} \mathbf{Q}$.

This ad hoc lemma shows how the theory at "higher level" recovers the theory at a "lower level". It is essentially this lemma which will allow us to construction representations attached to newforms even at low levels (such as Ramanujan's Δ at level 1). An alternative point of view, more in the spirit of [**10**] and adelic generalizations, is to consider full level structures and then form subspaces of invariants (which we cannot do with $\Gamma_1(N)$-structures). However, our later geometric analysis of pieces of étale cohomology makes essential use of the fact that various cohomology modules are free of rank 2 over suitable commutative Hecke rings. This forces us to work with $\Gamma_1(N)$-structures rather than $\Gamma(N)$-structures. One needs to formulate the theory in more adelic terms to deal more easily with general congruence subgroups in a uniform manner.

PROOF. The methods used to prove Theorem 2.3.4.1 carry over (using the formula $(E,P) \mapsto (E, \frac{N}{d}P)$) to prove compatibility of the map ${}^k_dW \to {}^k_NW$ with respect to $\langle n\rangle$ and T_p for all p not dividing N and all p dividing d. The commutativity of (2.3.5.2), which is *visibly* injective on the bottom row and therefore implies the injectivity of the map ${}^k_dW \to {}^k_NW$ of torsion free $\mathbf{Z}$-modules, reduces immediately to the commutativity of

(2.3.5.3)

$$\begin{tikzcd}
S_{k+2}(\Gamma_1(d),\mathbf{C}) \arrow[r, hook, "\mathrm{sh}_d"] \arrow[d] & \mathrm{H}^1(Y_1(d), \mathrm{Sym}^k \mathrm{R}^1 f_{d*}\underline{\mathbf{C}}) \arrow[d] \\
S_{k+2}(\Gamma_1(N),\mathbf{C}) \arrow[r, hook, "\mathrm{sh}_N"'] & \mathrm{H}^1(Y_1(N), \mathrm{Sym}^k \mathrm{R}^1 f_{N*}\underline{\mathbf{C}})
\end{tikzcd}$$

Since the Shimura isomorphism is defined in terms of the (relative) Hodge filtration, the Kodaira-Spencer map, and coboundary maps in sheaf cohomology, all of which are compatible with base change, the commutativity of (2.3.5.3) follows from the fact that (2.3.5.1) is cartesian. $\square$

Define

$$^{k}W_{\mathbf{Q}} = \varinjlim_{N \geq 5} \left({}^{k}_{N}W \otimes_{\mathbf{Z}} \mathbf{Q}\right)$$

(where all the transition maps are injective by Lemma 2.3.5.1) It makes sense to form intersections

$$\left({}^{k}_{N_1}W \otimes_{\mathbf{Z}} \mathbf{Q}\right) \cap \left({}^{k}_{N_2}W \otimes_{\mathbf{Z}} \mathbf{Q}\right) \subseteq {}^{k}W_{\mathbf{Q}}$$

for any $N_1, N_2 \geq 5$, and this recovers ${}^{k}_{d}W \otimes_{\mathbf{Z}} \mathbf{Q}$ if $d \geq 5$ divides N_1 and N_2 and if, in addition, d, N_1/d, N_2/d are all pairwise relatively prime. This leads us to make the definition

$${}^{k}_{N}W_{\mathbf{Q}} = \left({}^{k}_{Nn_1}W \otimes_{\mathbf{Z}} \mathbf{Q}\right) \cap \left({}^{k}_{Nn_2}W \otimes_{\mathbf{Z}} \mathbf{Q}\right) \subseteq {}^{k}W_{\mathbf{Q}}$$

for *any* $N \geq 1$, where $n_1, n_2 \geq 5$ and n_1, n_2, N are pairwise relatively prime. To see that this subspace is independent of $n_1, n_2 \geq 5$, we may extend scalars to $\mathbf{C}$, where it is clear (by Lemma 2.3.5.1).

By the same reasoning, we may now define Hecke operators on ${}^{k}_{N}W_{\mathbf{Q}}$ in terms of Hecke operators at higher level. We can define $\langle n \rangle$ for all $n \in (\mathbf{Z}/N)^{\times}$ since the map $(\mathbf{Z}/Nm)^{\times} \to (\mathbf{Z}/N)^{\times}$ is surjective, and operators T_p can be defined for all p dividing N by choosing $n_1, n_2 \geq 5$ relatively prime to p. Once again the comparison with the classical picture over $\mathbf{C}$ shows that these definitions are *well-defined* and recover the previous definitions on ${}^{k}_{N}W \otimes_{\mathbf{Z}} \mathbf{Q}$ when $N \geq 5$. We also obtain a well-defined isomorphism

$$\mathrm{sh}_N : S_{k+2}(\Gamma_1(N), \mathbf{C}) \oplus \overline{S_{k+2}(\Gamma_1(N), \mathbf{C})} \xrightarrow{\simeq} {}^{k}_{N}W_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{C}$$

recovering the original Eichler-Shimura isomorphism for $N \geq 5$.

By Lemma 2.3.5.1 we can define a $\mathbf{Z}$-algebra of operators $\mathbf{T}_1(N)$ on ${}^{k}_{N}W_{\mathbf{Q}}$ which is naturally identified with the classical Hecke ring via sh_N; in particular $\mathbf{T}_1(N)$ *is* commutative. From the *definition* it is clear that $\mathbf{T}_1(N)$ is generated by a finite $\mathbf{Z}$-algebra together with several other elements integral over $\mathbf{Z}$, so $\mathbf{T}_1(N)$ is *finite* flat over $\mathbf{Z}$. We also know that sh_N is induced by an $\mathbf{R} \otimes \mathbf{T}_1(N)$-linear isomorphism

$$S_{k+2}(\Gamma_1(N), \mathbf{C}) \xrightarrow{\simeq} {}^{k}_{N}W_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{R}.$$

In particular, when $N < 5$, then ${}^{k}_{N}W_{\mathbf{Q}} = 0$ when $k = 0$ (or even $k \leq 2$).

We now wish to define a *perfect* $\mathbf{Q}$-bilinear pairing on ${}^{k}_{N}W_{\mathbf{Q}}$ with values in $\mathbf{Q}(-k-1)$ for $N \leq 4$ and with the expected adjointness properties and Petersson compatibility (as in Theorem 2.3.3.1). In order to do this, we use the pairing ${}^{k}_{nN}(\cdot, \cdot)$ on ${}^{k}_{nN}W \otimes_{\mathbf{Z}} \mathbf{Q}$ multiplied by the index $[\overline{\Gamma}_1(N) : \overline{\Gamma}_1(nN)]^{-1}$, with $\overline{\Gamma}$ denoting the image of $\Gamma \subseteq \mathrm{SL}_2(\mathbf{Z})$ in $\mathrm{PSL}_2(\mathbf{Z})$. The reason this works is that the *non-degenerate* Petersson product for low levels is still defined via integration over $\Gamma \backslash \mathfrak{h}$ and we have the following compatibility.

LEMMA 2.3.5.2. *For all d dividing N with $d \geq 5$ and any $w, w' \in {}^{k}_{d}W$, we have*

$${}^{k}_{d}(w, w') = \frac{1}{[\Gamma_1(N) : \Gamma_1(d)]} {}^{k}_{N}(\iota(w), \iota(w'))$$

where $\iota : {}^{k}_{d}W \to {}^{k}_{N}W$ is the canonical injective pullback with respect to (2.3.5.1).

PROOF. Since cup product is compatible with pullback and $\pi : Y_1(N) \to Y_1(d)$ is a covering of degree $[\Gamma_1(N) : \Gamma_1(d)]$, we just have to show that the diagram

$$\begin{tikzcd} \mathrm{H}^2_{\mathrm{c}}(Y_1(d), \underline{\mathbf{Z}}(1)) \arrow[r, "\pi^*"] \arrow[d, "\simeq"'] & \mathrm{H}^2_{\mathrm{c}}(Y_1(N), \underline{\mathbf{Z}}(1)) \arrow[d, "\simeq"] \\ \mathbf{Z} \arrow[r, "\deg \pi"] & \mathbf{Z} \end{tikzcd}$$

commutes. This is a standard fact from algebraic topology (which can be seen easily by integrating C^∞ differential forms). ☐

2.3.6. The Atkin-Lehner involution. We turn now to the Atkin-Lehner involution on modular curves and the associated action (which is *not* an involution) on modular forms. Fix a primitive Nth root of unity $\zeta \in \mathbf{C}^\times$ with $N \geq 5$. Define the map

$$w_\zeta : Y_1(N) \to Y_1(N)$$

by sending a pair (E, P) to the pair $(E/\langle P\rangle, P')$ where $P' \in E[N]$ is such that $\langle P, P'\rangle_N = \zeta$. The existence and uniqueness of such $P' \in (E/\langle P\rangle)(S)$ follows from the local constancy of $E[N] \to S$. More precisely, the perfectness of the relative Weil pairing defines an isomorphism $E[N]/\langle P\rangle \simeq \mu_N$ and P' corresponds to ζ. Using [**35**, §2.8.2] it is not difficult to check that if $NQ = P$, then

$$\langle P' \bmod \langle P\rangle, Q \bmod \langle P\rangle\rangle_N = \zeta^{-1},$$

so

$$w_\zeta^2(E,P) = (E/E[N], -Q) \simeq (E, -P) \overset{-1}{\simeq} (E,P). \tag{2.3.6.1}$$

Thus, $w_\zeta^2 = 1$. It is easy to check for all $n \in (\mathbf{Z}/N)^\times$ that $w_\zeta \circ \langle n\rangle = \langle n^{-1}\rangle \circ w_\zeta$ as endomorphisms of $Y_1(N)$, and likewise $w_{\zeta^j} \circ w_\zeta = \langle j\rangle$ for $j \in (\mathbf{Z}/N)^\times$. When $\zeta = e^{\frac{2\pi i}{N}}$, then

$$\left(\mathbf{C}/\left[1, z, \tfrac{1}{N}\right], \tfrac{z}{N}\right) \simeq \left(\mathbf{C}/\left[1, -\tfrac{1}{Nz}\right], \tfrac{1}{N}\right)$$

via multiplication by $1/z$. Thus, w_ζ on $Y_1(N)$ is the "standard" Atkin-Lehner involution induced by the map $z \mapsto -1/(Nz)$ on $\mathfrak{h}$. On the level of universal elliptic curves and modular forms, these relations will become a little more subtle (e.g., the sign in (2.3.6.1) can't be ignored).

For the universal family $(E_1(N), \mathcal{P})$ over $Y_1(N)$, there is a unique cartesian diagram

$$\begin{tikzcd} \left(E_1(N)/\langle \mathcal{P}\rangle, \mathcal{P}'\right) \arrow[r] \arrow[d, "g_N"'] & \left(E_1(N), \mathcal{P}\right) \arrow[d, "f_N"] \\ Y_1(N) \arrow[r, "w_\zeta"'] & Y_1(N) \end{tikzcd} \tag{2.3.6.2}$$

where $\langle \mathcal{P}, \mathcal{P}'\rangle_N = \zeta$; here $\mathcal{P}'$ is the unique generator of

$$\mu_N \simeq E_1(N)[N]/\langle \mathcal{P}\rangle \hookrightarrow E_1(N)/\langle \mathcal{P}\rangle$$

determined by the Weil pairing condition, so the top row in (2.3.6.2) *does* depend on our Weil pairing sign convention.

We are interested in the diagram

$$
\begin{tikzcd}
E_1(N) \arrow[r] \arrow[rr, bend left, "\varphi_\zeta"] \arrow[dr, "f_N"'] & E_1(N)/\langle\mathcal{P}\rangle \arrow[r, "\simeq"'] \arrow[d, "g_N"] & E_1(N) \arrow[d, "f_N"] \\
& Y_1(N) \arrow[r, "w_\zeta"'] & Y_1(N)
\end{tikzcd}
\tag{2.3.6.3}
$$

where $\varphi_\zeta : E_1(N) \to E_1(N)$ is the composite map across the top row; it is a map over w_ζ with degree N. We obtain from this a composite pullback

$$w_\zeta^* \mathrm{R}^1 f_{N*}\underline{\mathbf{Z}} \to \mathrm{R}^1 g_{N*}\underline{\mathbf{Z}} \to \mathrm{R}^1 f_{N*}\underline{\mathbf{Z}}$$

depending up to sign on our Weil pairing sign convention. We therefore get a map

$$\widetilde{\mathrm{H}}^1\big(Y_1(N), \mathrm{Sym}^k \mathrm{R}^1 f_{N*}\underline{\mathbf{Z}}\big) \to \widetilde{\mathrm{H}}^1\big(Y_1(N), w_\zeta^* \mathrm{Sym}^k \mathrm{R}^1 f_{N*}\underline{\mathbf{Z}}\big) \to \widetilde{\mathrm{H}}^1\big(Y_1(N), \mathrm{Sym}^k \mathrm{R}^1 f_{N*}\underline{\mathbf{Z}}\big)$$

depending up to a sign $(-1)^k$ on our Weil pairing sign convention. We denote this element of $\mathrm{End}_{\mathbf{Z}}({}^k_N W)$ by w_ζ as well.

It is straightforward to check that $w_\zeta \circ \langle n\rangle = \langle n^{-1}\rangle \circ w_\zeta$ in $\mathrm{End}_{\mathbf{Z}}({}^k_N W)$ by writing down the appropriate diagram of elliptic curves. In addition, there is the following lemma, a contrast with the simpler situation on the base $Y_1(N)$.

LEMMA 2.3.6.1. *We have $w_{\zeta^j} \circ w_\zeta = (-N)^k \langle j\rangle$ in $\mathrm{End}_{\mathbf{Z}}({}^k_N W)$ for any $j \in (\mathbf{Z}/N)^\times$. In particular, $w_\zeta^2 = (-N)^k$. Also, $\langle -1\rangle = (-1)^k$ in $\mathrm{End}_{\mathbf{Z}}({}^k_N W)$.*

PROOF. Consider the commutative diagram

$$
\begin{tikzcd}
E_1(N) \arrow[r, "\varphi_\zeta"] \arrow[d] & E_1(N) \arrow[r, "\varphi_{\zeta^j}"] \arrow[d] & E_1(N) \arrow[d] \\
Y_1(N) \arrow[r, "w_\zeta"'] & Y_1(N) \arrow[r, "w_{\zeta^j}"'] & Y_1(N)
\end{tikzcd}
$$

where the bottom composite is $\langle j\rangle$. For $x = (E_0, P_0) \in Y_1(N)$, the map across the top is the map on *elliptic curves* induced by the "rule"

$$(E_0, P_0) \mapsto (E_0/P_0, P_0') \mapsto \left(E_0/E_0[N], -\frac{j}{N}P_0\right) \overset{-N}{\simeq} (E_0, jP_0)$$

where $\langle P_0, P_0'\rangle_N = \zeta$ (so P_0, P_0' generate $E_0[N]$ and the N-torsion Weil pairing $\langle P_0', \frac{1}{N}P_0\rangle_N$ on E_0/P_0 is ζ^{-1}). Since multiplication by $-N$ on elliptic curves induces multiplication by $-N$ on the first cohomology group, this proves the assertions about w_ζ.

Now consider $\langle -1\rangle$. Since $(E, P) \simeq (E, -P)$ via multiplication by -1 on E, we see that $\langle -1\rangle$ is trivial on $Y_1(N)$ and over it we have a cartesian square

$$
\begin{tikzcd}
E_1(N) \arrow[r, "-1"] \arrow[d, "f_N"'] & E_1(N) \arrow[d, "f_N"] \\
Y_1(N) \arrow[r, equal] & Y_1(N)
\end{tikzcd}
.
$$

The top row induces multiplication by -1 on $\mathrm{R}^1 f_{N*}\underline{\mathbf{Z}}$ and hence multiplication by $(-1)^k$ on $\mathrm{Sym}^k\,\mathrm{R}^1 f_{N*}\underline{\mathbf{Z}}$. In particular, $\langle -1\rangle$ on ${}^k_N W = \widetilde{\mathrm{H}}^1(Y_1(N), \mathrm{Sym}^k\,\mathrm{R}^1 f_{N*}\underline{\mathbf{Z}})$ is multiplication by $(-1)^k$. $\square$

2.3.7. Adjoints of Hecke operators. We now turn to the study of adjointness properties of Hecke operators. In the classical theory [**44**, Theorem 4.5.5] it is shown that for all $N \geq 1$

$$w_N : f(z) \mapsto z^{-k} f\left(-\tfrac{1}{Nz}\right)$$

preserves $M_k(\Gamma_1(N), \mathbf{C})$, $S_k(\Gamma_1(N), \mathbf{C})$ and conjugates each $T \in \mathbf{T}_1(N)$ into its Petersson adjoint, where the adjoint of T_p is $\langle p^{-1}\rangle T_p$ if p does not divide N, but is usually *not* in $\mathbf{T}_1(N)$ for T_p with p dividing N. Note that $w_N^2 = (-N)^k$. (The classical theory often uses the normalization $f(z) \mapsto N^{-\frac{k}{2}} z^{-k} f\left(-\frac{1}{Nz}\right)$, but this scaling is harmless (e.g., it does not affect the formation of adjoints). The next lemma makes the connection between this classical operation and our cohomological viewpoint.

LEMMA 2.3.7.1. *For any $N \geq 5$, under the Shimura isomorphism*

$$\mathrm{sh}_N : S_{k+2}(\Gamma_1(N), \mathbf{C}) \oplus \overline{S_{k+2}(\Gamma_1(N), \mathbf{C})} \xrightarrow{\simeq} {}^k_N W \otimes_{\mathbf{Z}} \mathbf{C}$$

the modular form

$$\mathrm{sh}_N^{-1} \circ w_{e^{\frac{2\pi i}{N}}} \circ \mathrm{sh}_N(f) \in S_{k+2}(\Gamma_1(N), \mathbf{C})$$

is the function

$$z \mapsto \frac{1}{N} z^{-(k+2)} f\left(-\tfrac{1}{Nz}\right).$$

REMARK 2.3.7.2. We can formulate a version of this result for $N < 5$ once we have w_ζ operators for $N < 5$; this will be done shortly. Also, note that iterating the operation $z \mapsto \frac{1}{N} z^{-(k+2)} f(\frac{-1}{Nz})$ twice is multiplication by $\frac{1}{N^2}(-N)^{k+2} = (-N)^k$, as we expect from Lemma 2.3.6.1.

PROOF. For a pair (E, P) and the canonical projection $\pi : E \to E/\langle P\rangle$, consider the rule

$$\widetilde{w}_N(f) : (E, P) \mapsto \pi^* f(E/\langle P\rangle, P').$$

This associates to the pair $(\mathbf{C}/[1, z], \frac{1}{N})$ the element $\pi^*(f(-\frac{1}{Nz}) \cdot (2\pi i\,\mathrm{d}\tau)^{\otimes(k+2)})$ where

$$\pi : \left(\mathbf{C}/[1, z], \tfrac{1}{N}\right) \to \left(\mathbf{C}/\left[\tfrac{1}{N}, z\right], \tfrac{z}{N}\right) \overset{1/z}{\simeq} \left(\mathbf{C}/\left[1, -\tfrac{1}{Nz}\right], \tfrac{1}{N}\right).$$

Thus, we have

$$\widetilde{w}_N(f)(z) = \frac{1}{z^{k+2}} f\left(-\tfrac{1}{Nz}\right) \cdot (2\pi i\,\mathrm{d}\tau)^{\otimes(k+2)}.$$

We therefore need to show that the map

$$\mathrm{H}^0\left(Y_1(N), \omega^{\otimes(k+2)}\right) \overset{\underline{\mathrm{KS}}}{\simeq} \mathrm{H}^0\left(Y_1(N), \omega^{\otimes k} \otimes \Omega^1\right) \longrightarrow \mathrm{H}^1\left(Y_1(N), \mathrm{Sym}^k\,\mathrm{R}^1 f_{N*}\underline{\mathbf{C}}\right)$$

is compatible with $\frac{1}{N}\widetilde{w}_N$ on the left side and $w_{e^{\frac{2\pi i}{N}}}$ on the right side.

Since $\underline{\mathrm{KS}}$ is compatible with base change (i.e., cartesian squares), by going back to the definition of w_ζ and recalling the method of proof of Theorem 2.3.2.1 we just have to analyze the diagram

$$\begin{tikzcd} E_1(N) \arrow[rr, "\pi"] \arrow[dr, "f_N"'] & & E_1(N)/\langle\mathcal{P}\rangle \arrow[dl, "g_N"] \\ & Y_1(N) & \end{tikzcd}$$

More precisely, we need to show the commutativity of the diagram

$$\begin{tikzcd} \mathrm{H}^0\big(Y_1(N), \omega^{\otimes(k+2)}_{E_1(N)/\langle\mathcal{P}\rangle}\big) \arrow[r, "\underline{\mathrm{KS}}"'] \arrow[d] & \mathrm{H}^0\big(Y_1(N), \omega^{\otimes k}_{E_1(N)/\langle\mathcal{P}\rangle} \otimes \Omega^1\big) \arrow[r] \arrow[d] & \mathrm{H}^1\left(Y_1(N), \mathrm{Sym}^k \mathrm{R}^1 g_{N*}\underline{\mathbf{C}}\right) \arrow[d] \\ \mathrm{H}^0\big(Y_1(N), \omega^{\otimes(k+2)}_{E_1(N)}\big) \arrow[r, "\frac{1}{N}\underline{\mathrm{KS}}"'] & \mathrm{H}^0\big(Y_1(N), \omega^{\otimes k}_{E_1(N)} \otimes \Omega^1\big) \arrow[r] & \mathrm{H}^1\left(Y_1(N), \mathrm{Sym}^k \mathrm{R}^1 f_{N*}\underline{\mathbf{C}}\right) \end{tikzcd}$$

The right side commutes for the same functoriality reasons as in the proof of Theorem 2.3.2.1. The commutativity of the left square amounts to studying the Kodaira-Spencer map, and it suffices to show that the diagram

$$\begin{tikzcd} \omega^{\otimes 2}_{E_1(N)/\langle\mathcal{P}\rangle} \arrow[rr, "\pi^*"] \arrow[dr, "\underline{\mathrm{KS}}"] & & \omega^{\otimes 2}_{E_1(N)} \arrow[dl, "\frac{1}{N}\underline{\mathrm{KS}}"] \\ & \Omega^1_{Y_1(N)} & \end{tikzcd}$$

commutes.

For this, it suffices to work in a punctured neighborhood of the cusp corresponding to $(\underline{\mathrm{Tate}}^*_N, q)$. The quotient $\underline{\mathrm{Tate}}^*_n/\langle q\rangle$ is $\underline{\mathrm{Tate}}^*_1 \to \Delta^*$, and by Lemma 2.3.2.3 the quotient map $\pi : \underline{\mathrm{Tate}}^*_N \to \underline{\mathrm{Tate}}^*_1$ satisfies $\pi^*\mathrm{d}t/t = \mathrm{d}t/t$. Thus, we want to show that

$$\underline{\mathrm{KS}}_{\underline{\mathrm{Tate}}^*_1/\Delta^*}\,(\mathrm{d}t/t)^{\otimes 2} = (1/N)\underline{\mathrm{KS}}_{\underline{\mathrm{Tate}}^*_N/\Delta^*}\,(\mathrm{d}t/t)^{\otimes 2}.$$

This was shown in the proof of Theorem 2.3.2.1. □

It is important to have operators analogous to the w_ζ's in levels $N \leq 4$. To this end, we need to define more general operations w_ζ at level N where ζ is a primitive dth root of unity for d dividing N with $\gcd(d, N/d) = 1$.

For such ζ, define $w_\zeta : Y_1(N) \to Y_1(N)$ by

$$(E, P_d, P_{N/d}) \mapsto (E/\langle P_d\rangle, P'_d, P_{N/d})$$

where $\langle P_d, P'_d\rangle_d = \zeta$ and we use the canonical decomposition $\mathbf{Z}/N \simeq \mathbf{Z}/d \times \mathbf{Z}/(N/d)$ to take P to $(P_d, P_{N/d})$. It is straightforward to check that for $j \in (\mathbf{Z}/d)^\times$ we have $w_{\zeta^j} w_\zeta = \langle \varepsilon_{j,d}\rangle$, where $\varepsilon \in (\mathbf{Z}/N)^\times$ satisfies

$$\varepsilon_{j,d} \equiv -j \pmod{d}$$
$$\varepsilon_{j,d} \equiv d \pmod{\tfrac{N}{d}}.$$

In particular, all w_ζ are automorphisms of $Y_1(N)$.

If $N' \geq 5$ divides N and if d is also a divisor of N', then the diagram

$$\begin{tikzcd} Y_1(N) \arrow[r, "w_\zeta"] \arrow[d] & Y_1(N) \arrow[d] \\ Y_1(N') \arrow[r, "w_\zeta"'] & Y_1(N') \end{tikzcd}$$

commutes and is automatically cartesian. It follows that w_ζ on $Y_1(N')$ determines w_ζ on $Y_1(N)$.

Using [**35**, §2.8.2], one checks that if d' divides $\frac{N}{d}$ (and if d' and $\frac{N}{d'}$ are relatively prime), then for a primitive dth root of unity ζ' we have $w_{\zeta'^d} w_\zeta = w_{\zeta^{d'}} w_{\zeta'}$. We conclude that there is a well-defined operation w_ζ on ${}^k_d W$ for every $d \geq 1$ and primitive dth root of unity ζ, via the subspace

$$\varinjlim_{\substack{N \in d\mathbf{Z} \\ \gcd(d, N/d)=1 \\ N \geq 5}} {}^k_N W \otimes_{\mathbf{Z}} \mathbf{Q} \subseteq {}^k W_{\mathbf{Q}}.$$

Choose a matrix $\gamma \in \mathrm{SL}_2(\mathbf{Z})$ with

$$\gamma \equiv \begin{pmatrix} * & -1 \\ 1 & 0 \end{pmatrix} \pmod{d} \qquad \gamma \equiv \begin{pmatrix} d^{-1} & * \\ 0 & d \end{pmatrix} \pmod{\tfrac{N}{d}}$$

where $N \geq 5$ is fixed, d divides N and $\gcd(d, \frac{N}{d}) = 1$. The same method of proof used in Lemma 2.3.7.1 shows that for any $f \in S_{k+2}(\Gamma_1(N), \mathbf{C})$, we have

$$\mathrm{sh}_N^{-1} \circ w_{e^{\frac{2\pi i}{d}}} \circ \mathrm{sh}_N(f) = d^{k+1}(f|_{k+2}\gamma)(d \cdot z).$$

In particular, if $f \in S_{k+2}(\Gamma_1(d), \mathbf{C})$, then since $\gamma \in \Gamma_1(d) \cdot \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ we conclude that

$$\mathrm{sh}_d^{-1} \circ w_{e^{\frac{2\pi i}{d}}} \circ \mathrm{sh}_d(f) = \frac{1}{d} f\left(\tfrac{1}{dz}\right) z^{k+2}$$

for any $d \geq 1$. That is, Lemma 2.3.7.1 is valid for *any* $N \geq 1$.

THEOREM 2.3.7.3. *Fix $N \geq 1$ and a primitive Nth root of unity ζ. All $T \in \mathbf{T}_1(N) \subseteq \mathrm{End}_{\mathbf{Q}}({}^k_N W_{\mathbf{Q}})$ are self-adjoint with respect to the perfect pairing*

$${}^k_N[x, y] \stackrel{\mathrm{def}}{=} {}^k_N\big(x, w_\zeta(y)\big).$$

The adjoints of T_p and $\langle n \rangle$ with respect to ${}^k_N(\cdot, \cdot)$ are $\langle p^{-1} \rangle T_p$ and $\langle n^{-1} \rangle$ respectively.

REMARK 2.3.7.4. The hard part is the case of $T = T_p$ for p dividing N. This seems quite tedious using the cohomological viewpoint. The case of T_p for p not dividing N and $\langle n \rangle$ for $n \in (\mathbf{Z}/N)^\times$ can be handled using our cohomological viewpoint, but since this does not prove the entire result we omit these arguments.

PROOF. Using the identity $w_{\zeta^i} w_\zeta = \langle \varepsilon_{i,d} \rangle$ and comparison with higher levels (together with the commutativity of the Hecke rings), it suffices to consider a single choice of ζ. Thus we may take $\zeta = e^{\frac{2\pi i}{N}}$ and we can also check the result after tensoring up to $\mathbf{C}$. By Theorem 2.3.3.1 and Lemma 2.3.7.1, together with their generalizations for small level, we may reduce to the classical analogues [**44**, Lemma 4.3.2(2), Theorem 4.5.4, Theorem 4.5.5]. ∎

2.4. Applications to modular forms and Hecke rings

2.4.1. Eigenforms. Fix $N \geq 1$ and consider ${}^k_N W_{\mathbf{Q}}$ as a module for the finite $\mathbf{Q}$-algebra $\mathbf{T}_1(N)_{\mathbf{Q}} \stackrel{\text{def}}{=} \mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{T}_1(N)$. Recall that a finite algebra A over a field F is Gorenstein precisely when $\mathrm{Hom}_F(A, F)$ is isomorphic to A as an A-module.

LEMMA 2.4.1.1. *The ring $\mathbf{T}_1(N)_{\mathbf{Q}}$ is Gorenstein and ${}^k_N W_{\mathbf{Q}}$ is free of rank 2 over $\mathbf{T}_1(N)_{\mathbf{Q}}$.*

PROOF. To prove that ${}^k_N W_{\mathbf{Q}}$ is free of rank 2 over $\mathbf{T}_1(N)_{\mathbf{Q}}$ it is necessary and sufficient to check after the faithfully flat base change $\mathbf{Q} \to \mathbf{C}$. Thus, by using sh_N it suffices to show that $S_{k+2}(\Gamma_1(N), \mathbf{C})$ is free of rank 1 over $\mathbf{T}_1(N)_{\mathbf{C}}$. For this, consider the pairing of *finite* dimensional $\mathbf{C}$-vector spaces

$$\begin{aligned} S_{k+2}(\Gamma_1(N), \mathbf{C}) \times \mathbf{T}_1(N)_{\mathbf{C}} &\to \mathbf{C} \\ (f, T) &\mapsto a_1(Tf). \end{aligned} \tag{2.4.1.1}$$

We claim that this pairing is perfect. Indeed, if (f, T) pairs to 0 for some f and all T, then in particular $a_n(f) = a_1(T_n f) = 0$, so $f = 0$. Similarly, if (f, T) pairs to 0 for some T and for all f, then $(T_n f, T)$ pairs to 0 for any fixed f, so $a_n(Tf) = 0$ for all f and n. This forces $Tf = 0$ for all f. Since $\mathbf{T}_1(N)_{\mathbf{C}}$ *injects* into $\mathrm{End}_{\mathbf{C}}(S_{k+2}(\Gamma_1(N), \mathbf{C}))$, this implies that $T = 0$. Thus, (2.4.1.1) is perfect and induces an isomorphism

$$S_{k+2}(\Gamma_1(N), \mathbf{C}) \simeq \mathrm{Hom}_{\mathbf{C}}(\mathbf{T}_1(N)_{\mathbf{C}}, \mathbf{C}) \tag{2.4.1.2}$$

of $\mathbf{T}_1(N)_{\mathbf{C}}$-modules.

Since $\mathbf{T}_1(N)_{\mathbf{C}}$ is a finite product of local artinian $\mathbf{C}$-algebras with residue field $\mathbf{C}$, we are reduced to the assertion that $\mathbf{T}_1(N)_{\mathbf{C}}$ is Gorenstein (which is visibly equivalent to $\mathbf{T}_1(N)_{\mathbf{Q}}$ being Gorenstein, either by [**42**, Thm 23.4] or a direct argument).

Using the pairing ${}^k_N[\cdot, \cdot]$ in Theorem 2.3.7.3, we have an isomorphism of $\mathbf{T}_1(N)_{\mathbf{Q}}$-modules

$${}^k_N W_{\mathbf{Q}} \simeq \mathrm{Hom}_{\mathbf{Q}}({}^k_N W_{\mathbf{Q}}, \mathbf{Q}).$$

Extending scalars to $\mathbf{C}$ and using (2.4.1.2) and sh_N we conclude that

$$\mathrm{Hom}_{\mathbf{C}}(\mathbf{T}_1(N)_{\mathbf{C}}, \mathbf{C})^{\oplus 2} \simeq (\mathbf{T}_1(N)_{\mathbf{C}})^{\oplus 2}$$

as $\mathbf{T}_1(N)_{\mathbf{C}}$-modules. Since a direct summand of a flat module is flat and a flat module over a local ring is free, by checking separately over each point of the artinian $\mathrm{Spec}\, \mathbf{T}_1(N)_{\mathbf{C}}$ we conclude that $\mathrm{Hom}_{\mathbf{C}}(\mathbf{T}_1(N)_{\mathbf{C}}, \mathbf{C})$ is free of rank 1 over $\mathbf{T}_1(N)_{\mathbf{C}}$, as claimed. $\square$

REMARK 2.4.1.2. When $N = 1$, $k = 10$, then $S_{k+2}(\Gamma_1(N), \mathbf{C}) = \mathbf{C} \cdot \Delta$ and $\mathbf{T}_1(N) = \mathbf{Z}$. This is the reason why the focus on Δ in [**10**] does not require any non-trivial considerations with Hecke modules or old forms.

COROLLARY 2.4.1.3. *The $\mathbf{Q}$-algebra $\mathbf{T}_1(N)_{\mathbf{Q}}$ is generated by the T_n's.*

PROOF. This follows immediately from the fact that the pairing $(f, T) \mapsto a_1(Tf)$ between $\mathbf{T}_1(N)_{\mathbf{C}}$ and $S_{k+2}(\Gamma_1(N), \mathbf{C})$ is perfect, together with the fact that if $a_(f) = a_1(T_n f)$ vanishes for all n then $f = 0$. $\square$

Under the isomorphism of $\mathbf{T}_1(N)_{\mathbf{C}}$-modules

$$S_{k+2}(\Gamma_1(N), \mathbf{C}) \simeq \mathrm{Hom}_{\mathbf{C}}(\mathbf{T}_1(N)_{\mathbf{C}}, \mathbf{C})$$

a map $\theta : \mathbf{T}_1(N)_{\mathbf{C}} \to \mathbf{C}$ corresponds to the modular form $\sum_{n\geq 1} \theta(T_n)e^{2\pi i n z}$ and a modular form $f = \sum_{n\geq 1} a_n(f)e^{2\pi i n z}$ corresponds to the map $\theta : \mathbf{T}_1(N)_{\mathbf{C}} \to \mathbf{C}$ sending T_n to $a_n(f)$. If $\theta : \mathbf{T}_1(N)_{\mathbf{C}} \to \mathbf{C}$ is a homomorphism of $\mathbf{C}$-*algebras*, then by Corollary 2.4.1.3 we see that for any $T \in \mathbf{T}_1(N)_{\mathbf{C}}$ we have

$$T \sum_{n\geq 1} \theta(T_n)e^{2\pi i n z} = \sum_{n\geq 1} \theta(TT_n)e^{2\pi i n z},$$

and this in turn is equal to

$$\theta(T) \sum_{n\geq 1} \theta(T_n)e^{2\pi i n z}$$

since θ is $\mathbf{C}$-algebra map. Thus, the form corresponding to θ is an eigenform for all $T \in \mathbf{T}_1(N)_{\mathbf{C}}$; in fact, for all such θ we have $\theta(T_1) = 1$, so this is necessarily a *normalized eigenform*. The fact that normalized eigenforms conversely give rise to θ which are $\mathbf{C}$-algebra maps is easily checked. We conclude that normalized eigenforms in $S_{k+2}(\Gamma_1(N), \mathbf{C})$ correspond to $\mathbf{C}$-algebra homomorphisms in $\mathrm{Hom}_{\mathbf{C}}(\mathbf{T}_1(N)_{\mathbf{C}}, \mathbf{C})$. Note also that

$$\mathrm{Hom}_{\mathbf{C}-\mathrm{alg}}(\mathbf{T}_1(N)_{\mathbf{C}}, \mathbf{C}) = \mathrm{Hom}_{\mathbf{Z}-\mathrm{alg}}(\mathbf{T}_1(N), \mathbf{C}).$$

Since $\mathbf{T}_1(N)$ is a *finite* flat $\mathbf{Z}$-algebra, it follows by linear algebra that for any normalized eigenform $f = \sum_{n\geq 1} a_n(f)e^{2\pi i n z}$, the eigenvalue $a_n(f) = a_1(T_n f)$ of T_n on f is an *algebraic integer* and the $a_n(f)$'s generate a *number field* $K_f \subseteq \mathbf{C}$. By Corollary 2.4.1.3, this number field automatically contains the values of the Dirichlet character $\chi_f : (\mathbf{Z}/N)^\times \to \mathbf{C}^\times$ giving the eigenvalues of $\langle n\rangle$ against f.

Let $\overline{\mathbf{Q}} \subseteq \mathbf{C}$ denote the algebraic closure of $\mathbf{Q}$ and define $G_{\mathbf{Q}} = \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. The action of $G_{\mathbf{Q}}$ on

$$\mathrm{Hom}_{\mathbf{Z}-\mathrm{alg}}(\mathbf{T}_1(N), \overline{\mathbf{Q}}) = \mathrm{Hom}_{\mathbf{C}-\mathrm{alg}}(\mathbf{T}_1(N)_{\mathbf{C}}, \mathbf{C})$$

gives an action of $G_{\mathbf{Q}}$ on the set of normalized eigenforms in $S_{k+2}(\Gamma_1(N), \mathbf{C})$ which corresponds exactly to the action of $G_{\mathbf{Q}}$ on q-expansion coefficients. By Corollary 2.4.1.3, this is the expected action on the associated Dirichlet characters. We can identify minimal primes of the finite flat $\mathbf{Z}$-algebra $\mathbf{T}_1(N)$ with $G_{\mathbf{Q}}$-conjugacy classes of such eigenforms: a minimal prime $\mathfrak{p}$ determines a map $\theta : \mathbf{T}_1(N) \to \mathbf{C}$ of rings (determined up to an automorphism of the image field) and thus determines a normalized eigenform $\sum_{n\geq 1} \theta(T_n)e^{2\pi i n z}$ up to $G_{\mathbf{Q}}$-equivalence. Conversely, for a $G_{\mathbf{Q}}$-conjugacy class of normalized eigenforms the kernels of the corresponding maps $\theta : \mathbf{T}_1(N) \to \mathbf{C}$ coincide, yielding a minimal prime of $\mathbf{T}_1(N)$. Note that this prime also arises as the $\mathbf{T}_1(N)$-annihilator of any form in the $G_{\mathbf{Q}}$-conjugacy class.

2.4.2. Newforms. Since the subring $\mathbf{T}_1(N)^{(N)}$ generated by the $\langle n\rangle$ and the T_p for p not dividing N consists of commuting operators with respect to the *Hermitian* Petersson product, we see that

$$\mathbf{T}_1(N)^{(N)} \otimes_{\mathbf{Z}} \mathbf{C} \hookrightarrow \mathrm{End}_{\mathbf{C}}(S_{k+2}(\Gamma_1(N), \mathbf{C}))$$

is a *semisimple* algebra. That is, the $T \in \mathbf{T}_1(N)^{(N)}$ can be simultaneously diagonalized. In particular, for any p not dividing N, any eigenvalue of T_p on $S_{k+2}(\Gamma_1(N), \mathbf{C})$ occurs as the eigenvalue of some eigenform f for $\mathbf{T}_1(N)^{(N)}$.

In addition, the theory of newforms tells us that given a $\mathbf{T}_1(N)^{(N)}$-eigenform f, there exists a d dividing N and some $g \in S_{k+2}(\Gamma_1(d), \mathbf{C})$ with the same T_p-eigenvalues as f for all p not dividing N, but with g a normalized eigenform for $\mathbf{T}_1(d)$ *and* for the adjoints of all $T \in \mathbf{T}_1(d)$ [**44**, 4.6.13, 4.6.14]. The "weak multiplicity one" theorem asserts for any $L \in d\mathbf{Z}$ and any d' dividing d, if $h \in S_{k+2}(\Gamma_1(d'), \mathbf{C})$ is an eigenform for T_p for all p not dividing L, with eigenvalue $a_p(g)$, then $d' = d$ and $h = g$ [**44**, Thm 4.6.19]. There is also a "strong multiplicity one" theorem which drops the assumption that d' divides d. This is much more difficult to prove; it is amusing to note that case 4 in the non-representation theoretic proof in [**44**, Thm 4.6.19] requires the Ramanujan-Petersson conjecture! We shall only need weak multiplicity one.

Since Main Theorems 1 and 2 only specify conditions at almost all primes, we conclude that in order to prove them we may focus our attention on *newforms* $f \in S_{k+2}(\Gamma_1(N), \mathbf{C})$ at level N. Write

$$f = \sum_{n \geq 1} a_n(f) e^{2\pi i n z}$$

with $a_1(f) = 1$. The weak multiplicity one theorem ensures that

$$\bigcap_{p \nmid N} \ker\big(T_p - a_p(f)|_{S_{k+2}(\Gamma_1(N), \mathbf{C})}\big) = \mathbf{C} \cdot f. \tag{2.4.2.1}$$

For any $\sigma \in \mathrm{Aut}(\mathbf{C})$, the normalized eigenform

$$f^\sigma = \sum_{n \geq 1} \sigma\big(a_n(f)\big) e^{2\pi i n z} \in S_{k+2}(\Gamma_1(N), \mathbf{C})$$

must be a newform as well. We conclude by Galois theory that the number field $\mathbf{Q}(\{a_p(f); p \nmid N\})$ contains the values of χ_f and $a_n(f)$ for all $n \geq 1$. This is the field K_f.

Since f^σ is an eigenform as well and the Petersson pairing is Hermitian, the eigenvalue $\chi_{f^\sigma}(p)^{-1}\sigma(a_p(f))$ of f^σ against the Petersson adjoint of T_p is equal to $\overline{\sigma(a_p(f))}$. Thus,

$$\overline{\sigma\big(a_p(f)\big)} = \chi_{f^\sigma}(p)^{-1}\sigma\big(a_p(f)\big) = \sigma\big(\chi_f(p)^{-1}a_p(f)\big) = \sigma\big(\overline{a_p(f)}\big).$$

But $\overline{a_p(f)} = \chi_f(p)^{-1}a_p(f)$ lies in K_f, so K_f is either totally real or CM. That is, "complex conjugation" on K_f is *intrinsic* to K_f (independent of the choice of embedding $K_f \hookrightarrow \mathbf{C}$). Thus, the formation of the newform

$$f^c = \sum_{n \geq 1} \overline{a_n(f)} e^{2\pi i n z}$$

is also intrinsic to K_f. Note that f^c is a holomorphic function; don't confuse it with the conjugate function $z \mapsto \overline{f(z)}$ which is not holomorphic. We have the following:

THEOREM 2.4.2.1. *Let $\mathfrak{p}_f = \mathrm{ann}_{\mathbf{T}_1(N)}(f)$ be the minimal prime of $\mathbf{T}_1(N)$ corresponding to a newform f at level N. The artin local ring $\mathbf{T}_1(N)_{\mathfrak{p}_f}$ is reduced, hence isomorphic to K_f via the map $T_n \mapsto a_n(f)$, $\langle n \rangle \mapsto \chi_f(n)$.*

PROOF. Consider the cartesian diagram

$$\begin{tikzcd}
& \operatorname{Spec}\mathbf{T}_1(N)_{\mathbf{C}} \arrow[dl] \arrow[d] \\
\operatorname{Spec}\mathbf{T}_1(N)_{\mathbf{Q}} \arrow[d] & \operatorname{Spec}\mathbf{C} \arrow[dl] \\
\operatorname{Spec}\mathbf{Q} &
\end{tikzcd}$$

induced by faithfully flat base change. The prime ideal $\mathfrak{P}_f = \mathrm{ann}_{\mathbf{T}_1(N)_{\mathbf{C}}}(f)$ of $\mathbf{T}_1(N)_{\mathbf{C}}$ lies over $\mathfrak{p}_f$, so it suffices to show that $(\mathbf{T}_1(N)_{\mathbf{C}})_{\mathfrak{P}_f}$ has $\mathbf{C}$-dimension at most 1 (for then it is isomorphic to $\mathbf{C}$ and is in particular reduced). Since $S_{k+2}(\Gamma_1(N),\mathbf{C})$ is a free $\mathbf{T}_1(N)_{\mathbf{C}}$-module of rank 1, it is enough to show that $\dim_{\mathbf{C}}(S_{k+2}(\Gamma_1(N),\mathbf{C})_{\mathfrak{P}_f}) \leq 1$.

Let $\{g_i\}$ be a basis of $S_{k+2}(\Gamma_1(N),\mathbf{C})$ consisting of simultaneous eigenforms for the T_p with p not dividing N. By (2.4.2.1), if $g_i \neq f$ then $a_p(g_i) \neq a_p(f)$ for some p not dividing N. Thus $T_p - a_p(g_i)$ does not lie in $\mathfrak{P}_f$, since $T_p - a_p(f)$ does lie in $\mathfrak{P}_f$, so $T_p - a_p(g_i)$ is a unit in $\mathbf{T}_1(N)_{\mathfrak{P}_f}$. This forces g_i to vanish in $S_{k+2}(\Gamma_1(N),\mathbf{C})_{\mathfrak{P}_f}$. Thus, $S_{k+2}(\Gamma_1(N),\mathbf{C})_{\mathfrak{P}_f} \subseteq \mathbf{C}\cdot f$ has $\mathbf{C}$-dimension at most 1, as claimed. □

2.4.3. Representation spaces. Let $\mathfrak{p}$ be a minimal prime of $\mathbf{T}_1(N)$, corresponding to a $G_{\mathbf{Q}}$-conjugacy class of newforms in $S_{k+2}(\Gamma_1(N),\mathbf{C})$. Define $V_{\mathfrak{p}} = ({}^k_N W_{\mathbf{Q}})_{\mathfrak{p}}$; by Lemma 2.4.1.1 and Theorem 2.4.2.1 this is a 2-dimensional vector space over the CM field $\mathbf{T}_1(N)_{\mathfrak{p}}$. Note that $V_{\mathfrak{p}}$ is a *direct summand* of ${}^k_N W_{\mathbf{Q}}$ since $\mathbf{T}_1(N)_{\mathfrak{p}}$ is a direct factor of the semilocal ring $\mathbf{T}_1(N)_{\mathbf{Q}}$. For any newform f in the conjugacy class of $\mathfrak{p}$, by Theorem 2.4.2.1 we have an isomorphism

$$\mathbf{T}_1(N)_{\mathfrak{p}} \xrightarrow{\simeq} K_f$$

sending $T_n \mapsto a_n(f)$ and $\langle n\rangle \mapsto \chi_f(n)$. We define $V_f = V_{\mathfrak{p}} \otimes_{\mathbf{T}_1(N)_{\mathfrak{p}}} K_f$ and we view V_f as a "concrete" version of $V_{\mathfrak{p}}$ insofar as it uses an identification of the abstract CM field $\mathbf{T}_1(N)_{\mathfrak{p}}$ with a subfield $K_f \subseteq \mathbf{C}$.

The $\mathbf{T}_1(N)_{\mathbf{C}}$-module $V_{\mathfrak{p}} \otimes_{\mathbf{Q}} \mathbf{C}$ is a direct summand of ${}^k_N W_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{C}$, and we have a decomposition

$$V_{\mathfrak{p}} \otimes_{\mathbf{Q}} \mathbf{C} \simeq \prod_{\sigma:\mathbf{T}_1(N)_{\mathfrak{p}}\hookrightarrow\mathbf{C}} V_{f^\sigma} \otimes_{K_{f^\sigma}} \mathbf{C},$$

where $f_\sigma(z) = \sum_{n\geq 1}\sigma(T_n)e^{2\pi i n z}$ and $\chi_{f_\sigma}(n) = \sigma(\langle n\rangle)$. Intrinsically, $V_{\mathfrak{p}} \otimes_{\mathbf{Q}} \mathbf{C}$ is a module over

$$\mathbf{T}_1(N)_{\mathfrak{p}} \otimes_{\mathbf{Q}} \mathbf{C} \simeq \prod_\sigma K_{f_\sigma},$$

with $V_{f_\sigma} \otimes_{K_{f_\sigma}} \mathbf{C}$ the factor corresponding to K_{f_σ}. The *Hecke compatible*

$$\mathrm{sh}_N : {}^k_N W_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{C} \xrightarrow{\simeq} S_{k+2}(\Gamma_1(N),\mathbf{C}) \oplus \overline{S_{k+2}(\Gamma_1(N),\mathbf{C})}$$

therefore induces equalities of direct factors

$$V_f \otimes_{K_f} \mathbf{C} = \mathbf{C}\cdot f \oplus \mathbf{C}\cdot \overline{f}^c$$

$$V_{f^c} \otimes_{K_{f^c}} \mathbf{C} = \mathbf{C}\cdot f^c \oplus \mathbf{C}\overline{f}$$

(recall $K_f = K_{f^c}$ as subfield of $\mathbf{C}$); here we recall that $\overline{f}$ denotes the element

$$1\otimes f \in \mathbf{C}\otimes_{\mathbf{C}} S_{k+2}(\Gamma_1(N),\mathbf{C}) = \overline{S_{k+2}(\Gamma_1(N),\mathbf{C})}$$

using extension of scalars by the complex conjugation map $\mathbf{C} \to \mathbf{C}$. Note that $f \neq f^c$ if and only if $\chi_f \neq 1$. We emphasize that $\overline{f}$ is not a modular form and does not have a q-expansion, while f^c is a cusp form. The elements f^c and $\overline{f}$ live in separated pieces of the direct sum decomposition of ${}^k_N W_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{C}$ under sh_N.

Since $\mathbf{T}_1(N)_{\mathfrak{p}} \otimes_{\mathbf{Q}} K_f$ contains $K_f = K_{f^c}$ as factor fields for the two maps $\mathbf{T}_1(N)_{\mathfrak{p}} \to K_f$ corresponding to f and f^c, we see that V_f and V_{f^c} are naturally $K_f[\mathbf{T}_1(N)]$-module direct summands of ${}^k_N W_{\mathbf{Q}} \otimes_{\mathbf{Q}} K_f$, with T_n (resp. $\langle n \rangle$) acting as $a_n(f)$ and $\overline{a_n(f)}$ (resp. $\chi_f(n), \overline{\chi}_f(n)$). Equivalently, the two factor rings $K_f \otimes_{\mathbf{Q}} K_f \twoheadrightarrow K_f$ defined by $x \otimes y \mapsto xy, \overline{x}y$ realize V_f, V_{f^c} as K_f-module direct summands of ${}^k_N W_{\mathbf{Q}} \otimes_{\mathbf{Q}} K_f$.

LEMMA 2.4.3.1. *The $K_f = K_{f^c}$-bilinear pairing*

$$ {}^k_N(\cdot,\cdot) : ({}^k_N W_{\mathbf{Q}} \otimes_{\mathbf{Q}} K_f) \otimes_{K_f} ({}^k_N W_{\mathbf{Q}} \otimes_{\mathbf{Q}} K_f) \to K_f(-k-1) $$

from (2.3.4.1) *restricts to a perfect pairing*

$$ (\cdot,\cdot)_f : V_f \otimes_{K_f} V_{f^c} \to K_f(-k-1). \tag{2.4.3.1} $$

That is, V_f (resp. V_{f^c}) annihilates a subspace complimentary to V_{f^c} (resp. V_f). Finally, $(x,y)_f = (-1)^{k+1}(y,x)_{f^c}$.

PROOF. The last part follows from the fact that ${}^k_\Gamma \langle \cdot,\cdot \rangle$ in (2.3.3.4) is symmetric for k odd and alternating for k even. To check perfectness of (2.4.3.1), we may tensor with $\mathbf{C}$ to arrive at the pairing

$$ \left(\mathbf{C} \cdot f \oplus \mathbf{C} \cdot \overline{f^c}\right) \otimes_{\mathbf{C}} \left(\mathbf{C} \cdot f^c \oplus \mathbf{C} \cdot \overline{f}\right) \longrightarrow \mathbf{C}(k-1) $$

determined by the formula in Theorem 2.3.3.1. Using this formula, we just have to observe that the Petersson product $\langle f, f \rangle_{\Gamma_1(N)}$ is non-zero. □

We can state Lemma 2.4.3.1 in a slightly more intrinsic form without using subfields of $\mathbf{C}$. Let $\overline{V}_{\mathfrak{p}}$ denote the base change of $V_{\mathfrak{p}}$ by the (possibly trivial) complex conjugation on the CM field $\mathbf{T}_1(N)_{\mathfrak{p}}$. Then $V_{\mathfrak{p}}$ and $\overline{V}_{\mathfrak{p}}$ are naturally direct summands of ${}^k_N W_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{T}_1(N)_{\mathfrak{p}}$ as $\mathbf{T}_1(N)_{\mathfrak{p}}$-vectors spaces, via the two left $\mathbf{T}_1(N)_{\mathfrak{p}}$-algebra maps

$$ \mathbf{T}_1(N)_{\mathfrak{p}} \otimes_{\mathbf{Q}} \mathbf{T}_1(N)_{\mathfrak{p}} \to \mathbf{T}_1(N)_{\mathfrak{p}} $$

defined by $T \otimes T' \mapsto TT', T\overline{T'}$ respectively. In this way, the $\mathbf{T}_1(N)$-action on ${}^k_N W_{\mathbf{Q}}$ goes over to $T \in \mathbf{T}_1(N)$ acting on the $\mathbf{T}_1(N)_{\mathfrak{p}}$-vector space $V_{\mathfrak{p}}$ (resp. $\overline{V}_{\mathfrak{p}}$) as T (resp. $\overline{T}$), where $\overline{T}$ is computed in the CM field $\mathbf{T}_1(N)_{\mathfrak{p}}$. Lemma 2.4.3.1 says that the $\mathbf{T}_1(N)_{\mathfrak{p}}$-bilinear pairing

$$ \left({}^k_N W_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{T}_1(N)_{\mathfrak{p}}\right) \otimes_{\mathbf{T}_1(N)_{\mathfrak{p}}} \left({}^k_N W_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{T}_1(N)_{\mathfrak{p}}\right) \to \mathbf{T}_1(N)_{\mathfrak{p}}(-k-1) $$

restricts to a *perfect* pairing

$$ V_{\mathfrak{p}} \otimes_{\mathbf{T}_1(N)_{\mathfrak{p}}} \overline{V}_{\mathfrak{p}} \to \mathbf{T}_1(N)_{\mathfrak{p}}(-k-1). $$

LEMMA 2.4.3.2. *Fix a primitive Nth root of unity ζ. The K_f-linear isomorphism $w_\zeta : {}^k_N W_{\mathbf{Q}} \otimes_{\mathbf{Q}} K_f \xrightarrow{\simeq} {}^k_N W_{\mathbf{Q}} \otimes_{\mathbf{Q}} K_f$ takes V_f over to V_{f^c} as K_f-vector spaces (but not as $\mathbf{T}_1(N)$-modules through the $\mathbf{T}_1(N)$-action on ${}^k_N W_{\mathbf{Q}}$).*

More intrinsically, Lemma 2.4.3.2 says that the $\mathbf{T}_1(N)_{\mathfrak{p}}$-linear $w_\zeta : {}^k_N W_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{T}_1(N)_{\mathfrak{p}} \xrightarrow{\simeq} {}^k_N W_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{T}_1(N)_{\mathfrak{p}}$ takes $V_{\mathfrak{p}}$ over to $\overline{V}_{\mathfrak{p}}$.

PROOF. By Theorem 2.3.7.3, conjugation by w_ζ takes T_p to $\langle p\rangle^{-1} T_p$ for p not dividing N, so upon extending scalars by $K_f \to \mathbf{C}$ we conclude from Lemma 2.3.7.1 that w_ζ takes $S_{k+2}(\Gamma_1(N), \mathbf{C})$ back to itself. This operation sends an eigenform f with $\langle p\rangle^{-1} T_p$ eigenvalue $\overline{a_f(p)}$ (for all p not dividing N) to a nonzero g which is a T_p-eigenform with eigenvalue $\overline{a_f(p)}$ (for all p not dividing N). It follows from weak multiplicity one and [**44**, Cor 4.6.14] that g *must* be a $\mathbf{C}^\times$-multiple of the newform f^c. Similarly, $w_\zeta(\overline{f})$ is a $\mathbf{C}^\times$-multiple of $\overline{f^c}$. $\square$

2.4.4. l-adic analogues. The l-adic analogue of all of this goes as follows. Let λ be a maximal ideal in the integer ring of the CM field $\mathbf{T}_1(N)_{\mathfrak{p}}$, and let $\mathbf{T}_1(N)_{\mathfrak{p},\lambda}$ denote the λ-adic completion. We define $V_{\mathfrak{p},\lambda} = V_{\mathfrak{p}} \otimes_{\mathbf{T}_1(N)_{\mathfrak{p}}} \mathbf{T}_1(N)_{\mathfrak{p},\lambda}$, and we similarly define

$$\overline{V}_{\mathfrak{p},\lambda} = \overline{V}_{\mathfrak{p}} \otimes_{\mathbf{T}_1(N)_{\mathfrak{p}}} \mathbf{T}_1(N)_{\mathfrak{p},\lambda}$$

in terms of the $\mathbf{T}_1(N)_{\mathfrak{p}}$-module $\overline{V}_{\mathfrak{p}}$. In particular, $\overline{V}_{\mathfrak{p},\lambda}$ is naturally the base change of $V_{\mathfrak{p},\overline{\lambda}}$ by the automorphism $\mathbf{T}_1(N)_{\mathfrak{p},\lambda} \simeq \mathbf{T}_1(N)_{\mathfrak{p},\overline{\lambda}}$ induced by the complex conjugation of the CM field $\mathbf{T}_1(N)_{\mathfrak{p}}$. Both $V_{\mathfrak{p},\lambda}$ and $\overline{V}_{\mathfrak{p},\lambda}$ are naturally direct summands of ${}^k_N W_l \otimes_{\mathbf{Q}_l} \mathbf{T}_1(N)_{\mathfrak{p},\lambda}$ as $\mathbf{T}_1(N)_{\mathfrak{p},\lambda}$-modules via the two maps

$$\mathbf{T}_1(N)_{\mathfrak{p}} \otimes_{\mathbf{Q}} \mathbf{T}_1(N)_{\mathfrak{p},\lambda} \to \mathbf{T}_1(N)_{\mathfrak{p},\lambda}$$

defined by $x \otimes y \mapsto xy, \overline{x}y$, with ${}^k_N W_l = {}^k_N W_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{Q}_l$.

If f is a newform in the conjugacy class determined by $\mathfrak{p}$, then $\mathbf{T}_1(N)_{\mathfrak{p}} \simeq K_f$ identifies $V_{\mathfrak{p},\lambda}$ with $V_{f,\lambda} = V_f \otimes_{K_f} K_{f,\lambda}$ as a direct summand of ${}^k_N W_l \otimes_{\mathbf{Q}_l} K_{f,\lambda} \simeq ({}^k_N W_{\mathbf{Q}} \otimes_{\mathbf{Q}} K_f) \otimes_{K_f} K_{f,\lambda}$, and $\overline{V}_{\mathfrak{p},\lambda}$ is identified with $V_{f^c,\lambda}$ (*not* $V_{f^c,\overline{\lambda}}$).

We have a *perfect* $\mathbf{T}_1(N)_{\mathfrak{p},\lambda}$-bilinear pairing

$$(\cdot,\cdot)_{\mathfrak{p},\lambda} : V_{\mathfrak{p},\lambda} \otimes_{\mathbf{T}_1(N)_{\mathfrak{p},\lambda}} \overline{V}_{\mathfrak{p},\lambda} \to \mathbf{T}_1(N)_{\mathfrak{p},\lambda}(-k-1)$$

induced by ${}^k_N(\cdot,\cdot)_l = {}^k_N(\cdot,\cdot) \otimes_{\mathbf{Q}} \mathbf{Q}_l$ via the extension of scalars $\mathbf{Q}_l \to \mathbf{T}_1(N)_{\mathfrak{p},\lambda}$, and w_ζ induces an isomorphism of vector spaces $V_{\mathfrak{p},\lambda} \simeq \overline{V}_{\mathfrak{p},\lambda}$ (by Lemma 2.4.3.2). Equivalently, for f as above there is a perfect $K_{f,\lambda} = K_{f^c,\lambda}$-bilinear pairing

$$(\cdot,\cdot)_{f,\lambda} : V_{f,\lambda} \otimes_{K_{f,\lambda}} V_{f^c,\lambda} \to K_{f,\lambda}(-k-1)$$

and $w_\zeta : V_{f,\lambda} \xrightarrow{\simeq} V_{f^c,\lambda}$; the perfectness follows from (2.4.3.1) and the commuativity of

$$\begin{tikzcd} \mathbf{Q} \arrow[r] \arrow[d] & \mathbf{T}_1(N)_{\mathfrak{p}} \arrow[d] \\ \mathbf{Q}_l \arrow[r] & \mathbf{T}_1(N)_{\mathfrak{p},\lambda} \end{tikzcd}$$

To have compatibility with the algebraic theory later on, we also note that there is a natural isomorphism of groups

$$\mathbf{Z}_l \otimes_{\mathbf{Z}} \mathbf{Z}(1) \simeq \varprojlim \mu_{l^n}(\mathbf{C}) \tag{2.4.4.1}$$

induced by

$$\exp\left(\tfrac{1}{l^n}(\cdot)\right) : \mathbf{Z}/l^n \otimes_{\mathbf{Z}} \mathbf{Z}(1) \simeq \mu_{l^n}(\mathbf{C})$$

for all $n \geq 1$.

Comparison theorems with étale cohomology will identify everything we have just constructed l-adically above with analogous purely algebraic constructions in étale cohomology (with ${}^k_N(\cdot,\cdot)_l$ arising from Poincaré duality). It will then follow that ${}^k_N W_l$ has a natural *continuous* action of $G_{\mathbf{Q}}$, compatible with the action of $\mathbf{T}_1(N)$, so that the spaces $V_{f,\lambda}$ are naturally $K_{f,\lambda}[G_{\mathbf{Q}}]$-modules. But we need the *analytic* theory to know that these are 2-dimensional over $K_{f,\lambda}$! That is, ${}^k_N W_l$ can be constructed as a $\mathbf{T}_1(N) \otimes \mathbf{Q}_l$-module purely algebraically, so by localization we can construct $V_{\mathfrak{p},\lambda} \simeq V_{f,\lambda}$ purely algebraically, but we *cannot* see the 2-dimensionality without appealing to the *analytic* theory of newforms, particularly the multiplicity one theory. The algebraic theory that provides the $G_{\mathbf{Q}}$-action will enable us to determine the characteristic polynomial of Frobenii, using algebraic incarnations of the constructions ${}^k_N(\cdot,\cdot)_l$ and w_ζ, whose l-adic properties were deduced above from the *topological* Lemma 2.4.3.1 and Lemma 2.4.3.2.

CHAPTER 3

Proof in weight 2

We want to specialize the constructions of Chapter 2 to the case $k = 0$. This will enable us to give the construction of Galois representations in weight 2 via Jacobians, as in Theorem 0.0.0.5. Recall that we already saw in the remarks following Theorem 0.0.0.5 that this implies the Ramanujan-Petersson conjecture in weight 2, via the Weil conjectures for abelian varieties. Our arguments for weight 2 do not require the theory of newforms, nor do they require étale cohomology beyond the incarnation of étale cohomology in the form of Tate modules of abelian varieties (which we can study without any étale cohomology apparatus). Since $S_2(\Gamma_1(N), \mathbf{C}) = 0$ when $N \leq 4$, throughout this chapter we can and will assume $N \geq 5$.

3.1. Formulation via Jacobians

Due to the autoduality of the Jacobian of a curve, there are two natural ways to define Hecke operators on the Jacobian of $X_1(N)$. Only one of these approaches agrees with the construction in Chapter 2 when identifying the cotangent space on the Jacobian of $X_1(N)$ with the global 1-forms on $X_1(N)$ (which is canonically identified with $S_2(\Gamma_1(N), \mathbf{C})$); the other construction gives dual maps. This can cause a lot of confusion. Our first order of business is to clear up this matter so that we can correctly compute maps on weight 2 cusp forms in terms of maps induced on cotangent spaces at 0 by explicitly defined endomorphisms of Jacobians. We can then use l-adic Tate modules of Jacobians with their Weil pairing autoduality in our study of weight 2 cusp forms; the natural generalization of these considerations in higher weight will involve étale cohomology equipped with its cup product structures and Poincaré duality.

3.1.1. Cup product and the Weil pairing. Before we analyze structures on modular curves and their Jacobians, it will be convenient to first work out some rather general considerations for arbitrary compact Riemann surfaces. Our aim is to identify certain intrinsic algebraic constructions with intrinsic analytic constructions. Note that if we identify $\mathbf{Z}(1) \simeq \mathbf{Z}$ via the basis $2\pi i$, we also obtain an identification $\mathbf{Z}_l \simeq \varprojlim \mu_{l^n}(\mathbf{C})$ via the exponential as in (2.4.4.1).

THEOREM 3.1.1.1. *Let X be a compact Riemann surface and define an isomorphism $\mathrm{H}^2(X, \underline{\mathbf{Z}}) \simeq \underline{\mathbf{Z}}$ via the i-orientation. Then the diagram*

$$\begin{tikzcd} \mathrm{H}^1(X, \underline{\mathbf{Z}}_l) \times \mathrm{H}^1(X, \underline{\mathbf{Z}}_l) \arrow[r, "\cup"] \arrow[d, "\simeq"] & \mathbf{Z}_l \arrow[d, "\simeq"] \\ T_l(\mathrm{Pic}^0_X) \times T_l(\mathrm{Pic}^0_X) \arrow[r] & \varprojlim \mu_{l^n}(\mathbf{C}) \end{tikzcd}$$

anti-commutes; here the bottom row is the l-adic Weil pairing (*with respect to the principal polarization* $\mathrm{Pic}^0_X \simeq \widehat{\mathrm{Pic}^0_X}$ *for the "second" factor above*).

REMARK 3.1.1.2. The proof of this is nothing more than a long argument in unwinding the analytic and algebraic theories of Pic^0 as in [**45**]. Since we have chosen the Weil pairing convention agreeing with that in [**45**], in the end the anti-commutativity depends upon the "Weil pairing" sign on [**45**, bottom of p. 237]. Note that this sign does *not* affect formation of adjoints. The proof also rests on the analysis in [**7**, Appendix B].

PROOF. In [**45**, §24, p.237] it is checked that by using the isomorphism $\mathbf{Z}(1) \simeq \mathbf{Z}$ as above as well as Mumford's Weil pairing sign conventions (which we also use), the Weil pairing associated to a polarization of an abelian variety over $\mathbf{C}$ (as in our setup above) is the *negative* of the pairing induced by the imaginary part of the Hermitian form associated to the polarization. Thus, the theorem amounts to the assertion that the *canonical* principal polarization of

$$\mathrm{Pic}^0_X \simeq \mathrm{H}^1(X, \mathcal{O}_X)/\mathrm{H}^1(X, \underline{\mathbf{Z}})$$

is induced by the (unique) Hermitian form

$$H_0 : \mathrm{H}^1(X, \mathcal{O}_X) \times \mathrm{H}^1(X, \mathcal{O}_X) \to \mathbf{C}$$

for which the restriction

$$E_0 = \mathrm{Im}(H_0) : \mathrm{H}^1(X, \underline{\mathbf{Z}}) \times \mathrm{H}^1(X, \underline{\mathbf{Z}}) \to \mathbf{C}$$

is Poincaré duality (via cup product and the i-orientation isomorphism $\mathrm{H}^2(X, \underline{\mathbf{Z}}) \simeq \mathbf{Z}$).

Consider the isomorphism

$$\mathrm{H}^1(X, \mathcal{O}_X) \simeq \mathrm{H}^0(X, \Omega^1_X)^\vee, \tag{3.1.1.1}$$

which is $2\pi i$ times the residue map (see Corollary 1.2.3.1 and Lemma 1.5.2.1 for the explication used below). Consider also the inverse of the composite isomorphism

$$\mathrm{H}^0(X, \overline{\Omega}^1_X) \to \mathrm{H}^1(X, \underline{\mathbf{C}}) \to \mathrm{H}^1(X, \mathcal{O}_X) \tag{3.1.1.2}$$

of Hodge theory. These allows us to define a pairing

$$H : \mathrm{H}^1(X, \mathcal{O}_X) \times \mathrm{H}^1(X, \mathcal{O}_X) \xrightarrow{\simeq} \mathrm{H}^0(X, \Omega^1_X)^\vee \times \mathrm{H}^0(X, \overline{\Omega}^1_X) \to \mathbf{C}$$

where the right map is $(\varphi, \overline{\omega}) \mapsto 2i\varphi(\omega)$. This is $\mathbf{C}$-linear on the left and $\mathbf{C}$-antilinear on the right.

Let $E = \mathrm{Im}(H)$. We claim that

$$E : \mathrm{H}^1(X, \underline{\mathbf{Z}}) \times \mathrm{H}^1(X, \underline{\mathbf{Z}}) \to \mathbf{C}$$

is *exactly* Poincaré duality; since $\mathrm{H}^1(X, \underline{\mathbf{Z}}) \hookrightarrow \mathrm{H}^1(X, \mathcal{O}_X)$ is a *lattice*, this will also imply that H is Hermitian.

Choose $c \in \mathrm{H}^1(X, \underline{\mathbf{Z}}) \subseteq \mathrm{H}^1(X, \underline{\mathbf{C}})$; we can write $c = \omega + \overline{\eta}$ via the isomorphism

$$\mathrm{H}^1(X, \underline{\mathbf{C}}) \simeq \mathrm{H}^0(X, \Omega^1_X) \oplus H^0(X, \overline{\Omega}^1_X). \tag{3.1.1.3}$$

We have $\omega = \eta$ since c lies in $\mathrm{H}^1(X, \underline{\mathbf{R}})$ and is therefore stable under complex conjugation.

It follows that the map (3.1.1.2) sends $\overline{\omega}$ to c. Thus, (3.1.1.1) sends c to the functional

$$\int_X \overline{\omega} \wedge (\cdot) = \int_X (\omega + \overline{\omega}) \wedge (\cdot)$$

on holomorphic 1-forms. By de Rham theory, this is just cup product $c\cup(\cdot)$ against c. We conclude that (3.1.1.1) induces the map $\mathrm{H}^1(X,\underline{\mathbf{Z}})\simeq \mathrm{H}_1(X,\mathbf{Z})$ sending c to the homology class σ_c corresponding to c under the Poincaré self-duality of $\mathrm{H}^1(X,\underline{\mathbf{C}})$ and the standard identification of homology with the dual of cohomology.

Similarly,

$$\mathrm{H}^1(X,\underline{\mathbf{Z}})\to \mathrm{H}^1(X,\mathcal{O}_X)\simeq \mathrm{H}^0(X,\overline{\Omega}^1_X)$$

sends $c'\in \mathrm{H}^1(X,\underline{\mathbf{Z}})$ to $\overline{\omega'}$, where $\omega'+\overline{\omega'}\in \mathrm{H}^1(X,\underline{\mathbf{C}})$ is the Hodge decomposition of c'. Thus, H induces the map

$$\mathrm{H}^1(X,\underline{\mathbf{Z}})\times \mathrm{H}^1(X,\underline{\mathbf{Z}})\to \underline{\mathbf{C}}$$
$$(c,c')\mapsto 2i\int_{\sigma_c}\overline{\omega'}.$$

Therefore E is just given by

$$\begin{aligned}E(c,c') &= \mathrm{Im}\left(2i\int_{\sigma_c}\overline{\omega'}\right)\\ &= 2\,\mathrm{Re}\left(\int_{\sigma_c}\overline{\omega'}\right)\\ &= \int_{\sigma_c}\omega'+\overline{\omega'}\\ &= c\cup(\omega'+\overline{\omega'})\\ &= c\cup c',\end{aligned}$$

so E is just Poincaré duality, as claimed.

The Hermitian pairing H induces an isomorphism

$$\alpha:\mathrm{Pic}^0_X\simeq(\mathrm{Pic}^0_X)^\vee$$

and it remains to show that this is the canonical principal polarization. By [**43**, Lemma 6.9], the canonical principal polarization is the *negative* of the map induced by functoriality from $i_P: X\to \mathrm{Pic}^0_X$ sending $x\mapsto \mathcal{O}(x)\otimes\mathcal{O}(P)^{-1}$ for a fixed $P\in X$. Checking that two endomorphisms of Pic^0_X coincide can be done by looking at the tangent space at the identity, so we want to show that

$$\mathrm{H}^1\big(\mathrm{Pic}^0_X,\mathcal{O}\big)\xrightarrow{\simeq} T_0\big((\mathrm{Pic}^0_X)^\vee\big)\xrightarrow{T_0(\alpha)} T_0(\mathrm{Pic}^0_X)\xleftarrow{\simeq}\mathrm{H}^1(X,\mathcal{O}_X)$$

is $-i_P^*$, where the outer maps are the "algebraic" ones defined via $T_0(\mathrm{Pic}^0_V)\simeq \mathrm{H}^1(V,\mathcal{O}_V)$ as in the definition of (1.3.3.2); of course, universally scaling this isomorphism by an element of $\mathbf{C}^\times$ does *not* affect the above composite.

If one unwinds the dictionary between the analytic and algebraic constructions of the dual abelian variety following [**45**, pp. 86–87], or more precisely one uses Lemma 1.3.3.1, the above "algebraic" isomorphism

$$\mathrm{H}^1\big(\mathrm{Pic}^0_X,\mathcal{O}\big)\simeq(\mathrm{Pic}^0_X)(\mathrm{Sp}\,\mathbf{C}[\varepsilon])=T_0\big((\mathrm{Pic}^0_X)^\vee\big)\tag{3.1.1.4}$$

is the composite of multiplication by $\frac{1}{2\pi i}$ on $\mathrm{H}^1(\mathrm{Pic}^0_X,\mathcal{O})$ and the "analytic" exponential map

$$\mathbf{e}:\mathrm{H}^1\big(\mathrm{Pic}^0_X,\mathcal{O}\big)\to T_0\big((\mathrm{Pic}^0_X)^\vee\big)$$

obtained from the general exponential sequence

$$0\to\underline{\mathbf{Z}}\to\mathcal{O}_V\xrightarrow{e^{2\pi i(\cdot)}}\mathcal{O}_V^\times\to 1.$$

By the functoriality of the exponential sequence with respect to $i_P : X \hookrightarrow \mathrm{Pic}^0_X$, we have a commutative diagram of *analytic* isomorphisms

$$\begin{tikzcd}
(\mathrm{Pic}^0_X)^\vee \arrow[rr, "\simeq"] & & \mathrm{Pic}^0_X \\
\mathrm{H}^1(\mathrm{Pic}^0_X, \mathcal{O})/\mathrm{H}^1(\mathrm{Pic}^0_X, \underline{\mathbf{Z}}) \arrow[u, "\simeq", "\mathrm{H}^1(i_P)"'] \arrow[rr, "\simeq"] & & \mathrm{H}^1(X, \mathcal{O}_X)/\mathrm{H}^1(X, \underline{\mathbf{Z}}) \arrow[u, "\simeq"]
\end{tikzcd}$$

We therefore get a *commutative* diagram

$$\begin{tikzcd}
\mathrm{H}^1(\mathrm{Pic}^0_X, \mathcal{O}) \arrow[r, "\frac{1}{2\pi i}"] \arrow[d, "\mathrm{H}^1(i_P)"'] & \mathrm{H}^1(\mathrm{Pic}^0_X, \mathcal{O}) \arrow[r, two heads, "\mathbf{e}"] \arrow[d, "\mathrm{H}^1(i_P)"'] & T_0((\mathrm{Pic}^0_X)^\vee) \arrow[d, "-T_0(\mathrm{can})"] \\
\mathrm{H}^1(X, \mathcal{O}_X) \arrow[r, "\frac{1}{2\pi i}"'] & \mathrm{H}^1(X, \mathcal{O}_X) \arrow[r, two heads, "\mathbf{e}"'] & T_0(\mathrm{Pic}^0_X)
\end{tikzcd} \tag{3.1.1.5}$$

with the top row the "algebraic" isomorphism (3.1.1.4) and $T_0(\mathrm{can})$ coming from the canonical principal polarization (the sign in the right column of (3.1.1.5) is due to the above description of this canonical principal polarization in terms of $-i_P$). By functoriality, the bottom row must also be the "algebraic" isomorphism (coming from the algebraic exponential sequence on the compact Riemann surface X).

To show that α is the canonical principal polarization, we are now reduced by (3.1.1.5) to checking the commutativity of

$$\begin{tikzcd}
\mathrm{H}^1(\mathrm{Pic}^0_X, \mathcal{O}) \arrow[r, two heads, "\mathbf{e}"] \arrow[d, "\mathrm{H}^1(i_P)"'] & T_0((\mathrm{Pic}^0_X)^\vee) \arrow[d, "-T_0(\alpha)=T_0(-\alpha)"] \\
\mathrm{H}^1(X, \mathcal{O}_X) \arrow[r, two heads, "\mathbf{e}"'] & T_0(\mathrm{Pic}^0_X)
\end{tikzcd} \tag{3.1.1.6}$$

One subtle point is that the top row is defined via the "exponential sequence" construction of the dual abelian variety of Pic^0_X and the map α in the right side is defined using the formulation of the analytic theory of the dual abelian variety in terms of Hermitian forms. As noted on [**45**, p. 87], these two ways of viewing the dual abelian variety are "off" by a factor of $2i$! In more explicit terms, this is just restating the commutativity of the lower left rectangle in (1.3.3.8) for the complex torus $A = \mathrm{Pic}^0_X$. Due to this fact, and recalling how α is defined in terms of H, (3.1.1.6) is identified with

$$\begin{tikzcd}
\mathrm{H}^1(\mathrm{Pic}^0_X, \mathcal{O}) \arrow[r, "\simeq"] \arrow[d, "i_P^*"'] & \mathrm{H}^0(\mathrm{Pic}^0_X, \overline{\Omega}^1_{\mathrm{Pic}^0_X}) \arrow[d, "\simeq"] \\
\mathrm{H}^1(X, \mathcal{O}_X) \arrow[r, "\simeq"] & \overline{\mathrm{H}^1(X, \mathcal{O}_X)}^\vee
\end{tikzcd} \tag{3.1.1.7}$$

where the top row is the isomorphism coming from Hodge theory multiplied by $2i$, the right column is defined using the isomorphism $\mathrm{Pic}^0_X \simeq \mathrm{H}^1(X, \mathcal{O}_X)/\mathrm{H}^1(X, \underline{\mathbf{Z}})$, and the bottom map sends s to $-H(s, \cdot)$. We can move the $2i$ into a $1/2i$ in front of the $H(s, \cdot)$ term, and this *cancels* the $2i$ in the definition of H.

We can enlarge (3.1.1.7) to

$$\begin{tikzcd}
\mathrm{H}^1(\mathrm{Pic}^0_X, \mathcal{O}) \arrow[r, "\mathrm{Hodge}", "\simeq"'] \arrow[d, "i_P^*"', "\simeq"] & \mathrm{H}^0(\mathrm{Pic}^0_X, \overline{\Omega}^1) \arrow[d, "i_P^*"', "\simeq"] & \overline{\mathrm{H}^1(X, \mathcal{O}_X)}^\vee \arrow[l, "\simeq"'] \arrow[d, "\simeq"', "\mathrm{Hodge}"] \\
\mathrm{H}^1(X, \mathcal{O}_X) \arrow[r, "\simeq", "\mathrm{Hodge}"'] & \mathrm{H}^0(X, \overline{\Omega}^1_X) \arrow[r, "\simeq"] & \mathrm{H}^0(X, \Omega^1_X)^\vee
\end{tikzcd}$$

$$\overline{\omega} \longmapsto -\int_X \overline{\omega} \wedge (\cdot)$$

where the composite along the bottom row and (inverse of) the right column is $s \mapsto -\frac{1}{2i} H(s, \cdot)$, due to Lemma 1.5.2.1 and the *definition* of H! Thus we need to prove commutativity of

(3.1.1.8)
$$\begin{tikzcd}
\mathrm{H}^0(\mathrm{Pic}^0_X, \overline{\Omega}^1) \arrow[d, "\mathrm{H}^0(i_P)"'] & \overline{\mathrm{H}^1(X, \mathcal{O}_X)}^\vee \arrow[l, "\simeq"'] \arrow[d, "\mathrm{Hodge}"] \\
\mathrm{H}^0(X, \overline{\Omega}^1_X) \arrow[r, "\simeq"'] & \mathrm{H}^0(X, \Omega^1_X)^\vee
\end{tikzcd}$$

where the top row is defined via the isomorphism $\mathrm{Pic}^0_X \simeq \mathrm{H}^1(X, \mathcal{O}_X)/\mathrm{H}^1(X, \underline{\mathbf{Z}})$ and the bottom row is defined by $\overline{\omega} \mapsto -\int_X \overline{\omega} \wedge (\cdot)$.

Using the Hodge theory isomorphism $\mathrm{H}^0(X, \Omega^1_X) \simeq \overline{\mathrm{H}^1(X, \mathcal{O}_X)}$, the composite along the right column and the bottom row of (3.1.1.8) corresponds to the pairing

$$(\overline{\omega}, \eta) \mapsto -\int_X \overline{\omega} \wedge \eta = \overline{\int_X \overline{\eta} \wedge \omega} \tag{3.1.1.9}$$

where $\eta \in \overline{\mathrm{H}^1(X, \mathcal{O}_X)} \simeq \mathrm{H}^0(X, \Omega^1_X)$.

In [**7**, Thms B.2.2, B.4.1] (see Remark 1.3.3.2 for clarification), it is shown that the isomorphism

$$\mathrm{H}^0(X, \Omega^1_X)^\vee \overset{i_P}{\to} \mathrm{H}^0(\mathrm{Pic}^0_X, \Omega^1)^\vee \simeq T_0(\mathrm{Pic}^0_X) \tag{3.1.1.10}$$

fits into a commutative diagram

(3.1.1.11)
$$\begin{tikzcd}
\mathrm{H}^1(X, \mathcal{O}_X) \arrow[r, "\mathrm{alg}"] \arrow[d, "\mathrm{res}"', "\simeq"] & T_0(\mathrm{Pic}^0_X) \\
\mathrm{H}^0(X, \Omega^1_X)^\vee \arrow[ur, "\simeq"'] &
\end{tikzcd}$$

with the top map the "algebraic" isomorphism and the left column the negative of the residue isomorphism. But the compatibility of the analytic and algebraic theories of the Jacobian asserts that (3.1.1.10) makes the map

$$\mathrm{H}^0(X, \Omega^1_X)^\vee \simeq T_0(\mathrm{Pic}^0_X) \overset{\exp}{\twoheadrightarrow} \mathrm{Pic}^0_X$$

have kernel equal to the image of $\mathrm{H}_1(X, \mathbf{Z})$ in $\mathrm{H}^0(X, \Omega^1_X)^\vee$ via integration. It also identifies $i_P : X \hookrightarrow \mathrm{Pic}^0_X \simeq \mathrm{H}^0(X, \Omega^1_X)^\vee/\mathrm{H}_1(X, \mathbf{Z})$ with the map $Q \mapsto \int_P^Q$.

Return now to (3.1.1.8). Extending scalars by complex conjugation and using (3.1.1.9), we are reduced to showing the commutativity of

$$\begin{tikzcd} \mathrm{H}^0(\mathrm{Pic}^0_X, \Omega^1) \arrow[r, "\simeq"] \arrow[d, "\mathrm{H}^0(i_P)"'] & \mathrm{H}^1(X, \mathcal{O}_X)^\vee \\ \mathrm{H}^0(X, \Omega^1_X) \arrow[ur, "\simeq"'] & \end{tikzcd}$$

where the top row comes from the isomorphism $\mathrm{Pic}^0_X \simeq \mathrm{H}^1(X, \mathcal{O}_X)/\mathrm{H}^1(X, \underline{\mathbf{Z}})$ and the right diagonal corresponds to $2\pi i \cdot \mathrm{res}$, thanks to Corollary 1.2.3.1 and Lemma 1.5.2.1. Dualizing (3.1.1.11) gives a commutative diagram

(3.1.1.12)
$$\begin{tikzcd} \mathrm{H}^1(X, \mathcal{O}_X)^\vee & T_0(\mathrm{Pic}^0_X)^\vee \arrow[l, "\simeq"'] \arrow[d, "\simeq"', "\mathrm{can}"] \\ \mathrm{H}^0(X, \Omega^1_X) \arrow[u, "\mathrm{res}", "\simeq"'] & \mathrm{H}^0(\mathrm{Pic}^0_X, \Omega^1) \arrow[l, "\simeq", "i_P"'] \end{tikzcd}$$

with the top row the dual of the "algebraic" isomorphism. Composing with multiplication by $2\pi i \in \mathbf{C}^\times$ on $\mathrm{H}^1(X, \mathcal{O}_X^\times)$ in (3.1.1.12), we finally have to show the commutativity of

$$\begin{tikzcd} \mathrm{H}^1(X, \mathcal{O}_X)^\vee & T_0(\mathrm{Pic}^0_X)^\vee \arrow[l, "2\pi i(\mathrm{alg})^\vee"'] \arrow[d, "\simeq"', "\mathrm{can}"] \\ & \mathrm{H}^0(\mathrm{Pic}^0_X, \Omega^1) \arrow[ul, "\simeq"] \end{tikzcd}$$

with the diagonal map coming from $\mathrm{Pic}^0_X \simeq \mathrm{H}^1(X, \mathcal{O}_X)/\mathrm{H}^1(X, \underline{\mathbf{Z}})$. There is an analogous such diagram which one can contemplate by replacing the Riemann surface X with a complex torus. Using the (easily verified) *functoriality* of this diagram with respect to the map $i_P : X \hookrightarrow \mathrm{Pic}^0_X$ we reduce to the commutativity of (1.3.3.6) for $A = \mathrm{Pic}^0_X$.

□

3.1.2. Picard functoriality. When $k = 0$, we have $\mathrm{Sym}^k_{\underline{\mathbf{Z}}} \mathcal{F} = \underline{\mathbf{Z}}$ for any local system $\mathcal{F}$ of finite free $\underline{\mathbf{Z}}$-modules. Thus,

$${}^0_N W = \widetilde{H}^1(Y_1(N), \underline{\mathbf{Z}}) = \mathrm{H}^1(X_1(N), \underline{\mathbf{Z}})$$

and the ring $\mathbf{T}_1(N) \subseteq \mathrm{End}_{\mathbf{Z}}(\mathrm{H}^1(X_1(N), \underline{\mathbf{Z}}))$ is generated by elements T_p and $\langle n \rangle$ as defined in §2.3.1. In the case $k = 0$ these definitions specialize as follows: T_p is

$$\mathrm{H}^1(X_1(N), \underline{\mathbf{Z}}) \xrightarrow{\pi_1^*} \mathrm{H}^1(X_1(N, p), \underline{\mathbf{Z}}) \xrightarrow{\pi_{2*}} \mathrm{H}^1(X_1(N), \underline{\mathbf{Z}})$$

and $\langle n \rangle$ is

$$\mathrm{H}^1(X_1(N), \underline{\mathbf{Z}}) \xrightarrow{I_n^*} \mathrm{H}^1(X_1(N), \underline{\mathbf{Z}}).$$

Also, the Atkin-Lehner involution w_ζ from §2.3.6 is

$$\mathrm{H}^1(X_1(N), \underline{\mathbf{Z}}) \xrightarrow{w_\zeta^*} \mathrm{H}^1(X_1(N), \underline{\mathbf{Z}}).$$

For any compact Riemann surface X, there is an isomorphism of complex Lie groups

(3.1.2.1)
$$\mathrm{Pic}^0_X \simeq \mathrm{H}^1(X, \mathcal{O}_X)/\mathrm{H}^1(X, \underline{\mathbf{Z}})$$

via the exponential sequence

$$0 \to \underline{\mathbf{Z}} \to \mathcal{O}_X \xrightarrow{e^{2\pi i(\cdot)}} \mathcal{O}_X^\times \to 1.$$

It may seem more canonical here to use $\underline{\mathbf{Z}}(1)$ rather than $\underline{\mathbf{Z}}$ (and exp rather than $e^{2\pi i(\cdot)}$), but we use $\underline{\mathbf{Z}}$ in order to remain consistent with the use of $\mathrm{H}^1(X_1(N), \underline{\mathbf{Z}})$ above (as it is the cohomology of $\underline{\mathbf{Z}}$ and not $\underline{\mathbf{Z}}(1)$ that occurs in the weight 2 case of the Eichler-Shimura isomorphism). For example, (3.1.2.1) yields an isomorphism

$$\mathrm{H}^1(X, \mathcal{O}_X) \simeq T_0(\mathrm{Pic}^0_X)$$

which is $2\pi i$ times the "algebraic" isomorphism arising from

$$0 \to \mathcal{O}_X \to \mathcal{O}_{X[\varepsilon]}^\times \to \mathcal{O}_X^\times \to 1. \tag{3.1.2.2}$$

This fundamental fact is analogous to (1.3.3.5) and was proven when we looked at the diagram (3.1.1.5) in the proof of Theorem 3.1.1.1.

LEMMA 3.1.2.1. *Let $f : X \to Y$ be a finite map of compact Riemann surfaces. The diagram*

$$\begin{tikzcd} \mathrm{H}^1(Y, \mathcal{O}_Y) \arrow[r, "f^*"] & \mathrm{H}^1(X, \mathcal{O}_X) \\ \mathrm{H}^1(Y, \underline{\mathbf{Z}}) \arrow[u] \arrow[r, "f^*"'] & \mathrm{H}^1(X, \underline{\mathbf{Z}}) \arrow[u] \end{tikzcd}$$

commutes; the induced analytic map $\mathrm{Pic}^0_Y \to \mathrm{Pic}^0_X$ on quotients along the columns coincides with the map $\mathrm{Pic}^0(f)$ induced by Pic^0-functoriality.

PROOF. The commutativity of the square is obvious. We may compare the two candidate maps $\mathrm{Pic}^0_Y \to \mathrm{Pic}^0_X$ by studying their effect on tangent spaces at the identity. The desired compatibility is then trivial in view of the preceding remarks concerning $T_0(\mathrm{Pic}^0_X)$.

□

3.1.3. Albanese functoriality. Let $f : X \to Y$ be as in Lemma 3.1.2.1. Since f is finite *flat*, there is a natural trace map

$$\mathrm{tr}_f : \mathrm{H}^1(X, \mathcal{O}_X) \simeq \mathrm{H}^1(Y, f_*\mathcal{O}_X) \to \mathrm{H}^1(Y, \mathcal{O}_Y).$$

Similarly, since finite coverings of the disk ramified only over 0 must locally look like $z \mapsto z^n$ for some positive integer n, it is easy to see that there is a well-defined trace map $f_*\underline{\mathbf{Z}} \to \underline{\mathbf{Z}}$ determined by the property than on stalks over $y \in Y$ it is just

$$\prod_{f(x)=y} \mathbf{Z} \to \mathbf{Z}$$

$$(a_x) \mapsto \sum e_x a_x$$

where e_x is the ramification degree of f at x. Note that $f_*\underline{\mathbf{Z}}$ is *not* a local system near the finite non-étale locus on Y for f. It is easy to see that the diagram of trace maps

$$\begin{tikzcd} f_*\mathcal{O}_X \arrow[r] & \mathcal{O}_Y \\ f_*\underline{\mathbf{Z}} \arrow[u] \arrow[r] & \underline{\mathbf{Z}} \arrow[u] \end{tikzcd}$$

commutes (as we may check this away from the branch locus), so we get a commutative diagram

$$\begin{tikzcd} \mathrm{H}^1(X,\mathcal{O}_X) \arrow[r, "\mathrm{tr}_f"] & \mathrm{H}^1(Y,\mathcal{O}_Y) \\ \mathrm{H}^1(X,\underline{\mathbf{Z}}) \arrow[u] \arrow[r, "\mathrm{tr}_f"'] & \mathrm{H}^1(Y,\underline{\mathbf{Z}}) \arrow[u] \end{tikzcd}$$

and hence a map

$$\mathrm{Pic}^0_X \to \mathrm{Pic}^0_Y\,. \tag{3.1.3.1}$$

THEOREM 3.1.3.1. *The map* (3.1.3.1) *is the map* $\mathrm{Alb}(f)$ *induced by Albanese functoriality of the Jacobian; equivalently, it is the dual of* $\mathrm{Pic}^0(f)$ *with respect to the canonical autoduality of Jacobians.*

PROOF. We again compare the two candidate maps $\mathrm{Pic}^0_X \to \mathrm{Pic}^0_Y$ on tangent spaces at the identity. Choosing base points $x \in X$, $y = f(x) \in Y$, the commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook, "i_x"'] & Y \arrow[d, hook, "i_y"] \\ \mathrm{Pic}^0_X \arrow[r, "\mathrm{Alb}(f)"'] & \mathrm{Pic}^0_Y \end{tikzcd}$$

(with vertical columns $i_x(x') = \mathcal{O}(x') \otimes \mathcal{O}(x)^{-1}$, $i_y(y') = \mathcal{O}(y') \otimes \mathcal{O}(y)^{-1}$) yields a commutative diagram

$$\begin{tikzcd} \mathrm{H}^0(X,\Omega^1_X) & \mathrm{H}^0(Y,\Omega^1_Y) \arrow[l, "f^*"'] \\ \mathrm{H}^0(\mathrm{Pic}^0_X,\Omega^1) \arrow[u, "\simeq"] \arrow[d, "\simeq"] & \mathrm{H}^0(\mathrm{Pic}^0_Y,\Omega^1) \arrow[l, "\mathrm{Alb}(f)^*"'] \arrow[u, "\simeq"'] \arrow[d, "\simeq"'] \\ T_0(\mathrm{Pic}^0_X)^\vee & T_0(\mathrm{Pic}^0_Y)^\vee \arrow[l, "\mathrm{Alb}(f)^\vee"] \end{tikzcd} \tag{3.1.3.2}$$

On the other hand, our candidate map (3.1.3.1) induces a commutative diagram

$$\begin{tikzcd} T_0(\mathrm{Pic}^0_X) \arrow[r] \arrow[d, "\simeq"'] & T_0(\mathrm{Pic}^0_Y) \arrow[d, "\simeq"] \\ \mathrm{H}^1(X,\mathcal{O}_X) \arrow[r, "\mathrm{tr}_f"'] & \mathrm{H}^1(Y,\mathcal{O}_Y) \end{tikzcd} \tag{3.1.3.3}$$

where we can take the columns to be the *algebraic* identifications via (3.1.2.2) (since linear maps commute with $2\pi i$).

In [**7**, Thms B.2.2, B.4.1], it is shown that the canonical duality between $T_0(\mathrm{Pic}^0_X)$ and $T_0(\mathrm{Pic}^0_X)^\vee$ is identified via the columns in (3.1.3.3) with the residue pairing between $\mathrm{H}^1(X,\mathcal{O}_X)$ and $\mathrm{H}^0(X,\Omega^1_X)$; this is *not* obvious (also see Remark

1.3.3.2). Given this and using (3.1.3.2) and (3.1.3.3), we are reduced to checking that

$$\begin{tikzcd}
\mathrm{H}^1(X,\mathcal{O}_X)\times\mathrm{H}^0(X,\Omega^1_X) \arrow[r] \arrow[d, "\mathrm{tr}_f"'] & \mathrm{H}^1(X,\Omega^1_X) \arrow[r, "\simeq"] & \mathbf{C} \arrow[d, equal] \\
\mathrm{H}^1(Y,\mathcal{O}_Y)\times\mathrm{H}^0(Y,\Omega^1_Y) \arrow[r] \arrow[u, shift left=8, "f^*"'] & \mathrm{H}^1(Y,\Omega^1_Y) \arrow[r, "\simeq"] & \mathbf{C}
\end{tikzcd}$$

commutes, using the residue map for the second maps of both rows.

Using the analytic description of the residue map and the fact that locally on X the map f looks like a map $\Delta_1 \to \Delta_2$ between open unit disks which is defined by $z \mapsto w = z^n$, it suffices to check that for holomorphic functions c on Δ_1^* and g on Δ_2^* we have

$$\mathrm{res}_{z=0}\big(c(z)g(z^n)nz^{n-1}\mathrm{d}z\big) \stackrel{?}{=} \mathrm{res}_{w=0}\big(\mathrm{tr}(c)g(w)\,\mathrm{d}w\big). \tag{3.1.3.4}$$

Expressing res as an integral and c, g as uniform limits of Laurent series, we may assume that $g(w) = w^i$ and $c(z) = z^j$. Then

$$\mathrm{tr}(c) = \begin{cases} 0 & n \nmid j; \\ w^{j/n} & n \mid j. \end{cases}$$

The calculation of the residues is now easy and confirms (3.1.3.4). $\square$

Combining Theorem 2.3.2.1 (with $k = 0$), Lemma 3.1.2.1, and Theorem 3.1.3.1, we conclude that

$$\begin{aligned} T_p^* &\stackrel{\mathrm{def}}{=} \mathrm{Alb}(\pi_1)\circ\mathrm{Pic}^0(\pi_2) \\ \langle n\rangle^* &\stackrel{\mathrm{def}}{=} \mathrm{Pic}^0(I_n) \\ w_\zeta^* &\stackrel{\mathrm{def}}{=} \mathrm{Pic}^0(w_\zeta) \end{aligned}$$

define operators on $\mathrm{Pic}^0_{X_1(N)}$ whose induced linear action on

$$T_0\big(\mathrm{Pic}^0_{X_1(N)}\big) \simeq \mathrm{H}^1\big(X_1(N),\mathcal{O}_{X_1(N)}\big) \simeq \mathrm{H}^0\big(X_1(N),\overline{\Omega}^1_{X_1(N)}\big)$$

is the *conjugate* of the classical action of the operators T_p, $\langle n\rangle$ and w_ζ on

$$S_2(\Gamma_1(N),\mathbf{C}) \simeq \mathrm{H}^0\big(X_1(N),\Omega^1_{X_1(N)}\big).$$

This uses the fact that the composite

$$\mathrm{H}^0\big(X_1(N),\overline{\Omega}^1_{X_1(N)}\big) \xrightarrow{\partial} \mathrm{H}^1\big(X_1(N),\underline{\mathbf{C}}\big) \longrightarrow \mathrm{H}^1\big(X_1(N),\mathcal{O}_{X_1(N)}\big).$$

is used to define the isomorphism $\mathrm{H}^0(X_1(N),\overline{\Omega}^1_{X_1(N)}) \simeq \mathrm{H}^1(X_1(N),\mathcal{O}_{X_1(N)})$.

By Theorem 3.1.3.1, the *dual* morphisms are

$$\begin{aligned} T_{p*} &= \mathrm{Alb}(\pi_2)\circ\mathrm{Pic}^0(\pi_1) \\ \langle n\rangle_* &= \mathrm{Alb}(I_n) \\ w_{\zeta*} &= \mathrm{Alb}(w_\zeta). \end{aligned}$$

Since I_n and w_ζ are automorphisms of $X_1(N)$ with inverses I_n^{-1} and w_ζ, respectively, we have $\langle n\rangle^* = \langle n^{-1}\rangle_*$ and $w_\zeta^* = w_{\zeta*}$. In particular, $\langle n\rangle^*$ commutes with T_{p*}, and $\langle n\rangle_*$ commutes with T_p^*. For p not dividing N, we also have $T_p^* = \langle p\rangle^* T_{p*} = \langle p^{-1}\rangle_* T_{p*}$. There are several ways to see this last fact; we will explain it shortly.

Using (1.5.2.2) and Corollary 1.2.3.1, together with [**7**, Appendix B] as before, we conclude that the maps induced on

$$T_0\big(\mathrm{Pic}^0_{X_1(N)}\big)^\vee \simeq \mathrm{H}^0\big(X_1(N), \Omega^1_{X_1(N)}\big) \simeq S_2(\Gamma_1(N), \mathbf{C})$$

by T_p^* and $\langle n\rangle^*$ are the *adjoints* of the usual Hecke action on $S_2(\Gamma_1(N), \mathbf{C})$ with respect to the (unnormalized) Petersson product

$$(f, g) \mapsto \int_{Y_1(N)} f\overline{g}\,\mathrm{d}z\mathrm{d}\overline{z}.$$

This reflects the pairing of $\mathrm{H}^0(X_1(N), \Omega^1_{X_1(N)})$ and $\mathrm{H}^0(X_1(N), \overline{\Omega}^1_{X_1(N)})$ via wedge products and integration. However, the induced action of T_p^* and $\langle n\rangle^*$ on the Tate module

$$T_l\big(\mathrm{Pic}^0_{X_1(N)}\big) \simeq \mathrm{H}^0\big(X_1(N), \underline{\mathbf{Z}}_l\big)$$

is *exactly* that induced by the earlier action of $T_p, \langle n\rangle \in \mathbf{T}_1(N)$ on $\mathrm{H}^1(X_1(N), \underline{\mathbf{Z}}) \simeq \mathrm{H}^1_c(Y_1(N), \underline{\mathbf{Z}}) = {}^0_N W$. Indeed, for any finite map $f : X_1 \to X_2$ of compact Riemann surfaces with f étale over the open complement $Y_2 \subseteq X_2$ of finitely many points, we have *commutative* diagrams

$$\begin{array}{ccc}
\mathrm{H}^1(X_1, \underline{\mathbf{Z}}) & \xrightarrow{\simeq} & \mathrm{H}^1_c(Y_1, \underline{\mathbf{Z}}) \\
{\scriptstyle \mathrm{tr}}\downarrow & & \downarrow{\scriptstyle \mathrm{tr}} \\
\mathrm{H}^1(X_2, \underline{\mathbf{Z}}) & \xrightarrow{\simeq} & \mathrm{H}^1_c(Y_2, \underline{\mathbf{Z}})
\end{array}$$

$$\begin{array}{ccc}
\mathrm{H}^1(X_2, \underline{\mathbf{Z}}) & \xrightarrow{\simeq} & \mathrm{H}^1_c(Y_2, \underline{\mathbf{Z}}) \\
{\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\
\mathrm{H}^1(X_1, \underline{\mathbf{Z}}) & \xrightarrow{\simeq} & \mathrm{H}^1_c(Y_1, \underline{\mathbf{Z}})
\end{array}$$

where $Y_1 = f^{-1}(Y_2)$ and the rows are the canonical isomorphisms. We emphasize that it is the $(\cdot)^*$ operations which recover the Hecke operators on $\mathrm{H}^1(X_1(N), \underline{\mathbf{Z}}) = {}^0_N W$ as defined cohomologically and geometrically in Chapter 2.

By the adjointness of pullback and trace with respect to cup product, we conclude that the operator T_{p*} on $\mathrm{H}^1(X_1(N), \underline{\mathbf{Z}})$ is the cup product adjoint of T_p^*. This is just the action of $\langle p\rangle^{-1} T_p \in \mathbf{T}_1(N)$ on $\mathrm{H}^1(X_1(N), \underline{\mathbf{Z}})$ (for p not dividing N), and thus indeed $T_{p*} = \langle p^{-1}\rangle^* T_p^*$, as desired.

DEFINITION 3.1.3.2. Set

$$V_l = \mathbf{Q}_l \otimes_{\mathbf{Z}_l} T_l(\mathrm{Pic}^0_{X_1(N)}) \simeq \mathrm{H}^1(X_1(N), \underline{\mathbf{Q}}_l).$$

By Lemma 2.4.1.1, V_l is a free rank 2 module over $\mathbf{T}_1(N)_{\mathbf{Q}_l}$, using the action of T_p^*, $\langle n\rangle^*$; V_l also has an action of w_ζ^*. We claim that conjugation by the involution $w_\zeta^* = w_{\zeta*}$ takes T_p^* to T_{p*} for *all* p, and $\langle n\rangle^*$ to $\langle n\rangle_*$ for all $n \in (\mathbf{Z}/N)^\times$. Since *dual* morphisms are *adjoints* with respect to the l-adic Weil pairing on $T_l(\mathrm{Pic}^0_{X_1(N)})$, where this self-pairing is defined via the canonical principal polarization on $\mathrm{Pic}^0_{X_1(N)}$, it suffices to use Theorem 2.3.7.3 and Theorem 3.1.1.1.

For any compact Riemann surface X and open subset Y which is the complement of finitely many points, the diagram

$$\begin{tikzcd} \mathrm{H}^1(X,\underline{\mathbf{Z}}) \times \mathrm{H}^1(X,\underline{\mathbf{Z}}) \arrow[r] \arrow[d, "\simeq"] & \mathrm{H}^2(X,\underline{\mathbf{Z}}) \arrow[r, "\simeq"] & \mathbf{Z} \arrow[d, equal] \\ \mathrm{H}^1_{\mathrm{c}}(Y,\underline{\mathbf{Z}}) \times \mathrm{H}^1_{\mathrm{c}}(Y,\underline{\mathbf{Z}}) \arrow[r] & \mathrm{H}^2_{\mathrm{c}}(Y,\underline{\mathbf{Z}}) \arrow[r, "\simeq"] & \mathbf{Z} \end{tikzcd}$$

commutes. Thus, Theorem 3.1.1.1 identifies the l-adic pairing ${}^0_N(\cdot,\cdot)_l$ with the negative of the l-adic Weil pairing $(\cdot,\cdot)_l$ on V_l. We therefore deduce:

COROLLARY 3.1.3.3. *With respect to the pairing*

$$[x,y]_l = (x, w_\zeta^* y)_l = (x, w_{\zeta *} y)_l$$

on V_l, all elements of $\mathbf{T}_1(N)$ are self-adjoint. With respect to $(\cdot,\cdot)_l$, the adjoint of T_p^* for p not dividing N is $\langle p\rangle^{*\,-1}T_p^* = T_{p*}$, and the adjoint of $\langle n\rangle^*$ is $\langle n^{-1}\rangle^* = \langle n\rangle_*$ for $n \in (\mathbf{Z}/N)^\times$. In particular, the dual morphisms to T_p^*, $\langle n\rangle^*$ are $\langle p\rangle^{*\,-1}T_p^*$, $\langle n\rangle^{*\,-1}$, respectively. The analogous formulas hold using $w_{\zeta *} = w_\zeta^*$, T_{p*}, $\langle n\rangle_*$ (i.e., with $\mathbf{T}_1(N)$ acting through the $(\cdot)_*$ operations).

3.2. Very brief survey of the arithmetic theory of modular curves

We will have much more to say on this topic in §4.2. Here we restrict our attention to stating some basic facts which are sufficient to handle weight 2. Further details and explanations will be given in §4.2.

3.2.1. Algebraic moduli schemes. A *generalized elliptic curve* $E \to S$ over a scheme S is defined in the same manner as in the analytic case, except that we replace “fiber” by “geometric fiber” in the definition. The algebraic theory of generalized elliptic curves is developed in [**14**]. We can likewise define the notion of *semistable curve of genus* 1 and *marked semistable curve of genus* 1; Theorem 1.6.2.3, Lemma 1.6.2.4 and Lemma 1.6.4.5 carry over to the algebraic setting, using geometric fibers rather than fibers. If S is a $\mathbf{Z}[1/N]$-scheme, we define the notion of a $\Gamma_1(N)$-structure on a generalized elliptic curve as in the analytic case (again imposing a condition on geometric fibers). Since $N \in \mathrm{H}^0(S, \mathcal{O}_S^\times)$ for such an S, it follows from a study of geometric fibers that $N : E^{\mathrm{sm}} \to E^{\mathrm{sm}}$ is quasi-finite étale.

The methods of [**14**, Chapter IV] show that for $N \geq 5$ there is a universal generalized elliptic curve with $\Gamma_1(N)$ structure over a base which is a projective smooth $\mathbf{Z}[1/N]$-scheme with geometric fibers of pure dimension 1. We write $E_1(N) \to X_1(N)$ for this family and so from now on we write $E_1(N)^{\mathrm{an}} \to X_1(N)^{\mathrm{an}}$ for its analytification. By Theorem 2.2.2.1, $E_1(N)^{\mathrm{an}} \to X_1(N)^{\mathrm{an}}$ is the analogous universal object in the analytic category which we studied in Chapter 1. The non-smooth locus in $X_1(N)$ is the base of a Cartier divisor which if finite étale over $\mathbf{Z}[1/N]$. This locus contains a $\mathbf{Z}[1/N]$-point called 0 which induces the analytic cusp 0. By comparison with the analytic theory (c.f. Theorem 2.2.2.1), the geometric fiber in characteristic 0 is connected. Recall that our proof of Theorem 2.2.2.1 did *not* require any (geometric) connectedness results on the algebraic side. We conclude from Stein factorization that all geometric fibers of $E_1(N) \to X_1(N)$ are connected. Let $Y_1(N) \subseteq X_1(N)$ be the open locus over which $E_1(N)$ is smooth.

In [**35**], a universal object $E_1(N)_{\mathrm{KM}} \to Y_1(N)_{\mathrm{KM}}$ is constructed in the category of (smooth) elliptic curves with $\Gamma_1(N)$-structures over $\mathbf{Z}[1/N]$-schemes, and it is shown that there is an open immersion $Y_1(N)_{\mathrm{KM}} \hookrightarrow X_1(N)_{\mathrm{KM}}$ into a projective smooth $\mathbf{Z}[1/N]$-scheme with geometrically connected fibers of dimension 1. This scheme $X_1(N)_{\mathrm{KM}}$ is the normalization of $\mathbf{P}^1_{\mathbf{Z}[1/N]}$ under the j-invariant map $j : Y_1(N)_{\mathrm{KM}} \to \mathbf{A}^1_{\mathbf{Z}[1/N]}$, which is shown to be *finite* flat (see Lemma 4.2.6.1). No object is constructed in [**35**] over $X_1(N)_{\mathrm{KM}}$.

LEMMA 3.2.1.1. *The canonical map*

$$Y_1(N)_{\mathrm{KM}} \to X_1(N) \tag{3.2.1.1}$$

is an isomorphism onto $Y_1(N)$. *It extends uniquely to an isomorphism*

$$X_1(N)_{\mathrm{KM}} \xrightarrow{\simeq} X_1(N).$$

PROOF. The first statement follows from the universal properties on both sides. It remains to extend (3.2.1.1) to a map $X_1(N)_{\mathrm{KM}} \to X_1(N)$, which will automatically be unique and an isomorphism by considerations of smoothness, properness, and the geometric fibers.

Certainly over $\mathbf{Q}$ the map (3.2.1.1) extends to an isomorphism, by the valuative criterion (i.e., the uniqueness of proper regular models of open curves over a field). Thus, for some large $M \in N\mathbf{Z}$, (3.2.1.1) extends to $X_1(N)_{\mathrm{KM}}[\frac{1}{M}] \xrightarrow{\simeq} X_1(N)[\frac{1}{M}]$. Similarly, by the valuative criterion (3.2.1.1) extends to the (codimension 1) generic points of the closed fibers.

To define the map everywhere one can argue conceptually by appealing to the uniqueness result in the minimal models theorem for regular proper curves over (excellent) Dedekind domains [**5**, Theorem 1.2]; this applies since the generic fibers do not have genus 0, as $N \geq 5$. Alternatively, the "contraction" construction in [**14**, Chapter IV, Section 1] leads to the definition of a map from $X_1(N)$ to the projective j-line. This map is proper and quasi-finite, thus finite, and also *flat* by [**42**, Thm 23.3]. It follows that $X_1(N)$ is the normalization of the projective j-line in the corresponding generic fiber function field extension. By *definition*, this gives the desired identification of $X_1(N)$ and $X_1(N)_{\mathrm{KM}}$. $\square$

3.2.2. Hecke operators. Similar methods allow one to construct (for all primes p) a projective smooth scheme $X_1(N,p)$ over $\mathbf{Z}[1/Np]$ with geometrically connected fibers and with analytification exactly as expected. The scheme $X_1(N,p)$ is canonically isomorphic to the analogous scheme constructed in [**35**]. We can functorially define degeneracy maps

$$\begin{tikzcd} & X_1(N,p) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X_1(N)[\frac{1}{p}] & & X_1(N)[\frac{1}{p}] \end{tikzcd}$$

over $\mathbf{Z}[1/Np]$ and

$$X_1(N) \xrightarrow{I_n} X_1(N)$$

over $\mathbf{Z}[1/N]$ as in the analytic case, recovering the analytic definitions after analytification (as can be checked on $\mathbf{C}$-points). Alternatively, one can define these maps in terms of the "open" models in [**35**] and extend via normalizations (though one needs to be careful since π_2 does *not* respect j-invariants). We omit a discussion of this alternative method.

Over $\mathbf{Z}[1/N, \zeta_N]$ we can also define w_ζ operators for all primitive Nth roots of unity $\zeta = \zeta_N^j$, $j \in (\mathbf{Z}/N)^\times$. We have $w_{\zeta^j} w_\zeta = \langle j \rangle$ for all $j \in (\mathbf{Z}/N)^\times$. In particular, $w_\zeta^2 = 1$.

Since $X_1(N) \to \operatorname{Spec} \mathbf{Z}[1/N]$ is a projective smooth curve with geometrically connected fibers, $\operatorname{Pic}^0_{X_1(N)/\mathbf{Z}[1/N]}$ is an abelian scheme over $\mathbf{Z}[1/N]$ and hence is the Néron model of its generic fiber. By the Néron mapping property we can therefore uniquely extend the map $\operatorname{Alb}(\pi_1) \circ \operatorname{Pic}^0(\pi_2)$ on $\operatorname{Pic}^0_{X_1(N)/\mathbf{Q}}$ to a map of abelian schemes

$$T_p^* : \operatorname{Pic}^0_{X_1(N)/\mathbf{Z}[1/N]} \to \operatorname{Pic}^0_{X_1(N)/\mathbf{Z}[1/N]}$$

over $\mathbf{Z}[1/N]$. We likewise define $\langle n \rangle^* = \operatorname{Pic}^0(I_n)$ over $\mathbf{Z}[1/N]$ and $w_\zeta^* = \operatorname{Pic}^0(w_\zeta)$ over $\mathbf{Z}[1/N, \zeta_N]$.

Passing to analytifications gives the previous analytic constructions, by Lemma 3.1.2.1, Theorem 3.1.3.1. Thus, the resulting subring

$$\mathbf{Z}[T_p^*, \langle n \rangle^*] \subseteq \operatorname{End}\big(\operatorname{Pic}^0_{X_1(N)/\mathbf{Z}[1/N]}\big) \hookrightarrow \operatorname{End}\big(\operatorname{Pic}^0_{X_1(N)^{\mathrm{an}}}\big)$$

is naturally identified with $\mathbf{T}_1(N)$. Using the isomorphism

$$\varprojlim_n \operatorname{Pic}^0_{X_1(N)/\mathbf{Z}[1/N]}[l^n](\overline{\mathbf{Q}}) \simeq T_l\big(\operatorname{Pic}^0_{X_1(N)^{\mathrm{an}}}\big)$$

(with respect to the algebraic closures $\overline{\mathbf{Q}} \subseteq \mathbf{C}$ of $\mathbf{Q}$) we can endow our previously defined analytic object

$$V_l \stackrel{\text{def}}{=} \mathbf{Q}_l \otimes_{\mathbf{Z}_l} T_l(\operatorname{Pic}^0_{X_1(N)^{\mathrm{an}}})$$

from Definition 3.1.3.2 with a canonical *continuous* action of $G_\mathbf{Q}$, unramified at all p not dividing Nl and *commuting* with the action of $\mathbf{T}_1(N)$.

Beware that for $\zeta \in \mu_N(\overline{\mathbf{Q}}) = \mu_N(\mathbf{C})$ a primitive Nth root of unity, the action of $G_\mathbf{Q}$ on V_l does not commute with w_ζ^*, as it is easy to see by base change considerations (over $\overline{\mathbf{Q}}$) that

$$(g^{-1})^* \circ w_{g(\zeta)}^* \circ g^* = w_\zeta^*,$$

where $g^* \in \operatorname{End}_{\mathbf{Q}_l}(V_l)$ is induced by the natural action of $g \in G_\mathbf{Q}$ on $\overline{\mathbf{Q}}$-points of the abelian variety $\operatorname{Pic}^0_{X_1(N)/\mathbf{Q}}$. Since $w_\zeta^* = w_{\zeta^{-1}}^*$ on V_l, it follows that

$$w_\zeta^* \in \operatorname{End}_{\mathbf{Q}(\zeta_N)}(\operatorname{Pic}^0_{X_1(N)/\mathbf{Q}(\zeta_N)})$$

is actually defined over the real subfield $\mathbf{Q}(\zeta_N)^+$ of $\mathbf{Q}(\zeta_N)$. It then follows by étale descent that the w_ζ are all defined over $\mathbf{Z}[1/N, \zeta_N]^+ \subseteq \mathbf{Z}[1/N, \zeta_N]$.

We may define T_{p*}, $\langle n \rangle_*$ and $w_{\zeta *} = w_\zeta^*$ operations on V_l in a similar manner, recovering the original $(\cdot)_*$-operations after analytification. For example, $T_{p*} = \operatorname{Alb}(\pi_2) \circ \operatorname{Pic}^0(\pi_1) \in \operatorname{End}_\mathbf{Q}(\operatorname{Pic}^0_{X_1(N)/\mathbf{Q}})$.

COROLLARY 3.2.2.1. *The representation*

$$\rho_{N,l} : G_\mathbf{Q} \to \operatorname{Aut}_{\mathbf{T}_1(N)_{\mathbf{Q}_l}}(V_l) \simeq \operatorname{GL}_2\big(\mathbf{T}_1(N)_{\mathbf{Q}_l}\big)$$

is a continuous representation unramified at p not dividing Nl (using either choice of $\mathbf{T}_1(N)$-action on $\operatorname{Pic}^0_{X_1(N)/\mathbf{Z}[1/N]}$, via the $(\cdot)_*$ or $(\cdot)^*$ constructions).

3.2.3. Galois representations. The main fact to be proven in the next section is the following result, which distinguishes the $(\cdot)_*$ and $(\cdot)^*$ constructions (cf. end of §3.3.3 as well as §4.1.2).

THEOREM 3.2.3.1. *Let $\mathbf{T}_1(N)$ act on $\mathrm{Pic}^0_{X_1(N)/\mathbf{Z}[1/N]}$ through the $(\cdot)_*$-action. For all p not dividing Nl, the characteristic polynomial of $\rho_{N,l}(\mathrm{Frob}_p)$ is*

$$X^2 - (T_{p*})X + p\,\langle p\rangle_* \,,$$

where Frob_p is an arithmetic Frobenius element at p.

Grant Theorem 3.2.3.1 for now. If we compose $\rho_{N,l}$ with a map

$$\mathbf{T}_1(N)_{\mathbf{Q}_l} \to K_{f,\lambda}$$

determined by a normalized *eigenform* $f \in S_2(\Gamma_1(N), \mathbf{C})$, we get the sought-after representations in weight 2. In order to get the more precise result of Shimura in Theorem 0.0.0.5, now let f be new *at level* N. Let $\mathfrak{p} \subseteq \mathbf{T}_1(N)$ be the minimal prime corresponding to f. We require $\mathbf{T}_1(N)$ to act on $\mathrm{Pic}^0_{X_1(N)/\mathbf{Z}[1/N]}$ via the $(\cdot)_*$-construction.

DEFINITION 3.2.3.2. Let A_f denote the quotient of $\mathrm{Pic}^0_{X_1(N)/\mathbf{Q}}$ by the action of $\mathfrak{p} \subseteq \mathbf{T}_1(N)$.

By construction, A_f has good reduction over $\mathbf{Z}[1/N]$ and the action of $\mathbf{T}_1(N)$ on $\mathrm{Pic}^0_{X_1(N)/\mathbf{Q}}$ induces an action of $\mathbf{T}_1(N)/\mathfrak{p}$ on A_f, and hence an action of $K_f \simeq (\mathbf{T}_1(N)/\mathfrak{p}) \otimes_{\mathbf{Z}} \mathbf{Q}$ on A_f in the "up-to-isogeny" category.

THEOREM 3.2.3.3 (Shimura). *A_f has dimension $[K_f : \mathbf{Q}]$ and $V_l(A_f)$ is free of rank 2 over $\mathbf{Q}_l \otimes_{\mathbf{Q}} K_f$. The characteristic polynomial of an arithmetic Frobenius Frob_p acting on $V_l(A_f)$ is*

$$X^2 - \bigl(1 \otimes a_p(f)\bigr)X + 1 \otimes p\chi_f(p)$$

for all p not dividing Nl.

PROOF. By Corollary 3.2.2.1 and Theorem 3.2.3.1, we just have to check that the natural map

$$V_l\bigl(\mathrm{Pic}^0_{X_1(N)/\mathbf{Q}}\bigr) \to V_l(A_f)$$

of $\mathbf{T}_1(N)_{\mathbf{Q}_l}$-modules identifies the right side with the quotient of the left side by $\mathfrak{p}$. More generally, for any exact sequence

$$B' \to B \to A \to 0$$

of abelian varieties over a field of characteristic prime to l, we claim that

$$V_l(B') \to V_l(B) \to V_l(A) \to 0$$

is exact. For this, we may assume that the base field is algebraically closed, and then we may appeal to Poincaré reducibility (i.e., abelian varieties over an algebraically closed field form a semisimple category when considered up to isogeny). $\square$

3.3. Proof of Theorem 3.2.3.1

3.3.1. Frobenius actions. For p not dividing N, let J_p denote

$$\mathrm{Pic}^0_{X_1(N)/\mathbf{F}_p} = \mathrm{Pic}^0_{X_1(N)/\mathbf{Z}[1/N]} \times_{\mathrm{Spec}\,\mathbf{Z}[1/N]} \mathrm{Spec}\,\mathbf{F}_p.$$

Clearly J_p is an abelian variety over $\mathbf{F}_p$ endowed with a natural action of T_{p*}. The canonical isomorphism

$$V_l\big(\mathrm{Pic}^0_{X_1(N)/\mathbf{Q}}\big) \simeq V_l\big(\mathrm{Pic}^0_{X_1(N)/\mathbf{F}_p}\big) = V_l\left(J_p\right)$$

identifies the arithmetic Frob_p action on $\overline{\mathbf{Q}}$-points on the left side with the arithmetic Frobenius action on $\overline{\mathbf{F}}_p$-points on the right side. We also know that $V_l(J_p)$ is free of rank 2 over $\mathbf{T}_1(N)_{\mathbf{Q}_l}$. For any $\mathbf{F}_p$-schemes Z, Z', the diagram

(3.3.1.1)
$$\begin{tikzcd} Z \arrow[r] \arrow[d, "F_Z"'] & Z' \arrow[d, "F_{Z'}"] \\ Z \arrow[r] & Z' \end{tikzcd}$$

commutes, where the columns are the absolute Frobenius morphisms. Taking $Z = \mathrm{Spec}\,\overline{\mathbf{F}}_p$, we see that the arithmetic Frobenius action on $V_l(J_p)$ is induced by functoriality from the intrinsic scheme morphism

$$F : J_p \to J_p$$

over $\mathbf{F}_p$. The key fact we need is the following.

THEOREM 3.3.1.1 (Eichler-Shimura). *In* $\mathrm{End}_{\mathbf{F}_p}(J_p)$,

$$T_{p*} = F + \langle p\rangle_* F^\vee \qquad w_\zeta^{-1} F w_\zeta = \langle p\rangle_*^{-1} F.$$

Admit this result for now, so

(3.3.1.2)
$$T_{p*}F = F^2 + \langle p\rangle_* FF^\vee.$$

The Frobenius F is induced by Albanese functoriality from $F_{X_1(N)/\mathbf{F}_p}$ since the diagram

$$\begin{tikzcd} X_1(N)_{/\mathbf{F}_p} \arrow[r, hook] \arrow[d, "F"'] & J_p \arrow[d, "F"] \\ X_1(N)_{/\mathbf{F}_p} \arrow[r, hook] & J_p \end{tikzcd}$$

must commute, where we define both horizontal embeddings by the same choice of $\mathbf{F}_p$-rational base point (e.g., coming from the cusp 0; i.e., the canonical $\Gamma_1(N)$-structure on the standard N-gon over $\mathbf{F}_p$). Using $J_p(\mathbf{F}_p) = \mathrm{Pic}^0(X_1(N)_{/\overline{\mathbf{F}}_p})$ and the interpretation of Albanese and Picard functoriality in terms of degree 0 divisors, we compute

(3.3.1.3)
$$\begin{aligned} FF^\vee &= (F_{X_1(N)})_*(F_{X_1(N)})^* \\ &= \deg F_{X_1(N)} \\ &= p. \end{aligned}$$

We now conclude from (3.3.1.2) that

$$F^2 - T_{p*}F + p\,\langle p\rangle_* = 0$$

on J_p, and hence on $V_l(J_p)$. Thus, $\rho_{N,l}(\mathrm{Frob}_p)$ satisfies the expected quadratic polynomial

$$X^2 - T_{p*}X + p\,\langle p\rangle_*\,.$$

3.3.2. Determination of the characteristic polynomial. Let us write $X^2 - aX + b$ for the *true* characteristic polynomial of $\rho_{N,l}(\mathrm{Frob}_p)$. We know that $\rho_{N,l}(\mathrm{Frob}_p)$ also satisfies this quadratic polynomial, by the Cayley-Hamilton theorem, so we need to prove that $a = T_{p*}$; given this, the equality $b = p\,\langle p\rangle_*$ is forced. The same dilemma will also arise in the case of higher weight. The following argument takes care of this problem (and naturally generalizes to higher weight), and is one of the reasons for introducing the operators w_ζ.

We want to show that

$$\mathrm{tr}_{\mathbf{T}_1(N)_{\mathbf{Q}_l}}\big(\rho_{N,l}(\mathrm{Frob}_p)\big) = T_{p*}$$

or equivalently that

$$\mathrm{tr}_{\mathbf{T}_1(N)_{\mathbf{Q}_l}}\big(V_l(F)\big) = T_{p*}.$$

The modified Weil pairing

$$[x,y]_l = \big(x, w_{\zeta*}(y)\big)_l$$

on V_l from Corollary 3.1.3.3 allows us to identify $V_l(J_p)$ with

$$V_l(J_p)^\vee = \mathrm{Hom}_{\mathbf{Q}_l}(V_l(J_p), \mathbf{Q}_l)$$

as $\mathbf{Q}_l$-vector spaces, since the isomorphism $V_l(J_p) \simeq V_l(\mathrm{Pic}^0_{X_1(N)/\mathbf{Q}})$ respects l-adic Weil pairings (check this using the relative Weil pairing as defined in [**48**], which coincides with the Weil pairing in [**45**] over an algebraically closed field). This identification is compatible with the $\mathbf{T}_1(N)$-action, but it takes the action of F on $V_l(J_p)$ to the action of $\langle p\rangle_* F^\vee$ on $V_l(J_p)^\vee$. This is because adjoints with respect to the Weil pairings correspond to dual morphisms of abelian varieties and $w_\zeta^{-1}F^\vee w_\zeta$ is the dual of $w_\zeta^{-1}Fw_\zeta = \langle p\rangle_*^{-1} F = F\,\langle p\rangle_*^{-1}$.

By Lemma 2.4.1.1, $V_l(J_p)$ is free of rank 2 over $\mathbf{T}_1(N)_{\mathbf{Q}_l}$ and

$$\mathrm{Hom}_{\mathbf{Q}_l}(\mathbf{T}_1(N)_{\mathbf{Q}_l}, \mathbf{Q}_l)$$

is free of rank 1 over $\mathbf{T}_1(N)_{\mathbf{Q}_l}$. Since the isomorphism $V_l(J_p) \simeq V_l(J_p)^\vee$ from Corollary 3.1.3.3 is $\mathbf{T}_1(N)_{\mathbf{Q}_l}$-equivariant, we conclude that $V_l(J_p)^\vee$ is free of rank 2 over $\mathbf{T}_1(N)_{\mathbf{Q}_l}$ and

$$\mathrm{tr}_{\mathbf{T}_1(N)_{\mathbf{Q}_l}}\big(F|_{V_l(J_p)}\big) = \mathrm{tr}_{\mathbf{T}_1(N)_{\mathbf{Q}_l}}\big(\langle p\rangle_* F^\vee|_{V_l(J_p)^\vee}\big). \tag{3.3.2.1}$$

We now invoke the following result, applied to the finite locally free $\mathbf{Q}_l$-algebra $\mathbf{T}_1(N)_{\mathbf{Q}_l}$ and the finite free $\mathbf{T}_1(N)_{\mathbf{Q}_l}$-module $V_l(J_p)$.

LEMMA 3.3.2.1. *Let $\mathcal{O}$ be a commutative ring and A a finite locally free $\mathcal{O}$-algebra with* $\mathrm{Hom}(A,\mathcal{O})$ *a locally free A-module (necessarily of rank 1). Let M be a finite locally free A-module, and set $M^* = \mathrm{Hom}_{\mathcal{O}}(M,\mathcal{O})$, so M^* is finite locally free over A with the same rank as M. For any A-linear map $f : M \to M$ with $\mathcal{O}$-dual $f^* : M^* \to M^*$ (which is automatically A-linear), the characteristic polynomials of f and f^* coincide in $A[T]$.*

PROOF. We may assume that $\mathcal{O}$ is local, so that A is semi-local. Making a faithfully flat base change to the Henselization of $\mathcal{O}$ (or the completion of $\mathcal{O}$ if $\mathcal{O}$ is noetherian), we may assume that A is a finite product of local rings. We can therefore assume that A is local.

Since A is local, $\mathrm{Hom}_{\mathcal{O}}(A, \mathcal{O})$ is free of rank 1 over A and M is finite free, so we can write $M = \oplus Ae_i$. Choose an isomorphism $h : A \simeq \mathrm{Hom}_{\mathcal{O}}(A, \mathcal{O})$ as A-modules. For each of the projections

$$\pi_i : M \to Ae_i \simeq A,$$

define $e_i^* = h(1) \circ \pi_i \in M^*$. These e_i^* are an A-basis of M^*, and the matrices of f and f^* over A satisfy

$$\mathrm{Mat}_{\{e_i\}}(f) = \mathrm{Mat}_{\{e_i^*\}}(f^*)^{\mathrm{t}}.$$

This proves the lemma. □

It follows from (3.3.2.1) and Lemma 3.3.2.1 that

$$\mathrm{tr}_{\mathbf{T}_1(N)_{\mathbf{Q}_l}}\big(F|_{V_l(J_p)}\big) = \mathrm{tr}_{\mathbf{T}_1(N)_{\mathbf{Q}_l}}\big(\langle p\rangle_* F^\vee|_{V_l(J_p)}\big).$$

Thus, by Theorem 3.3.1.1 we have

$$\begin{aligned} 2T_{p*} &= \mathrm{tr}\big(T_{p*}|_{V_l(J_p)}\big) \\ &= \mathrm{tr}\big(F + \langle p\rangle_* F^\vee|_{V_l(J_p)}\big) \\ &= 2\,\mathrm{tr}\big(F_{V_l(J_p)}\big). \end{aligned}$$

This proves that $\mathrm{tr}(F|_{V_l(J_p)}) = T_{p*}$, so indeed $X^2 - (T_{p*})X + p\,\langle p\rangle_*$ *is* the characteristic polynomial of Frob_p. This completes the proof of Theorem 3.2.3.1, and it remains to give the proof of Theorem 3.3.1.1.

3.3.3. Proof of the Eichler-Shimura relation.

PROOF OF THEOREM 3.3.1.1. It suffices to check that the pairs of maps in Theorem 3.3.1.1 coincide on a Zariski dense subset of $J_p(\overline{\mathbf{F}}_p) = \mathrm{Pic}^0_{X_1(N)/\overline{\mathbf{F}}_p}(\overline{\mathbf{F}}_p)$. If g is the genus of $X_1(N)_{/\overline{\mathbf{F}}_p}$ and if we fix an embedding $X_1(N)_{/\overline{\mathbf{F}}_p} \hookrightarrow J_p$ arising from an $\overline{\mathbf{F}}_p$-rational base point, then the induced map

$$X_1(N)^g_{/\overline{\mathbf{F}}_p} \to J_p$$

is finite. We conclude that for any nonempty open subscheme $U \subseteq X_1(N)_{/\overline{\mathbf{F}}_p}$, the map $U^g \to J_p$ hits a Zariski dense subset of $\overline{\mathbf{F}}_p$-points. Taking U to be the ordinary locus $Y_1(N)^{\mathrm{ord}}_{/\overline{\mathbf{F}}_p}$ (i.e., the set of points of $Y_1(N)_{/\overline{\mathbf{F}}_p}$ over which the universal elliptic curve has ordinary fibers), it suffices to study what happens to a divisor $(x) - (x')$ where $x, x' \in Y_1(N)^{\mathrm{ord}}(\overline{\mathbf{F}}_p)$ correspond to $\Gamma_1(N)$-structures (E, P), (E', P') over $\overline{\mathbf{F}}_p$.

The map $J_p(\overline{\mathbf{F}}_p) \to J_p(\overline{\mathbf{F}}_p)$ induced by F is exactly the result of applying the pth power map on $\overline{\mathbf{F}}_p$, by (3.3.1.1). By Picard functoriality, this corresponds to the base change of an invertible sheaf on $X_1(N)_{/\overline{\mathbf{F}}_p}$ by the endomorphism $1 \otimes \phi$ of $X_1(N) \times_{\mathbf{F}_p} \overline{\mathbf{F}}_p$, where ϕ is the arithmetic Frobenius on $\overline{\mathbf{F}}_p$. But by the definition of $X_1(N)_{/\overline{\mathbf{F}}_p}$ as a *universal object*, the map on $X_1(N)(\overline{\mathbf{F}}_p)$ induced by $1 \times \phi$ corresponds *exactly* to the "base change by arithmetic Frobenius" ϕ in terms of elliptic curves with $\Gamma_1(N)$-structures over $\overline{\mathbf{F}}_p$. We conclude that

$$F\big((x) - (x')\big) = \big(E^{(p)}, P^{(p)}\big) - \big(E'^{(p)}, P'^{(p)}\big)$$

where $(\cdot)^{(p)}$ denotes base change by arithmetic Frobenius ϕ on $\overline{\mathbf{F}}_p$.

Since $p = FF^\vee = F^\vee F$ and F is bijective on $\overline{\mathbf{F}}_p$-points, we have

$$\begin{aligned} F^\vee\big((x) - (x')\big) &= p \cdot F^{-1}\big((x) - (x')\big) \\ &= p \cdot \big((E^{(p)}, P^{(p)}) - (E'^{(p)}, P'^{(p)})\big). \end{aligned}$$

Thus,

$$\langle p\rangle_* F^\vee\big((x) - (x')\big) = p \cdot \big(E^{(p^{-1})}, p \cdot P^{(p^{-1})}\big) - p \cdot \big(E'^{(p^{-1})}, p \cdot P'^{(p^{-1})}\big),$$

so

$$\begin{aligned} \big(F + \langle p\rangle_* F^\vee\big) = \big(E^{(p)}, P^{(p)}\big) + p \cdot \big(E^{(p^{-1})}, p \cdot P^{(p^{-1})}\big) \\ - \big(E'^{(p)}, P'^{(p)}\big) - p \cdot \big(E'^{(p^{-1})}, p \cdot P'^{(p^{-1})}\big). \end{aligned} \tag{3.3.3.1}$$

Computing T_{p*} on $J_p = \mathrm{Pic}^0_{X_1(N)/\mathbf{Z}[1/N]} \times_{\mathbf{Z}[1/N]} \mathbf{F}_p$ is more subtle because T_{p*} was defined over $\mathbf{Z}[1/Np]$ as $\pi_{2*}\pi_1^*$ and was *extended* to $\mathbf{Z}[1/N]$ by the Néron mapping property. That is, we do *not* have a direct definition of T_p in characteristic p, but rather we need to lift to characteristic 0 to compute things.

Let R be a strictly henselian discrete valuation ring of characteristic 0 with residue field $\overline{\mathbf{F}}_p$. For example, R can be any of the rings $\mathbf{Z}_{(p)}^{\mathrm{sh}}$ (the strict Henselization of the localization $\mathbf{Z}_{(p)}$ of $\mathbf{Z}$), $\mathbf{Z}_p^{\mathrm{un}}$ (the ring of integers of the maximal unramified extension of $\mathbf{Q}_p$) or $W(\overline{\mathbf{F}}_p)$ (the Witt vectors of $\overline{\mathbf{F}}_p$); it does not matter which we use, but the argument below will require us to replace R by an unspecified finite extension ring, so it is important to allow such generality in the choice of R. Note that R is a $\mathbf{Z}[1/N]$-algebra.

Let K be the fraction field of R. Since R is a $\mathbf{Z}[1/N]$-algebra and $X_1(N)$ is *smooth* over $\mathbf{Z}[1/N]$, we conclude from the fact that R is strictly Henselian that

$$X_1(N)(R) \to X_1(N)(\overline{\mathbf{F}}_p)$$

is *surjective*. Of course, this can be proven by over $Y_1(N)$: if (E, P) is given over $\overline{\mathbf{F}}_p$, choose a Weierstrass model $\mathcal{E} \hookrightarrow \mathbf{P}^2_R$ to lift E. Since R is strictly Henselian, the finite étale $\mathcal{E}[N]$ is constant, so there is a unique closed immersion $\underline{\mathbf{Z}/N} \hookrightarrow \mathcal{E}[N]$ lifting $P : \underline{\mathbf{Z}/N} \hookrightarrow E[N]$.

Let $(\mathcal{E}, \mathcal{P})$, $(\mathcal{E}', \mathcal{P}')$ over R lift x and x'. Using the reduction map

$$\mathrm{Pic}^0_{X_1(N)/\mathbf{Z}[1/N]}(R) \to J_p(\overline{\mathbf{F}}_p)$$

and the *definition* of T_{p*}, we see that $T_{p*}((x) - (x'))$ is the image of

$$T_{p*}((\mathcal{E}, \mathcal{P}) - (\mathcal{E}', \mathcal{P}')).$$

Though R is *not* a $\mathbf{Z}[1/Np]$-algebra, K is such an algebra. Also, we have an injection (in fact, an isomorphism)

$$\mathrm{Pic}^0_{X_1(N)/\mathbf{Z}[1/N]}(R) \hookrightarrow \mathrm{Pic}^0_{X_1(N)/\mathbf{Z}[1/N]}(K) \tag{3.3.3.2}$$

since $\mathrm{Pic}^0_{X_1(N)/\mathbf{Z}[1/N]} \to \operatorname{Spec} \mathbf{Z}[1/N]$ is separated (in fact, proper).

We first compute T_{p*} by working with $\overline{K}$-points for an algebraic closure $\overline{K}$ of K. Since p does not divide N,

$$\pi_{2*}\pi_1^*\big((\mathcal{E}, \mathcal{P})_{/\overline{K}}\big) = \sum_{\substack{C \subseteq \mathcal{E}_{/\overline{K}}[p] \\ \#C = p}} \big(\mathcal{E}_{\overline{K}}/C, \mathcal{P}_{\overline{K}} \bmod C\big), \tag{3.3.3.3}$$

where the sum is take over all order p subgroups of $\mathcal{E}(\overline{K})$. Since $\mathcal{E} \to \operatorname{Spec} R$ has *ordinary* reduction and R is strictly Henselian, there is a short exact sequence

$$0 \to \mu_p \to \mathcal{E}[p] \to \underline{\mathbf{Z}/p} \to 0 \tag{3.3.3.4}$$

of finite flat group schemes on $\operatorname{Spec} R$; this is just the connected-étale sequence of $\mathcal{E}[p]$. Enlarging R to a finite extension does not change the residue field $\overline{\mathbf{F}}_p$, so we may assume

$$\mathcal{E}[p]_{/K} \simeq \underline{\mathbf{Z}/p} \times \underline{\mathbf{Z}/p}.$$

Taking the scheme-theoretic closure in $\mathcal{E}[p]$ of the $p+1$ distinct subgroups of $\mathcal{E}[p]_{/K}$ gives $p+1$ distinct finite flat subgroup schemes $\mathcal{C} \subseteq \mathcal{E}$ realizing the $p+1$ distinct subgroups over $\overline{K}$. *Exactly one* of these subgroups is killed by the map $\mathcal{E}[p] \to \underline{\mathbf{Z}/p}$ in (3.3.3.4), as this can be checked on the generic fiber. This subgroup must be $\mu_p \hookrightarrow \mathcal{E}[p]$ by (3.3.3.4), and the rest of the subgroups admit a map $\mathcal{C} \to \underline{\mathbf{Z}/p}$ which is an isomorphism on generic fibers. These maps *must* be isomorphisms! Indeed, if $\mathcal{C}$ is étale then this is clear. If $\mathcal{C}$ is not étale, since $\mathcal{C} \hookrightarrow \mathcal{E}[p]$ is finite flat of order p we can check on the closed fiber that $\mathcal{C}$ is multiplicative. But $\mathcal{E}[p]$ has a *unique* multiplicative subgroup scheme of order p since $\mathcal{E}[p]^\vee \simeq \mathcal{E}[p]$ by Cartier-Nishi duality and $\mathcal{E}[p]$ has a *unique* order p étale quotient (as any such quotient must kill the μ_p we have already exhibited inside of $\mathcal{E}[p]$).

We can therefore rewrite (3.3.3.3) as

$$\pi_{2*}\pi_1^*\big((\mathcal{E},\mathcal{P})_{/\overline{K}}\big) = \sum_{\mathcal{C}} \big(\mathcal{E}/\mathcal{C}, \mathcal{P} \bmod \mathcal{C}\big)_{/\overline{K}}.$$

We conclude by (3.3.3.2) that

$$\sum_{\mathcal{C}} \big(\mathcal{E}/\mathcal{C}, \mathcal{P} \bmod \mathcal{C}\big) - \sum_{\mathcal{C}'} \big(\mathcal{E}'/\mathcal{C}', \mathcal{P} \bmod \mathcal{C}'\big) \in \operatorname{Pic}^0_{X_1(N)/\mathbf{Z}[1/N]}(R)$$

coincides with

$$T_{p*}\big((\mathcal{E},\mathcal{P}) - (\mathcal{E}',\mathcal{P}')\big),$$

since they both induce the same $\overline{K}$-point. Passing to closed fibers we find that

$$\begin{aligned} T_{p*}\big((x)-(x')\big) = {} & \big(E/\mu_p, P \bmod \mu_p\big) + p\cdot\big(E/(\underline{\mathbf{Z}/p}), P \bmod \underline{\mathbf{Z}/p}\big) - \\ & \big(E'/\mu_p, P' \bmod \mu_p\big) - p\cdot\big(E'/(\underline{\mathbf{Z}/p}), P' \bmod \underline{\mathbf{Z}/p}\big). \end{aligned}$$

Here $E[p] \simeq \mu_p \times \underline{\mathbf{Z}/p}$, $E'[p] \simeq \mu_p \times \underline{\mathbf{Z}/p}$ are the canonical splittings of the connected étale sequences over the perfect field $\overline{\mathbf{F}}_p$.

Consider now the *relative* Frobenius morphism over $\overline{\mathbf{F}}_p$

$$F_{E/\overline{\mathbf{F}}_p} : E \to E^{(p)}$$

which takes 0 to 0 (and P to $P^{(p)}$) and thus is a map of elliptic curves over $\overline{\mathbf{F}}_p$. This map has degree p (since $E \to \operatorname{Spec} \overline{\mathbf{F}}_p$ is smooth of relative dimension 1) and is bijective on $\overline{\mathbf{F}}_p$-points. In particular, it has a connected kernel of order p. The only such subgroup scheme of E is the given μ_p, so we conclude that $E/\mu_p \simeq E^{(p)}$, with $P \bmod \mu_p$ corresponding to $P^{(p)}$.

Similarly, we have a commutative diagram

$$\begin{tikzcd} E \arrow[r, "F_{E/\overline{\mathbf{F}}_p}"'] \arrow[rr, bend left, "p"] & E^{(p)} \arrow[r, "F^\vee_{E/\overline{\mathbf{F}}_p}"'] & E \end{tikzcd}$$

so $F^\vee_{E/\overline{\mathbf{F}}_p}$ is *étale* of order p by the *ordinariness* hypothesis. Base change by $\mathrm{Frob}_p^{-1} : \overline{\mathbf{F}}_p \to \overline{\mathbf{F}}_p$ yields

$$\begin{tikzcd} E^{(p^{-1})} \arrow[r] \arrow[rr, bend left, "p"] & E \arrow[r] & E^{(p^{-1})} \\ P^{(p^{-1})} \arrow[r, mapsto] & P \arrow[r, mapsto] & p \cdot P^{(p^{-1})} \end{tikzcd}$$

and the second map is étale of degree p. Thus,

$$\big(E/(\underline{\mathbf{Z}/p}), P \bmod \underline{\mathbf{Z}/p}\big) \simeq \big(E^{(p^{-1})}, p \cdot P^{(p^{-1})}\big).$$

We conclude that

$$\begin{aligned} T_{p*}\big((x) - (x')\big) = \big(E^{(p)}, P^{(p)}\big) + p \cdot \big(E^{(p^{-1})}, p \cdot P^{(p^{-1})}\big) - \\ \big(E'^{(p)}, P'^{(p)}\big) - p \cdot \big(E'^{(p^{-1})}, p \cdot P'^{(p^{-1})}\big). \end{aligned}$$

By (3.3.3.1), this completes the proof of the first relation in Theorem 3.3.1.1.

Of course, the result of the above calculation is more precise; since we even obtained literal equality of *divisors* on $X_1(N)_{\overline{\mathbf{F}}_p}$ without needing to invoke linear equivalence. (This is fortunate, as it is hard to imagine being able to give such a proof if we didn't have equality on the level of divisors.)

It remains to check that

$$w_\zeta^{-1} F w_\zeta = \langle p \rangle_*^{-1} F$$

or equivalently that

$$F w_\zeta = w_\zeta \langle p \rangle_*^{-1} F.$$

All maps here are induced by maps on $X_1(N)_{\overline{\mathbf{F}}_p}$, with $F = \mathrm{Alb}(F_{X_1(N)})$, $w_\zeta = \mathrm{Alb}(w_\zeta|_{X_1(N)})$ and $\langle p \rangle_*^{-1} = \mathrm{Alb}(\langle p^{-1} \rangle|_{X_1(N)})$. Thus, it suffices to prove

$$F_{X_1(N)} w_\zeta = w_\zeta \langle p \rangle_*^{-1} F_{X_1(N)}$$

on $X_1(N)_{/\mathbf{F}_p}$. We can even check this on $\overline{\mathbf{F}}_p$-points of $Y_1(N)$.

Let $x = (E, P)$ be such a point, so

$$\begin{aligned} F_{X_1(N)} w_\zeta(x) &= F_{X_1(N)}\big(E/P, P'\big) \\ &= \big(E^{(p)}/\langle P^{(p)} \rangle, P'\big) \end{aligned} \tag{3.3.3.5}$$

where $\langle P, P' \rangle_N = \zeta$. In particular, $\langle P^{(p)}, P'^{(p^{-1})} \rangle_N = \zeta^p$ by the compatibility of the Weil pairing with base change.

Meanwhile, we compute

$$\begin{aligned} w_\zeta \langle p \rangle_*^{-1} F_{X_1(N)}(x) &= w_\zeta\big(E^{(p)}, p^{-1} P^{(p)}\big) \\ &= \big(E^{(p)}/\langle p^{-1} P^{(p)} \rangle, Q \bmod \langle p^{-1} P^{(p)} \rangle\big) \end{aligned}$$

where Q is any N-torsion point on $E^{(p)}$ satisfying $\langle p^{-1} P^{(p)}, Q \rangle_N = \zeta$, or equivalently $\langle P^{(p)}, Q \rangle_N = \zeta^p$. Since $Q = P'^{(p)}$ is such a point, by (3.3.3.5) we are done. $\square$

The comparison isomorphism

$$\mathrm{H}^1_{\text{ét}}(X_1(N)_{\overline{\mathbf{Q}}}, \mathbf{Q}_l(1)) \simeq V_l,$$

analogous to the canonical isomorphism

$$V_l(\mathrm{Pic}^0_X) \simeq V_l(\mathrm{H}^1(X, \mathcal{O}_X)/\mathrm{H}^1(X, \underline{\mathbf{Z}}(1))) \simeq \mathrm{H}^1(X, \underline{\mathbf{Q}}_l(1))$$

for a compact Riemann surface X, allows us to translate the above results into the language of étale cohomology. Our study of étale cohomology will involve generalizations of $\mathrm{H}^1_{\text{ét}}(X_1(N), \mathbf{Q}_l)$ rather than $\mathrm{H}^1_{\text{ét}}(X_1(N), \mathbf{Q}_l(1))$, so the absence of the "Tate twist" $\mathbf{Q}_l(1)$ will lead to some changes in the analogues of the formulas in Theorem 3.3.1.1. In particular, it turns out that the results in étale cohomology are best expressed in terms of the analogues of the T_p^* and $\langle n \rangle^*$ operators. The precise dictionary will be explained in §4.1.2.

CHAPTER 4

Preparations for étale cohomology

We begin by giving an l-adic formulation of the main results and constructions over $\mathbf{C}$. We also translate our weight 2 results into the same language, bypassing Tate modules of abelian varieties. The l-adic formulation over $\mathbf{C}$ will allow us to reduce the proof of Main Theorem 2 to an analogue of Theorem 3.3.1.1 for arbitrary weight. Then we discuss the arithmetic models of modular curves and various geometric preparations in characteristic p. In Chapter 5 we will discuss the facts that we will need from étale cohomology, and how they are used to prove the higher weight generalization of Theorem 3.3.1.1 which leads to proofs of Main Theorems 1 and 2, building upon on the algebro-geometric framework we set up in this chapter.

4.1. l-adic reformulation

4.1.1. Geometric set-up. Let us recall the basic geometric setup from Chapter 2. For $N \geq 5$ we defined analytic spaces and maps fitting into the two diagrams:

$$\begin{tikzcd}
 & E_1(N,p) \arrow[rr, "\varphi"] \arrow[dl] \arrow[dr, "u"] & & E_1(N,p)/\mathcal{C} \arrow[dl, "v"'] \arrow[dr] & \\
E_1(N) \arrow[dr, "f_N"'] & & Y_1(N,p) \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"'] & & E_1(N) \arrow[dl, "f_N"] \\
 & Y_1(N) & & Y_1(N) &
\end{tikzcd}$$

$$\begin{tikzcd}
E_1(N) \arrow[r, "\simeq"] \arrow[d, "f_N"] & E_1(N) \arrow[d, "f_N"'] \\
Y_1(N) \arrow[r, "I_n", "\simeq"'] & Y_1(N)
\end{tikzcd}$$

We have also defined

$$ {}^k_N W_{\mathbf{Q}} = \widetilde{\mathrm{H}}^1(Y_1(N), \mathrm{Sym}^k \, \mathrm{R}^1 f_{N*}\underline{\mathbf{Z}}) \otimes_{\mathbf{Z}} \mathbf{Q}, $$

with operators

$$ T_p = \mathrm{tr}_{\pi_1} \circ \varphi^* \circ \pi_2^* \qquad \langle n \rangle = I_n^* $$

generating a finite $\mathbf{Z}$-algebra $\mathbf{T}_1(N) \subseteq \mathrm{End}_{\mathbf{Q}}({}^k_N W_{\mathbf{Q}})$. The $\mathbf{T}_1(N)$-module ${}^k_N W_{\mathbf{Q}}$ is free of rank 2 and we have a perfect bilinear pairing

$$ {}^k_N(\cdot,\cdot) : {}^k_N W_{\mathbf{Q}} \otimes_{\mathbf{Q}} {}^k_N W_{\mathbf{Q}} \to \mathbf{Q}(-k-1) $$

satisfying

$$ {}^k_N(x,y) = (-1)^{k+1} \cdot {}^k_N(y,x). $$

For a primitive Nth root of unity $\zeta \in \mu_N(\mathbf{C})$, we have a diagram

$$\begin{tikzcd} E_1(N) \arrow[r] \arrow[dr] & E_1(N)/\langle \mathcal{P} \rangle \arrow[r] \arrow[d] & E_1(N) \arrow[d, "f_N"] \\ & Y_1(N) \arrow[r, "w_\zeta"'] & Y_1(N) \end{tikzcd}$$

and $\mathbf{T}_1(N)$ acts in a self-adjoint manner with respect to the modified pairing

$$^k_N[x, y] = {}^k_N(x, w_\zeta y).$$

We also gave (non-geometric) constructions of $^k_N W_{\mathbf{Q}}$, $\mathbf{T}_1(N)$, $^k_N(,)$ and w_ζ for $N \leq 4$, having the same algebraic properties, and we established compatibility with respect to change in level for all $N \geq 1$.

Let us recall the l-adic analogues from the end of Chapter 2. Define $^k_N W_l = {}^k_N W_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{Q}_l$. By Lemma 1.7.7.2, for $N \geq 5$ we have an isomorphism

$$^k_N W_l \simeq \widetilde{\mathrm{H}}^1(Y_1(N), \mathrm{Sym}^k \mathrm{R}^1 f_{N*} \underline{\mathbf{Q}}_l)$$

and the entire geometric/cohomological discussion over $\mathbf{C}$ can be carried out with $\mathbf{Q}_l$ replacing $\mathbf{Z}$ everywhere; this recovers exactly $^k_N W_l$ with the pairing $^k_N(\cdot,\cdot)_l$ obtained by extension of scalars from $^k_N(\cdot,\cdot)$ over $\mathbf{Z}$. Moreover, we can view the pairing $^k_N(\cdot,\cdot)_l$ as taking values in $\mathbf{Q}_l(1)^{\otimes(-k-1)}$, where $\mathbf{Q}_l(1) = \mathbf{Q}_l \otimes_{\mathbf{Z}_l} \mathbf{Z}_l(1)$ with

$$\mathbf{Z}_l(1) \stackrel{\mathrm{def}}{=} \varprojlim \ \mu_{l^n}(\mathbf{C}) \simeq \varprojlim \ (\mathbf{Z}(1) \otimes \mathbf{Z}/l^n) \simeq \mathbf{Z}(1) \otimes_{\mathbf{Z}} \mathbf{Z}_l$$

via

$$\exp\left(\tfrac{1}{l^n}(\cdot)\right) : \mathbf{Z}(1) \otimes \mathbf{Z}/l^n \xrightarrow{\simeq} \mu_{l^n}(\mathbf{C});$$

this ensures compatibility with our earlier notation $M(1) = M \otimes_{\mathbf{Z}} \mathbf{Z}(1)$.

The l-adic structures at $N \leq 4$ can be intrinsically recovered from the l-adic structures at levels ≥ 5 by the same procedure as in the case of $\mathbf{Q}$-coefficients.

Fix $N \geq 1$ and $\mathfrak{p} \subseteq \mathbf{T}_1(N)$ a minimal prime corresponding to a $G_{\mathbf{Q}}$-conjugacy class of newforms. Recall that $\mathbf{T}_1(N)_{\mathfrak{p}}$ is a CM (or totally real) field. Let λ be a place of $\mathbf{T}_1(N)_{\mathfrak{p}}$ over l, so that $\mathbf{T}_1(N)_{\mathfrak{p},\lambda}$ is a factor ring of the finite étale $\mathbf{Q}_l$-algebra $\mathbf{T}_1(N)_{\mathfrak{p}} \otimes_{\mathbf{Q}} \mathbf{Q}_l$. In particular,

$$V_{\mathfrak{p},\lambda} = {}^k_N W_l \otimes_{\mathbf{T}_1(N)_{\mathbf{Q}_l}} \mathbf{T}_1(N)_{\mathfrak{p},\lambda} \tag{4.1.1.1}$$

is a 2-dimensional $\mathbf{T}_1(N)_{\mathfrak{p},\lambda}$-vector space and a direct summand of $^k_N W_l$.

If $\overline{\lambda}$ is the conjugate of λ with respect to the intrinsic complex conjugation on the CM (or totally real) field $\mathbf{T}_1(N)_{\mathfrak{p}}$, then we have a canonical linear isomorphism

$$V_{\mathfrak{p},\overline{\lambda}} \simeq V_{\mathfrak{p},\lambda} \otimes_{\mathbf{T}_1(N)_{\mathfrak{p},\lambda}} \mathbf{T}_1(N)_{\mathfrak{p},\overline{\lambda}}.$$

Via (4.1.1.1), the $\mathbf{T}_1(N)_{\mathbf{Q}_l}$-action on $^k_N W_l$ goes over to the the action of $\mathbf{T}_1(N)_{\mathfrak{p},\lambda}$ (resp. $\mathbf{T}_1(N)_{\mathfrak{p},\overline{\lambda}}$) on $V_{\mathfrak{p},\lambda}$ (resp. $V_{\mathfrak{p},\overline{\lambda}}$). Both $V_{\mathfrak{p},\lambda}$ and $V_{\mathfrak{p},\overline{\lambda}}$ are canonical $\mathbf{T}_1(N)_{\mathbf{Q}_l}$-module direct summands of

$$^k_N W_l \otimes_{\mathbf{Q}_l} \mathbf{T}_1(N)_{\mathfrak{p},\lambda}, \tag{4.1.1.2}$$

where the $\mathbf{T}_1(N)_{\mathfrak{p},\lambda}$-action on (4.1.1.2) goes over to the canonical action on $V_{\mathfrak{p},\lambda}$ and the action through $\mathbf{T}_1(N)_{\mathfrak{p},\lambda} \simeq \mathbf{T}_1(N)_{\mathfrak{p},\overline{\lambda}}$ on $V_{\mathfrak{p},\overline{\lambda}}$.

Moreover, under the $\mathbf{T}_1(N)_{\mathfrak{p},\lambda}$-bilinear pairing

$$
{}_N^k(\cdot,\cdot)_\lambda : ({}_N^kW_l \otimes_{\mathbf{Q}_l} \mathbf{T}_1(N)_{\mathfrak{p},\lambda}) \otimes_{\mathbf{T}_1(N)_{\mathfrak{p},\lambda}} ({}_N^kW_l \otimes_{\mathbf{Q}_l} \mathbf{T}_1(N)_{\mathfrak{p},\lambda}) \\ \to \mathbf{T}_1(N)_{\mathfrak{p},\lambda}(-k-1),
$$

the induced pairing between $V_{\mathfrak{p},\lambda}$ and $V_{\mathfrak{p},\overline{\lambda}}$ is *perfect*. Since the extension of scalars of w_ζ takes $V_{\mathfrak{p},\lambda}$ over to $V_{\mathfrak{p},\overline{\lambda}}$ as a direct summand of (4.1.1.2), the pairing

$$
{}_N^k[\cdot,\cdot]_\lambda = {}_N^k\big(\cdot, w_\zeta(\cdot)\big)_\lambda
$$

is a perfect bilinear pairing of $V_{\mathfrak{p},\lambda}$ with *itself*, so it respects the canonical $\mathbf{T}_1(N)$-action through the map $\mathbf{T}_1(N) \to \mathbf{T}_1(N)_{\mathfrak{p},\lambda}$.

Fix $N \geq 1$ and let $f \in S_{k+2}(\Gamma_1(N), \mathbf{C})$ be a newform with associated minimal prime $\mathfrak{p}$. From the above discussion we conclude that from the l-adic data of ${}_N^kW_l$ with its $\mathbf{T}_1(N)$-action and w_ζ-action, together with the pairing

$$
{}_N^k(\cdot,\cdot)_l : {}_N^kW_l \otimes_{\mathbf{Q}_l} {}_N^kW_l \to \mathbf{Q}_l(-k-1),
$$

we can construct a 2-dimensional vector space $V_{f,\lambda} \simeq V_{\mathfrak{p},\lambda}$ over $K_{f,\lambda} \simeq \mathbf{T}_1(N)_{\mathfrak{p},\lambda}$ for any prime λ of $K_f \simeq \mathbf{T}_1(N)_{\mathfrak{p}}$ over l. Moreover, $V_{f,\lambda}$ is a $\mathbf{T}_1(N)_{\mathbf{Q}_l}$-module direct summand of ${}_N^kW_l$ and the action of w_ζ sets up a perfect $K_{f,\lambda}$-bilinear pairing of $V_{f,\lambda}$ with itself, taking values in $K_{f,\lambda}(-k-1)$. For $N \geq 5$, we have also just seen that this l-adic data can be constructed in terms of the cohomology of "l-adic" sheaves and geometric maps of (open) analytic modular curves This entire l-adic picture will be reconstructed later as the complex analytic fiber of a construction in étale cohomology over $\mathbf{Z}[1/N]$.

4.1.2. Translation of weight 2 results. Roughly speaking, "all" that remains is to endow ${}_N^kW_l$ with a continuous $\mathbf{T}_1(N)_{\mathbf{Q}_l}$-linear action of $G_{\mathbf{Q}}$ satisfying relations analogous to those already proven in Theorem 3.3.1.1 (where $G_{\mathbf{Q}} = \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ for the algebraic closure $\overline{\mathbf{Q}}$ of $\mathbf{Q}$ in $\mathbf{C}$. To do this, we first want to translate Theorem 3.3.1.1 into the language of

$$
{}_N^0W_l = \widetilde{\mathrm{H}}^1(X_1(N), \mathbf{Q}_l).
$$

The key point is that the compatibility of the analytic and algebraic theories of Pic^0_X is "intrinsic" (i.e., is independent of the choice of i) when one uses the exact sequence

$$
0 \to \underline{\mathbf{Z}}(1) \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times \to 1
$$

rather than the exact sequence

$$
0 \to \underline{\mathbf{Z}} \to \mathcal{O}_X \xrightarrow{e^{2\pi i(\cdot)}} \mathcal{O}_X^\times \to 1.
$$

Here X is a compact Riemann surface. Put differently, the "better" analytic construction is

$$
\mathrm{Pic}^0_X \simeq \mathrm{H}^1(X, \mathcal{O}_X)/\mathrm{H}^1(X, \underline{\mathbf{Z}}(1))
$$

(with respect to which the induced isomorphism $\mathrm{H}^1(X, \mathcal{O}_X) \simeq T_0(\mathrm{Pic}^0_X)$ *is* exactly the algebraic one, rather than being off by a factor of $2\pi i$). In fact, the isomorphism

$$
\mathrm{H}^1(X, \underline{\mathbf{Z}}_l(1)) \simeq \mathrm{H}^1(X, \underline{\mathbf{Z}}(1)) \otimes_{\mathbf{Z}} \mathbf{Z}_l \simeq T_l(\mathrm{Pic}^0_X)
$$

coming from this description will be *exactly* the isomorphism that emerges from étale cohomology (via various comparison theorems). Thus it is more "correct" to work with the *canonical* isomorphism

$$\mathrm{H}^1(X, \underline{\mathbf{Q}}_l) \simeq V_l(\mathrm{Pic}^0_X) \otimes_{\mathbf{Q}_l} \mathbf{Q}_l(-1).$$

The translation of our weight 2 results is the following. For $N \geq 5$, we let $G_{\mathbf{Q}}$ act (continuously) on $\mathrm{H}^1(X_1(N), \underline{\mathbf{Q}}_l)$ via the isomorphism

(4.1.2.1)
$$\mathrm{H}^1(X_1(N), \underline{\mathbf{Q}}_l) \simeq V_l\big(\mathrm{Pic}^0_{X_1(N)/\mathbf{C}}\big) \otimes_{\mathbf{Q}_l} \mathbf{Q}_l(-1) \simeq V_l\big(\mathrm{Pic}^0_{X_1(N)/\overline{\mathbf{Q}}}\big) \otimes_{\mathbf{Q}_l} \mathbf{Q}_l(-1),$$

Here $\overline{\mathbf{Q}} \subseteq \mathbf{C}$ by definition and $G_{\mathbf{Q}}$ acts on $V_l(\mathrm{Pic}^0_{X_1(N)/\overline{\mathbf{Q}}})$ and $\mathbf{Q}_l(-1)$ in the usual manner. Let V denote the adjoint operator to the linear action of a *geometric* Frobenius element $F = \mathrm{Frob}_p^{-1}$ with respect to the pairing

$${}^0_N(\cdot,\cdot)_l : \mathrm{H}^1(X_1(N), \underline{\mathbf{Q}}_l) \otimes \mathrm{H}^1(X_1(N), \underline{\mathbf{Q}}_l) \to \mathbf{Q}_l(-1).$$

THEOREM 4.1.2.1. *The $G_{\mathbf{Q}}$-action on $\mathrm{H}^1(X_1(N), \underline{\mathbf{Q}}_l)$ via* (4.1.2.1) *is $\mathbf{T}_1(N)_{\mathbf{Q}_l}$-linear for the canonical $\mathbf{T}_1(N)_{\mathbf{Q}_l}$-action on $\mathrm{H}^1(X_1(N), \underline{\mathbf{Q}}_l) = {}^0_N W_l$. It is unramified at all p not dividing Nl, and for such p the action of a geometric Frobenius element $F = \mathrm{Frob}_p^{-1}$ satisfies*

$$\det{}_{\mathbf{T}_1(N)_{\mathbf{Q}_l}}\big(1 - FX\big) = 1 - T_pX + p\,\langle p \rangle\, X^2.$$

In fact,

$$T_p = F + \langle p \rangle\, V,\ \ FV = VF = p,\ \ w_\zeta^{-1} F w_\zeta = \langle p \rangle^{-1} F. \tag{4.1.2.2}$$

REMARK 4.1.2.2. Recall that the action of $\mathbf{T}_1(N)_{\mathbf{Q}_l}$ on ${}^0_N W_l = \mathrm{H}^1(X_1(N), \underline{\mathbf{Q}}_l)$ corresponds to the $(\cdot)^*$-action on $\mathrm{Pic}^0_{X_1(N)}$ (as was noted in §3.2.2) and ${}^0_N W_l$ is free of rank 2 as such a module. In particular, the $(\cdot)^*$-action of $\mathbf{T}_1(N)$ on $\mathrm{Pic}^0_{X_1(N)}$ makes (4.1.2.1) $\mathbf{T}_1(N)$-linear. It turns out that the $G_{\mathbf{Q}}$-action on $\mathrm{H}^1(X_1(N), \underline{\mathbf{Q}}_l)$ above *is* exactly that obtained from the isomorphism

$$\mathrm{H}^1\big(X_1(N), \underline{\mathbf{Q}}_l\big) \simeq \mathrm{H}^1_{\text{ét}}\big(X_1(N)_{\mathbf{C}}, \mathbf{Q}_l\big) \simeq \mathrm{H}^1_{\text{ét}}\big(X_1(N)_{\mathbf{Q}} \times_{\mathbf{Q}} \overline{\mathbf{Q}}, \mathbf{Q}_l\big).$$

PROOF. The proof amounts to unwinding the relation between the $(\cdot)_*$ and $(\cdot)^*$-actions on $\mathrm{Pic}^0_{X_1(N)}$ and keeping track of the twist by $\mathbf{Q}_l(-1)$ for the action of $G_{\mathbf{Q}}$. Since $\mathbf{T}_1(N)_{\mathbf{Q}_l}$-linearity is known, the $G_{\mathbf{Q}}$-compatibility of the $\mathbf{T}_1(N)_{\mathbf{Q}_l}$-action on $\mathrm{H}^1(X_1(N), \underline{\mathbf{Q}}_l)$ is immediate from the definition of (4.1.2.1). The key point is to compute the quadratic polynomial $\det_{\mathbf{T}_1(N)_{\mathbf{Q}_l}}(1 - FX)$ and to prove (4.1.2.2), which will essentially amount to using Theorems 3.2.3.1 and 3.3.1.1.

Since Frob_p acts on $\mathbf{Q}_l(-1)$ as p^{-1}, we see that

$$F = \mathrm{Frob}_p^{-1} \otimes p \tag{4.1.2.3}$$

on (4.1.2.1). By Theorem 3.2.3.1,

$$T_{p*} = \mathrm{Frob}_p + \langle p \rangle_*\, p\, \mathrm{Frob}_p^{-1}$$

on $V_l(\mathrm{Pic}^0_{X_1(N)})$, and we know that

$$T_{p*} = \langle p \rangle^{*\,-1} T_p^*$$

as in the discussion after the proof of Theorem 3.1.3.1. Thus,

$$\begin{aligned} T_p^* &= \langle p\rangle^* T_{p*} \\ &= \langle p\rangle^* \operatorname{Frob}_p + \langle p\rangle^* \langle p\rangle_* \, p \operatorname{Frob}_p^{-1} \\ &= p \langle p\rangle^* F^{-1} + F, \end{aligned}$$

where the final equality uses (4.1.2.3). Thus, to show that

$$T_p = F + \langle p\rangle V$$

on $\mathrm{H}^1(X_1(N), \underline{\mathbf{Q}_l})$ we just have to check (by the definition of V) that

$${}_N^0(Fx, y)_l = {}_N^0(x, pF^{-1}y)_l.$$

This will also force $FV = VF = p$.

By Theorem 3.1.1.1, $-{}_N^0(\cdot,\cdot)_l$ is obtained by twisting through the l-adic Weil pairing by $\mathbf{Q}_l(-2)$, with a twist of $\mathbf{Q}_l(-1)$ on each $V_l(\mathrm{Pic}^0_{X_1(N)})$. Thus, to establish our formula for T_p amounts to showing that

$$\left(p \cdot \operatorname{Frob}_p^{-1}(x), y\right)_l \stackrel{?}{=} (x, \operatorname{Frob}_p y)_l$$

in $\mathbf{Q}_l(1)$, or equivalently that

$$p(x,y)_l \stackrel{?}{=} (\operatorname{Frob}_p(x), \operatorname{Frob}_p(y))_l.$$

The left side is $\operatorname{Frob}_p((x,y)_l)$, as $(x,y)_l \in \mathbf{Q}_l(1)$, so this is just the $G_{\mathbf{Q}}$-compatibility of the l-adic Weil pairing.

Since $w_\zeta^{-1} \operatorname{Frob}_p w_\zeta = \langle p\rangle_*^{-1} \operatorname{Frob}_p$ on $V_l(\mathrm{Pic}^0_{X_1(N)})$ by Theorem 3.3.1.1, we find that

$$w_\zeta^{-1} p F^{-1} w_\zeta = \langle p\rangle \, pF^{-1}$$

on $\mathrm{H}^1(X_1(N), \underline{\mathbf{Q}_l})$. Canceling the p and inverting both sides yields

$$w_\zeta^{-1} F w_\zeta = F \langle p\rangle^{-1} = \langle p\rangle^{-1} F.$$

Finally, the assertion about the characteristic polynomial says that $\mathrm{tr}(F) = T_p$, $\det(F) = p\langle p\rangle$. Since $F = p \operatorname{Frob}_p^{-1}$, we compute (with the help of Theorem 3.2.3.1)

$$\begin{aligned} \mathrm{tr}(F) &= p \, \mathrm{tr}(\operatorname{Frob}_p^{-1}) \\ &= p \frac{\mathrm{tr}(\operatorname{Frob}_p)}{\det(\operatorname{Frob}_p)} \\ &= \frac{pT_{p*}}{p\langle p\rangle_*} \\ &= \frac{\langle p\rangle^{*\,-1} T_p^*}{\langle p\rangle_*} \\ &= T_p. \end{aligned}$$

Similarly,

$$\begin{aligned} \det(F) &= p^2 (\det \operatorname{Frob}_p)^{-1} \\ &= p^2 (p\langle p\rangle_*)^{-1} \\ &= p\langle p\rangle^* \\ &= p\langle p\rangle. \end{aligned}$$

This completes the proof. $\square$

4.1.3. Reductions of the main theorems. Deligne's generalization of Theorem 4.1.2.1 is to identify $\mathrm{H}^1(X_1(N), \underline{\mathbf{Q}}_l)$ with $\mathrm{H}^1_{\text{ét}}(X_1(N) \times_{\mathbf{Q}} \overline{\mathbf{Q}}, \mathbf{Q}_l)$ and to calculate how the $G_{\mathbf{Q}}$-action and $\mathbf{T}_1(N)_{\mathbf{Q}_l}$-action on this étale cohomology group interact.

Let us reduce the construction of l-adic representations attached to newforms as in Main Theorem 2 to the problem of endowing $\mathrm{H}^1(X_1(N), \underline{\mathbf{Q}}_l)$ with the "right" $G_{\mathbf{Q}}$-structure. As in the discussion following Lemma 2.3.7.1, for any $d \mid N$ with $\gcd(d, N/d) = 1$ and $p \nmid d$ any prime, let $\varepsilon_{d,p} \in (\mathbf{Z}/N)^\times$ be the unique element congruent to p modulo d and 1 modulo N/d.

THEOREM 4.1.3.1. *Fix an embedding $\overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$. Suppose that for all $N \geq 5$ and $k \geq 0$ we can define a (continuous) $\mathbf{T}_1(N)_{\mathbf{Q}_l}$-linear action of $G_{\mathbf{Q}}$ on ${}^k_N W_l$ which is unramified at all p not dividing Nl and is compatible with inclusions ${}^k_N W_l \hookrightarrow {}^k_{Nd} W_l$ for all $d \geq 1$. Assume moreover that for p not dividing Nl, the action F of* Frob_p^{-1} *on ${}^k_N W_l$ satisfies (with $V = p^{k+1} F^{-1}$)*

$$T_p = F + \langle p \rangle V, \qquad w_\zeta^{-1} F w_\zeta = \langle \varepsilon_{d,p} \rangle^{-1} F$$

for some (hence any) primitive dth root of unity $\zeta \in \mu_N(\mathbf{C})$ with d dividing N and $\gcd(d, \frac{N}{d}) = 1$. Assume finally that

$${}^k_N(\cdot,\cdot)_l : {}^k_N W_l \otimes_{\mathbf{Q}_l} {}^k_N W_l \to \mathbf{Q}_l(-k-1)$$

is $G_{\mathbf{Q}}$-equivariant, so V is the ${}^k_N(\cdot,\cdot)_l$-adjoint of F. Then the same structures uniquely exist on ${}^k_N W_l$ for $N \leq 4$, compatible with change in level, and

$$\det(1 - FX) = 1 - T_p X + p^{k+1} \langle p \rangle X^2$$

for all $N \geq 1$.

REMARK 4.1.3.2. For a minimal prime $\mathfrak{p}$ attached to a newform, the induced $G_{\mathbf{Q}}$-action on $V_{\mathfrak{p},\lambda} = {}^k_N W_l \otimes_{\mathbf{T}_1(N)_{\mathbf{Q}_l}} \mathbf{T}_1(N)_{\mathfrak{p},\lambda}$ (which is a $\mathbf{T}_1(N)_{\mathbf{Q}_l}$-module direct summand of ${}^k_N W_l$) gives a 2-dimensional continuous representation of $G_{\mathbf{Q}}$ over the field $\mathbf{T}_1(N)_{\mathfrak{p},\lambda}$, unramified at p not dividing Nl. The (contragredient) representation of $G_{\mathbf{Q}}$ on

$$\mathrm{Hom}_{\mathbf{T}_1(N)_{\mathfrak{p},\lambda}}(V_{\mathfrak{p},\lambda}, \mathbf{T}_1(N)_{\mathfrak{p},\lambda})$$

answers Main Theorem 2, where the "k" there corresponds to our "$k+2$".

For weight 2 (i.e., $k = 0$), the connection to our discussion in Chapter 3 goes as follows. Using the trace map $\mathbf{T}_1(N)_{\mathfrak{p},\lambda} \to \mathbf{Q}_l$ gives an isomorphism of $\mathbf{T}_1(N)_{\mathfrak{p},\lambda}[G_{\mathbf{Q}}]$-modules

$$\mathrm{Hom}_{\mathbf{T}_1(N)_{\mathfrak{p},\lambda}}(V_{\mathfrak{p},\lambda}, \mathbf{T}_1(N)_{\mathfrak{p},\lambda}) \xrightarrow{\simeq} \mathrm{Hom}_{\mathbf{Q}_l}(V_{\mathfrak{p},\lambda}, \mathbf{Q}_l).$$

This is the $\mathbf{T}_1(N)_{\mathfrak{p},\lambda}$-component of the $\mathbf{T}_1(N)_{\mathbf{Q}_l}[G_{\mathbf{Q}}]$-module $\mathrm{Hom}_{\mathbf{Q}_l}({}^k_N W_l, \mathbf{Q}_l) \simeq {}^k_N W_l \otimes_{\mathbf{Q}_l} \mathbf{Q}_l(k+1)$, where this isomorphism is defined by ${}^k_N[\cdot,\cdot]_l = {}^k_N(\cdot, w_\zeta(\cdot))_l$ for $\zeta \in \mu_N(\mathbf{C})$ a primitive Nth root of unity. When $k = 0$ and $N \geq 5$ this is

$$(4.1.3.1) \qquad {}^0_N W_l \otimes_{\mathbf{Q}_l} \mathbf{Q}_l(1) = \mathrm{H}^1\big(X_1(N), \underline{\mathbf{Q}}_l(1)\big) \simeq V_l\big(\mathrm{Pic}^0_{X_1(N)/\mathbf{C}}\big)$$

as a $\mathbf{T}_1(N)_{\mathbf{Q}_l}$-module with the $(\cdot)^*$-action on $\mathrm{Pic}^0_{X_1(N)/\mathbf{C}}$. Also, as mentioned already, the study of ${}^k_N W_l$ via étale cohomology will make (4.1.3.1) an isomorphism of $G_{\mathbf{Q}}$-modules as well, via the identification of the right side with $V_l(\mathrm{Pic}^0_{X_1(N)/\mathbf{Q}})$ with respect to our initial choice of $\overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$.

PROOF OF THEOREM 4.1.3.1. The constructions at low level are (uniquely) made from higher level exactly as in our earlier constructions at low level.

We must compute $\det(1 - FX)$, which amounts to showing that F has characteristic polynomial (with respect to the $\mathbf{T}_1(N)_{\mathbf{Q}_l}$-structure)

$$X^2 - T_p X + p^{l+1} \langle p \rangle .$$

Since $T_p = F + \langle p \rangle p^{k+1} F^{-1}$, certainly F satisfies this quadratic polynomial. As in the weight 2 case, it remains to show that

$$\mathrm{tr}_{\mathbf{T}_1(N)_{\mathbf{Q}_l}}(F) = \mathrm{tr}_{\mathbf{T}_1(N)_{\mathbf{Q}_l}}\left(\langle p \rangle F^{\vee}\right)$$

where $F^{\vee} = p^{k+1} F^{-1}$ is the adjoint of F with respect to ${}^k_N(\cdot,\cdot)_l$. Using the $\mathbf{T}_1(N)_{\mathbf{Q}_l}$-equivariance of ${}^k_N[\cdot,\cdot]_l$, the same duality argument as in Section 3.3 carries over verbatim, again using Lemma 3.3.2.1 to finish the argument. □

COROLLARY 4.1.3.3. *In the setting of Theorem* 4.1.3.1*, assume that the eigenvalues of F on ${}^k_N W_l$ are algebraic integers with all complex absolute values equal to $p^{\frac{k+1}{2}}$. Then the Ramanujan-Petersson conjecture for weight k and level N is true. In particular, the first part of Main Theorem* 1 *is true.*

PROOF. Fix p for which we want to study the eigenvalues of T_p. By Remark 4.1.3.2, the contragredient (i.e., dual) representation attached to the one in Main Theorem 2 occurs as a subrepresentation of ${}^k_N W_l$ as $\mathbf{Q}_l[G_{\mathbf{Q}}]$-module, for some $N \geq 5$ relatively prime to p. Also, passing to the contragredient interchanges the actions of geometric/arithmetic Frobenius. Finally, for finite dimensional representations over finite extensions K of $\mathbf{Q}_l$, characteristic polynomials relative to the K-structure *divide* the characteristic polynomials relative to the $\mathbf{Q}_l$-structure. Thus our hypothesis on the action of F on the $\mathbf{Q}_l[G_{\mathbf{Q}}]$-module ${}^k_N W_l$ yields that the eigenvalue in $K_{f,\lambda}$ is the sum of *two* algebraic integers all of whose complex absolute values are $p^{\frac{k+1}{2}}$, so we're done. □

4.2. Arithmetic models of modular curves

In order to use étale cohomology, we will first need an algebraic model of the open Riemann surface $Y_1(N)$ as a smooth affine curve over $\mathbf{Z}[1/N]$. We will also need to construct the smooth proper curve $X_1(N)$ over $\mathbf{Z}[1/N]$, as this is crucial for showing that the Weil conjectures imply the Ramanujan conjecture.

4.2.1. Rigidity.

LEMMA 4.2.1.1. *Fix $N \geq 5$. If $E \to S$ is an elliptic curve over a $\mathbf{Z}[1/N]$-scheme and if P is a point of exact order N on E, then* $\mathrm{Aut}(E, P) = \{1\}$.

REMARK 4.2.1.2. We call (E, P) a $\Gamma_1(N)$*-structure* on $E \to S$.

PROOF. Let $\alpha : E \xrightarrow{\simeq} E$ be an automorphism with $\alpha(P) = P$. We want to show that α is the identity. Without loss of generality we can assume that $S = \operatorname{Spec} A$ for some $\mathbf{Z}[1/N]$-algebra A. By direct limit arguments from [**28**, Section 8ff], we may assume that A is noetherian. Using the Krull intersection theorem, it is easy to reduce from here to the case that A is artinian (this is exactly the argument used in the discussion preceding the statement of the analytic Theorem on Formal Functions). The argument from [**46**, §6] used near the end of the proof of Theorem 1.1.1.15 shows that smooth elliptic curves do not admit infinitesimal automorphisms. Thus, it suffices to consider the case where $S = \operatorname{Spec} k$ for a field k.

If $\alpha \neq 1$, then the map $\alpha - 1$ is an isogeny, necessarily of degree divisible by N since it contains P in its kernel. But

$$\deg(\alpha - 1) = (\alpha - 1)(\alpha^{\vee} - 1) = 1 - (\alpha + \alpha^{\vee}) + 1 = 2 - (\alpha + \alpha^{\vee})$$

with $\alpha^{\vee}$ the dual of α. Thus

$$\alpha + \alpha^{\vee} = 2 - \deg(\alpha - 1) \in \mathbf{Z}. \tag{4.2.1.1}$$

But we also have

$$|\alpha + \alpha^{\vee}|^2 \leq 4 \deg(\alpha) = 4, \tag{4.2.1.2}$$

since

$$n^2 - (\alpha + \alpha^{\vee})nm + m^2 = (n - \alpha m)(n - \alpha^{\vee} m) = \deg(n - \alpha m) \geq 0$$

for all $n, m \geq 0$. Combining (2.4.1.1), (4.2.1.2), and the fact that $\deg(\alpha - 1)$ is divisible by $N \geq 5$ yields a contradiction.

□

Since $3^2 > 4$, we can use the same method to prove the next lemma.

LEMMA 4.2.1.3. *Fix $N \geq 3$. Let $E \to S$ be an elliptic curve over a $\mathbf{Z}[1/N]$-scheme S and let $j : \underline{\mathbf{Z}/N} \times \underline{\mathbf{Z}/N} \xrightarrow{\simeq} E[N]$ be an isomorphism of group schemes. Then* $\operatorname{Aut}(E, j) = \{1\}$.

REMARK 4.2.1.4. We call (E, j) a $\Gamma(N)$-*structure* or a (naive) *full level N structure* on $E \to S$.

4.2.2. Weierstrass cubics. Using Grothendieck's theory of cohomology and base change, together with classical Riemann-Roch calculations on the geometric fibers, it is easy to check that if $f : C \to S$ is a proper smooth map with geometrically connected fibers of dimension 1 and genus 1, and if $\varepsilon : S \to C$ is a section, then

- the maps $\mathcal{O}_S \to f_*\mathcal{O}_C \to f_*\mathcal{L}(\varepsilon)$ are isomorphisms compatible with base change,
- for all $n \geq 2$, the sheaf $f_*\mathcal{L}(n\varepsilon)$ is locally free of rank n, of formation compatible with base change,
- for all $n \geq 2$, the higher direct images of $\mathcal{L}(n\varepsilon)$ vanish.

In particular, $f_*\mathcal{L}(n\varepsilon)$ is locally a direct summand of $f_*\mathcal{L}((n+1)\varepsilon)$, as this can be seen on geometric fibers. Since $\mathcal{L}(3\varepsilon)$ is *very ample* on fibers, it follows that there is a well-defined closed immersion

$$C \hookrightarrow \mathbf{P}\big(f_*\mathcal{L}(3\varepsilon)\big), \tag{4.2.2.1}$$

with the target a $\mathbf{P}^2$-bundle over S.

Zariski locally over S, we may choose a basis $\{1, x\}$ of $f_*\mathcal{L}(2\varepsilon)$ and a basis $\{1, x, y\}$ of $f_*\mathcal{L}(3\varepsilon)$. In particular, x induces a generator of the $\mathcal{O}_S$-module $\mathcal{L}(2\varepsilon)/\mathcal{L}(\varepsilon)$ and y induces a generator of the $\mathcal{O}_S$-module $\mathcal{L}(3\varepsilon)/\mathcal{L}(2\varepsilon)$ as invertible sheaves on $\varepsilon : S \hookrightarrow C$. Choosing a local generator t^{-1} of $\mathcal{L}(\varepsilon)/\mathcal{O}_C$ along $\varepsilon : S \hookrightarrow C$, we may rescale x, y by elements of $\mathcal{O}_S^{\times}$ so that $x = t^{-2}$ in $\mathcal{L}(2\varepsilon)/\mathcal{L}(\varepsilon)$ and $y = t^{-3}$ in $\mathcal{L}(3\varepsilon)/\mathcal{L}(2\varepsilon)$. Then $y^2 - x^3 \in \mathcal{L}(6\varepsilon)/\mathcal{L}(5\varepsilon)$ *vanishes*, so

$$y^2 - x^3 \in f_*\mathcal{L}(5\varepsilon).$$

But $\mathcal{L}(5\varepsilon)/\mathcal{L}(4\varepsilon)$ is generated by x^2y, so

$$y^2 - x^3 - ax^2y \in f_*\mathcal{L}(4\varepsilon)$$

for some $a \in \mathrm{H}^0(S, \mathcal{O}_S)$. Continuing in this way we arrive (locally) at a Weierstrass relation in $\mathbf{P}^2_S$ through which C factors. By considerations of flatness/properness and the classical theory on geometric fibers, this map to a Weierstrass cubic is an isomorphism, with $\varepsilon : S \hookrightarrow C$ going over to $[0, 1, 0]$ in $\mathbf{P}^2_S$.

We conclude that any pair (C, ε) can, Zariski locally on the base, be realized as a smooth Weierstrass cubic in $\mathbf{P}^2$; the converse is obvious. Using the canonical projectivity via (4.2.2.1), the following facts are proven in [**35**, 2.1, 2.4, 2.5].

THEOREM 4.2.2.1. *Let $C \to S$ be a proper smooth curve of genus 1 with a section $\varepsilon : S \to C$. There is a unique structure of S-group scheme on C with identity section ε, it is commutative, and any map $f : (C, \varepsilon) \to (C', \varepsilon')$ of such pointed curves respects these group structures.*

The fact that an explicit smooth Weierstrass curve has a unique group scheme structure is *crucial* for the construction of moduli schemes of level structures in [**35**]. One important consequence is the following, which follows from classical calculations.

COROLLARY 4.2.2.2. *An elliptic curve $E \to S$ determines a map $j : S \to \mathbf{A}^1_{\mathbf{Z}}$, compatible with base change on S, inducing the classical j-invariant on geometric fibers. The construction is uniquely determined by these properties.*

PROOF. The existence of the map is clear by the classical Weierstrass calculations. For uniqueness we may reduce to the case when S is noetherian, and then even artinian local. Any artinian local ring is the quotient of a regular local ring, and lifting a Weierstrass model shows that one can lift elliptic curves through surjections of local rings. Thus, we may assume that the base is a regular local ring. This injects into an algebraically closed field, where uniqueness is clear. This completes the proof. One can reduce to the field case by contemplating universal polynomial identities as well. $\square$

4.2.3. G-torsor families. We now review the approach to moduli problems in [**35**]. Let $\mathbf{E} \to \mathcal{M}$ be an elliptic curve and assume that a finite group G acts on $\mathbf{E}$ covering an action on $\mathcal{M}$. That is, assume that we have a cartesian diagram

(4.2.3.1)
$$\begin{tikzcd} \mathbf{E} \arrow[r, "\theta(g)"] \arrow[d] & \mathbf{E} \arrow[d] \\ \mathcal{M} \arrow[r, "{[g]}"'] & \mathcal{M} \end{tikzcd}$$

with g acting on the left on $\mathcal{M}$, such that the induced map $\mathbf{E} \to g^*\mathbf{E}$ over $\mathcal{M}$ is an isomorphism of elliptic curves with $\theta(g)\theta(h) = \theta(gh)$. In particular, we must have $\theta(1) = \mathrm{id}$. For example, if there is a universal full level N structure $\mathbf{E}_N \to Y(N)$ for some $N \geq 3$ (we'll see below that such a universal family does exist), then we have the above situation with $G = \mathrm{GL}_2(\mathbf{Z}/N)$ acting in the natural way. Indeed, if (P, Q) is the basis of the universal full level N-structure on $\mathbf{E}_N \to Y(N)$, then for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbf{Z}/N)$,

$$(gP, gQ) \stackrel{\text{def}}{=} (aP + cQ, bP + dQ)$$

is a full level N structure on $\mathbf{E}_N \to Y(N)$. Thus, by universality there exists a unique map $[g] : Y(N) \to Y(N)$ fitting into a cartesian square

$$\begin{CD} \mathbf{E}_N @>{\theta(g)}>> \mathbf{E}_N \\ @VVV @VVV \\ Y(N) @>>{[g]}> Y(N) \end{CD}$$

where $\mathbf{E}_N \to g^*\mathbf{E}_N$ is an isomorphism of elliptic curves and $\theta(g)(gP) = P \circ [g]$, $\theta(g)(gQ) = Q \circ [g]$. Since $N \geq 3$, this θ is uniquely determined (by Lemma 4.2.1.3). Of course, this example is just an explication of the left action of $\mathrm{GL}_2(\mathbf{Z}/N)$ on the $\Gamma(N)$-moduli problem. This makes it clear that $[h][g] = [hg]$ and $\theta(h)\theta(g) = \theta(hg)$.

Given data as in (4.2.3.1) and *any* elliptic curve $E \to S$, consider the S-scheme

$$\wp_{E/S} = \underline{\mathrm{Isom}}_{S\times\mathcal{M}}(p_1^*E, p_2^*\mathbf{E});$$

here the p_i are the two projections from $S \times \mathcal{M} = S \times_{\mathbf{Z}} \mathcal{M}$ to its factors. By the canonical projectivity of elliptic curves and the theory of Hilbert schemes, $\wp_{E/S}$ exists as an $S \times \mathcal{M}$-scheme which is quasi-projective Zariski locally over $S \times \mathcal{M}$. Thus, if $\mathcal{M}$ is separated over $\mathbf{Z}$ then the same is true for $\wp_{E/S} \to S$. In concrete terms, for an S-scheme T, $\wp_{E/S}(T)$ is the set of pairs (f, α) where $f : T \to \mathcal{M}$ is a map and $\alpha : E_{/T} \xrightarrow{\simeq} f^*\mathbf{E}$ is an isomorphism of elliptic curves over T.

Example 4.2.3.1. If $\mathbf{E} \to \mathcal{M}$ is a universal full level N structure for some $N \geq 3$, then $\wp_{E/S}(T)$ is *exactly* the set of full level N structures on E_T.

Returning to the general setup above, there is a *canonical* left action of G on the S-scheme $\wp_{E/S}$, compatible with base change on S. Indeed, for $(f, \alpha) \in \wp_{E/S}(T)$ we may view (f, α) as simply a *cartesian* diagram

$$\begin{CD} E_{/T} @>{\alpha}>> \mathbf{E} \\ @VVV @VVV \\ T @>>{f}> \mathcal{M} \end{CD}$$

with $\alpha \circ \varepsilon_{E_T} = \varepsilon_{\mathbf{E}} \circ f$ (see Theorem 4.2.2.1). Given this, we see that $(\theta(g) \circ \alpha, [g] \circ f)$ is another such pair, and this yields the desired action.

Definition 4.2.3.2. Let $(\mathbf{E} \to \mathcal{M}, G)$ be as above with G acting on a scheme $\mathcal{M}$ over a scheme S_0. We say that this is a *modular G-torsor family* (over S_0) if, for every S_0-scheme S and every elliptic curve $E \to S$, the scheme $\wp_{E/S} \to S$ is an étale G-torsor. That is, $\wp_{E/S} \to S$ is étale and the natural "action" map

$$\underline{G} \times_S \wp_{E/S} \to \wp_{E/S} \times_S \wp_{E/S}$$

is an isomorphism. (This forces the étale $\wp_{E/S} \to S$ to be finite surjective by fppf descent considerations.)

For example, suppose for some $N \geq 3$ that there exists a universal full level N structure $\mathbf{E}_N \to Y(N)$ over $\mathbf{Z}[1/N]$. We claim that this *is* a modular $\mathrm{GL}_2(\mathbf{Z}/N)$-torsor family over $\mathbf{Z}[1/N]$. Indeed, let $E \to S$ be an elliptic curve with S a $\mathbf{Z}[1/N]$-scheme. Then recall that $\wp_{E/S}(T)$ is the set of full level N structures on $E_{/T}$. It is clear that $\wp_{E/S}(\mathrm{Spec}\, A)$ is of formation compatible with direct limits in A, so by the *functorial* characterization of locally finitely presented maps [**28**, 8.14.2] (or facts about $\underline{\mathrm{Isom}}$ and $\underline{\mathrm{Hilb}}$ scheme constructions in the noetherian case) $\wp_{E/S} \to S$ is locally of finite presentation. Since $E[N] \to S$ is finite *étale*, it is clear that isomorphisms $\underline{\mathbf{Z}/N} \times \underline{\mathbf{Z}/N} \xrightarrow{\simeq} E[N]_{/T}$ uniquely lift through infinitesimal thickenings of T (this is just the "topological invariance" of the étale site [**28**, 18.1.2]), so by the functorial characterization of étale maps $\wp_{E/S} \to S$ is étale.

It is clear that if $\wp_{E/S}(T) \neq \emptyset$, then Zariski locally on T any two elements in $\wp_{E/S}(T)$ are related by the action of a *unique* $g \in \mathrm{GL}_2(\mathbf{Z}/N)$. Thus, the map

$$\underline{\mathrm{GL}_2(\mathbf{Z}/N)} \times_S \wp_{E/S} \to \wp_{E/S} \times_S \wp_{E/S}$$

is an isomorphism. This shows that a universal $\mathbf{E}_N \to Y(N)$ over $\mathbf{Z}[1/N]$ must be a modular $\mathrm{GL}_2(\mathbf{Z}/N)$-torsor family over $\mathbf{Z}[1/N]$.

LEMMA 4.2.3.3. *Let $E \to S$ be an elliptic curve over a $\mathbf{Z}[1/N]$-scheme S. Then the functors*

$$T \longrightarrow \{\Gamma_1(N)\text{-structures on } E_{/T}\};$$
$$T \longrightarrow \{\Gamma(N)\text{-structures on } E_{/T}\};$$

are representable by finite étale covers of S.

PROOF. The functors in question are visibly fppf sheaves, so their representability by a finite étale cover of S can be checked after base change to an fppf cover of S (since fppf descent is effective for *affine* maps, such as finite maps). We can find a (finite) étale cover $S' \to S$ with $E[N] \times_S S' \simeq \underline{\mathbf{Z}/N} \times_{S'} \underline{\mathbf{Z}/N}$, so that we may assume that $E[N] \simeq \underline{\mathbf{Z}/N} \times_S \underline{\mathbf{Z}/N}$. This case is obvious. ☐

4.2.4. Quotients of G-torsor families. Let $[E/S]_{\Gamma_1(N)}$ and $[E/S]_{\Gamma(N)}$ denote the finite étale covers of S constructed in Lemma 4.2.3.3. The method of construction of moduli schemes in [**35**] rests upon the next theorem. The basic idea, given a G-torsor family $\mathbf{E} \to \mathcal{M}$ for some G, is to show that the quotient of $[\mathbf{E}/\mathcal{M}]_\Gamma$ by G represents the desired functor, where $\Gamma = \Gamma_1(N)$ or $\Gamma = \Gamma(N)$ (for suitable N). The "simplest" way to carry this out would be with $G = 1$, but since no universal $\{1\}$-torsor family exists we instead are led to the following slightly roundabout method.

THEOREM 4.2.4.1. *Let S_0 be a $\mathbf{Z}[1/N]$-scheme and assume that for some finite group G there exists a modular G-torsor family $(\mathbf{E} \to \mathcal{M}, G)$ over S_0, with $\mathcal{M} \to S_0$ affine. Then the moduli problem of $\Gamma_1(N)$-structures is representable over S_0 for $N \geq 5$ and likewise for $\Gamma(N)$-structures for $N \geq 3$. The base of this universal family is affine over S_0, and is smooth of pure relative dimension 1 if $\mathcal{M} \to S_0$ is.*

REMARK 4.2.4.2. We have not yet shown that any modular families exist!

PROOF. Let $\Gamma = \Gamma_1(N)$ (resp. $\Gamma(N)$) with $N \geq 5$ (resp. $N \geq 3$). Using Lemma 4.2.3.3 we get a finite étale cover $[\mathbf{E}/\mathcal{M}]_\Gamma$ over $\mathcal{M}$, affine over S_0 and in addition smooth of pure relative dimension 1 if $\mathcal{M} \to S_0$ is. As an S_0-scheme, $[\mathbf{E}/\mathcal{M}]_\Gamma$ represents the functor which associates to any S_0-scheme T the set of pairs (f, α) where $f : T \to \mathcal{M}$ is a map over S_0 and α is a Γ-structure on $f^*\mathbf{E} \to T$. That is, α is either a closed immersion $\underline{\mathbf{Z}/N} \hookrightarrow f^*\mathbf{E}$ (when $\Gamma = \Gamma_1(N)$) or an isomorphism $\underline{\mathbf{Z}/N} \times \underline{\mathbf{Z}/N} \simeq f^*\mathbf{E}[N]$ (when $\Gamma = \Gamma(N)$). We will construct a free left action of G on $[\mathbf{E}/\mathcal{M}]_\Gamma$ as an S_0-scheme (*not* as an $\mathcal{M}$-scheme).

For $g \in G$, we can form the pair

$$\big([g] \circ f : T \to \mathcal{M}, f^*(\theta(g)) \circ \alpha\big),$$

where $\theta(g) : \mathbf{E} \to [g]^*\mathbf{E}$ is the given isomorphism and

$$f^*(\theta(g)) : f^*\mathbf{E} \xrightarrow{\simeq} f^*([g]^*\mathbf{E}) = ([g] \circ f)^*\mathbf{E}$$

is the canonical isomorphism over T. This is visibly a left action of G on $[\mathbf{E}/\mathcal{M}]_\Gamma$ over S_0, and by *rigidity* (Lemma 4.2.1.1 and Lemma 4.2.1.3) this action is free. That is, if $(f, \alpha) = ([g] \circ f, f^*(\theta(g)) \circ \alpha)$, then $g = 1$. Indeed, this makes $f^*(\theta(g))$ an *automorphism* of the elliptic curve $f^*\mathbf{E}$ *fixing* the Γ-structures $\alpha = f^*(\theta(g)) \circ \alpha$. The *rigidity* then forces $f^*(\theta(g)) = 1$. Pulling back to a geometric point of T, we have produced a geometric point $f : \operatorname{Spec} k \to \mathcal{M}$ which is fixed by $[g]$ and such that $\theta(g)$ is the identity over this point. That is, the canonical element in $\wp_{\mathbf{E}/\mathcal{M}}(k)$ corresponding to the identity on $f^*\mathbf{E}$ is *fixed* by g. Thus $g = 1$ since $\wp_{\mathbf{E}/\mathcal{M}}$ is a G-torsor over $\mathcal{M}$.

Since $[\mathbf{E}/\mathcal{M}]_\Gamma \to S_0$ is an *affine* map with a free action of G, it follows from [**25**, Expose V, 4.1] that the affine S_0-scheme $[\mathbf{E}/\mathcal{M}]_\Gamma/G$ (formed by taking subrings of G-invariants over open affines in S_0) has the property that $[\mathbf{E}/\mathcal{M}]_\Gamma \to [\mathbf{E}/\mathcal{M}]_\Gamma/G$ is a finite étale G-torsor. In particular, $[\mathbf{E}/\mathcal{M}]_\Gamma/G$ is smooth of pure relative dimension 1 over S_0 if the same is true for $\mathcal{M} \to S_0$.

Thus, for any S_0-scheme S, to give a map $S \to [\mathbf{E}/\mathcal{M}]_\Gamma/G$ is *equivalent* to giving an isomorphism class of pairs $(\widetilde{S}, \iota)$ where $\widetilde{S}$ is a finite étale G-torsor $\widetilde{S} \to S$ and $\iota : \widetilde{S} \to [\mathbf{E}/\mathcal{M}]_\Gamma$ is a map over S_0 compatible with G-actions. Passing to the quotient by G induces $\overline{\iota} : S \to [\mathbf{E}/\mathcal{M}]_\Gamma$; the resulting map $\widetilde{S} \to \overline{\iota}^*([\mathbf{E}/\mathcal{M}]_\Gamma)$ is a map of G-torsors and so must be an isomorphism. The data of ι is the same as the specification of an S_0-map $\iota' : \widetilde{S} \to \mathcal{M}$ compatible with the actions of G and a Γ-structure on $\iota'^*\mathbf{E} \to \widetilde{S}$ compatible with the action of G on $\iota'^*\mathbf{E}$ (via the $\theta(g)$) over the action of G on $\widetilde{S}$. Keep in mind that only the isomorphism class of $(\widetilde{S}, \iota)$ matters.

By the canonical projectivity of elliptic curves, the G-compatible data of $\iota'^*\mathbf{E}$ and its Γ-structures on the G-torsor $\widetilde{S}$ *descends* canonically (up to unique isomorphism) to an elliptic cruve $E \to S$ with a Γ-structure. The isomorphism $E \times_S \widetilde{S} \simeq \iota'^*\mathbf{E}$ respects Γ-structures and G-actions over the G-action on $\widetilde{S}$.

We now get a map $\widetilde{S} \to \wp_{E/S}$ over S *compatible with G-actions.* As both sides are G-torsors over S, this map is an isomorphism and must identify ι' and the isomorphism $E \times_S \widetilde{S} \simeq \iota'^*\mathbf{E}$ with the *canonical* map $\wp_{E/S} \to \mathcal{M}$ and the *universal* isomorphism over $\wp_{E/S}$.

Not only does the isomorphism class of $(\widetilde{S}, \iota)$ determine an isomorphism class of $E \to S$ with Γ-structure over S, but conversely, given any $E \to S$ with Γ-structure, the G-torsor $\wp_{E/S}$ with its canonical (G-compatible) map $\iota' : \wp_{E/S} \to \mathcal{M}$ has the property that the Γ-structure on $\iota'^*\mathbf{E}$ via the universal isomorphism $\iota'^*\mathbf{E} \simeq E \times_S \wp_{E/S}$ is *automatically* compatible with the G-action! Indeed, going back to the definition of the G-action on $\wp_{E/S}$, we see that the G-action on $\iota'^*\mathbf{E}$ (from the $\theta(g)$ on $\mathbf{E}$ over $[g]$ on $\mathcal{M}$, compatible via ι' with the G-action on $\wp_{E/S}$) translates over to $1 \times [g]$ on $E \times_S \wp_{E/S}$ (over $[g]$ on $\wp_{E/S}$). This *respects* the Γ-structure coming from $E \to S$.

We conclude from this that $\operatorname{Hom}_{S_0}(S, [\mathbf{E}/\mathcal{M}]_\Gamma/G)$ is in bijection with *isomorphism classes* of $E \to S$ with Γ-structure, functorially in S. Since Γ-structures are *rigid*, this implies that the functor of isomorphism classes of Γ-structures is represented by $[\mathbf{E}/\mathcal{M}]_\Gamma/G$, as claimed. ☐

4.2.5. Representability results. The main result we are after is the following.

THEOREM 4.2.5.1. *Fix* $N \geq 3$ (*resp.* $N \geq 5$) *and* $\Gamma = \Gamma(N)$ (*resp.* $\Gamma = \Gamma_1(N)$). *There exists a universal* Γ*-structure over* $\mathbf{Z}[1/N]$*, and the base* Y_Γ *of this universal family is a smooth affine* $\mathbf{Z}[1/N]$*-scheme with pure relative dimension* 1.

As in the analytic setting, we denote by $E_1(N) \to Y_1(N)$ the universal elliptic curve. In Theorem 4.2.6.2, we will see that this is compatible with the analytic theory for $\Gamma = \Gamma_1(N)$.

PROOF. By rigidity it suffices to work separately over $\mathbf{Z}[1/2N]$ and $\mathbf{Z}[1/3N]$. By Theorem 4.2.4.1, it suffices to construct modular G-torsor familes over $\mathbf{Z}[1/2]$ and $\mathbf{Z}[1/3]$, with the base of these families smooth affine of pure relative dimension 1. In view of Example 4.2.3.1 (which assumed the *existence* of universal full level structures) it suffices to prove representability in the case $\Gamma = \Gamma(4)$ over $\mathbf{Z}[\frac{1}{4}] = \mathbf{Z}[1/2]$ and the case $\Gamma = \Gamma(3)$ over $\mathbf{Z}[1/3]$. Following [**35**], both cases are done by explicit computations with Weierstrass equations. More precisely, if $E \to S$ is an elliptic curve with Γ-structure, Zariski locally we can view $E \to S$ as a Weierstrass cubic. The idea is to use the Γ-structure to "rigidify" the Weierstrass equations so that it globalizes, giving a universal family. One can give a computation-free proof using the method of algebraic stacks as in [**14**].

Before discussing the construction, we briefly explain how all of the desired properties *aside* from affineness (i.e., smoothness and being of pure relative dimension 1) can be seen by pure thought, though the actual *construction* process also makes this clear. The functorial criteria for being locally of finite presentation [**28**, 8.14.2] and smooth [**28**, 17.1.4] are obviously satisfied by the moduli problems of Γ-structures: we have already seen this for being locally of finite presentation, and for smoothness the point is that we can always deform Weierstrass equations through surjections of local artin rings, and then Theorem 4.2.2.1 and the étaleness of N-torsion over S allows us to (uniquely) lift the Γ-structure. Thus, if a univeral Γ-structure $E_\Gamma \to Y_\Gamma$ exists, then $Y_\Gamma \to \operatorname{Spec} \mathbf{Z}[1/N]$ *must* be smooth.

The fact that (geometric) fibers have pure dimension 1 can be seen by deformation theory. Namely, if k is an algebraically closed field of characteristic prime to N, (e.g., $\overline{\mathbf{F}}_p$ or $\overline{\mathbf{Q}}$) and if $(E, \alpha)_k$ is a Γ-structure over k, then the complete local ring of the corresponding (closed) point of $Y_{\Gamma/k}$ is *exactly* a universal deformation ring of $(E, \alpha)_k$, classifying deformations (up to isomorphism) of $(E, \alpha)_k$ to artin local k-algebras. The Γ-structure uniquely lifts to any such deformation of E, so to complete our dimension calculation we just need to show that the universal deformation ring for $E_{/k}$ over artin local k-algebras is $k[\![t]\!]$.

For any artin local k-algebra R, isomorphism classes of deformations of E to R as elliptic curves are the *same* as isomorphism classes of deformations of E to R as *flat* R-schemes (which are necessarily smooth curves). Indeed, if $\mathcal{E}$ is a flat deformation to R, then $\mathcal{E}(R) \to E(k)$ is surjective by smoothness, so by Lemma 4.2.3.3 we can give $\mathcal{E}$ the structure of an elliptic curve after choosing $\varepsilon \in \mathcal{E}(R)$ lifting $\varepsilon \in E(k)$. If $\varepsilon_1, \varepsilon_2 \in \mathcal{E}(R)$ are two such lifts, and if $+_1, +_2$ are the resulting unique group laws, then

$$\varepsilon_2 +_1 (\cdot) : \mathcal{E} \xrightarrow{\simeq} \mathcal{E}$$

is an isomorphism from $(\mathcal{E}, +_1)$ to $(\mathcal{E}, +_2)$ since it takes ε_1 to ε_2. This isomorphism lifts the identity on E. Since *elliptic curve* deformations of E have no non-trivial automorphisms (as shown in the proof of Theorem 1.1.1.15), we conclude that there is no obstruction to lifting automorphisms from one *flat* deformation to another. Thus we can just as well try to classify *flat* deformations of E to artin local k-algebras.

The Weierstrass theory (or [**23**, III, 6.7] for curves in general) shows that there is no obstruction to such a lifting. Combining this with the analysis of *flat* deformations of smooth proper schemes over a field in [**51**, Section 3.7, esp. Proposition 3.10(a)], we conclude that the universal deformation ring does exist and that it is a power series ring in

$$\dim_k \mathrm{H}^1\big(E, (\Omega^1_{E/k})^\vee\big) = 1$$

variables.

This analysis tells us that our only problem is to prove that the $\Gamma(3)$- and $\Gamma(4)$-moduli problems are representable by *affine* schemes. Of course, as was mentioned earlier, the smoothness and pure relative dimension 1 will also follow from the *constructions* below.

The case of $\Gamma(3)$ over $\mathbf{Z}[1/3]$ is explained in [**35**, 2.2.10] by a slightly involved calculation with Weierstrass equations. The universal family is

$$y^2 - a_1xy + a_3y = x^3 \qquad P_3 = (0,0) \qquad Q_3 = (C, B + C)$$

over

$$\mathbf{Z}[\tfrac{1}{3}, B, C]/\big(B^3 - (B+C)^3\big)\left[\frac{1}{(a_1^3 - 27a_3)a_3C}\right]$$

where

$$\begin{aligned} a_1 &= 3C - 1; \\ a_3 &= -3C^2 - B - 3BC. \end{aligned}$$

One can check directly that this is $\mathbf{Z}[1/3]$-flat with geometric fibers smooth of pure dimension 1.

Over $\mathbf{Z}[1/2]$, [**35**] uses the so-called Legendre problem, but this does *not* have the required properties. However, we find it convenient to use the Legendre problem as a first step toward the study of full level 4 structures, so let us define this moduli problem. We first need some preliminary discussion.

If $f : E \to S$ is an elliptic curve, then $f_*\Omega^1_{E/S}$ is invertible, of formation compatible with base change, and the canonical map $f^*f_*\Omega^1_{E/S} \to \Omega^1_{E/S}$ is an isomorphism. This follows from the fact that the analogues for $\mathcal{O}_E$ in place of $\Omega^1_{E/S}$ are clear, and Zariski locally *over* S we have $\Omega^1_{E/S} \simeq \mathcal{O}_E$ because of the quite general fact that for *any* group scheme $p : G \to S$, there is a canonical isomorphism of $\mathcal{O}_G$-modules

$$\Omega^1_{G/S} \simeq p^*\varepsilon^*\Omega^1_{G/S} \tag{4.2.5.1}$$

compatible with base change [**2**, 4.2/2].

If S is a $\mathbf{Z}[1/2]$-scheme and $\langle P, Q\rangle$ is a basis of $E[2]$, then there is a *unique* $x \in \mathrm{H}^0(E, \mathcal{L}(2\varepsilon))$ with $x(P) = 0$ and $x(Q) = 1$ (as this can be checked Zariski locally, where $f_*\mathcal{L}(2\varepsilon)$ is free of rank 2). On the other hand, all sections of $f_*\Omega^1_{E/S}$ are translation-invariant [**35**, 2.2.2], so by [**2**, 4.2/1] any $\omega \in \mathrm{H}^0(S, f_*\Omega^1_{E/S}) =$

$\mathrm{H}^0(E, \Omega^1_{E/S})$ is uniquely determined by the section $\varepsilon^*\omega$ of $\varepsilon^*\Omega^1_{E/S}$; conversely all sections of $\varepsilon^*\Omega^1_{E/S}$ have this form.

Any x as above induces a generator of $\mathcal{L}(2\varepsilon)/\mathcal{L}(\varepsilon)$, and any generator ω of $f_*\Omega^1_{E/S}$ induces a generator $\varepsilon^*\omega$ of $\varepsilon^*\Omega^1_{E/S} \simeq \mathcal{I}(\varepsilon)/\mathcal{I}(2\varepsilon)$, where, for a closed subscheme Z of a scheme X, $\mathcal{I}(Z)$ denotes the quasi-coherent ideal sheaf of Z. The isomorphism $\varepsilon^*\Omega^1_{E/S} \simeq \mathcal{I}(\varepsilon)/\mathcal{I}(2\varepsilon)$ is via relative differentiation. Considering also the isomorphism

$$(4.2.5.2) \qquad \left(\mathcal{I}(\varepsilon)/I(2\varepsilon)\right)^{\otimes 2} \simeq \mathcal{I}(2\varepsilon)/\mathcal{I}(3\varepsilon) \simeq \left(\mathcal{L}(2\varepsilon)/L(\varepsilon)\right)^{\vee},$$

given a choice of ω it makes sense to ask if x is *dual* to $(\varepsilon^*\omega)^{\otimes 2}$ under the isomorphism (4.2.5.2). If this is the case, we say that x is *adapted* to ω.

A *Legendre structure* on $f : E \to S$ over a $\mathbf{Z}[1/2]$-scheme is defined to be a full level 2-structure $\langle P, Q\rangle$ and a generator ω of $f_*\Omega^1_{E/S}$ such that the unique x as above with $x(P) = 0$ and $x(Q) = 1$ is adapted to ω. In [**35**, 2.2.8–2.2.9] it is shown that a universal Legendre structure is

$$y^2 = x(x-1)(x-\lambda) \qquad P = (0,0) \qquad Q = (1,0) \qquad \omega = -\frac{\mathrm{d}x}{2y}$$

over

$$\mathbf{Z}[\tfrac{1}{2}, \lambda]\left[\frac{1}{\lambda(\lambda+1)}\right].$$

Note that this base is obviously smooth of pure relative dimension 1 over $\mathbf{Z}[1/2]$. In particular, Legendre structures are *rigid*. This follows from the following facts:

- the automorphism scheme of a full level 2-structure is $\langle \pm 1\rangle$ [**35**, 2.7.3];
- multiplication by $n \in \mathbf{Z}$ on $E \to S$ induces multiplication by n on $\varepsilon^*\Omega^1_{E/S}$;
- $\omega = -\omega$ forces $\omega = 0$ over a $\mathbf{Z}[1/2]$-scheme.

In [**35**, 4.6.2] it is asserted that Legendre structures on elliptic curves define a $\mathrm{GL}_2(\mathbf{Z}/2) \times \langle -1\rangle$-torsor via Isom-schemes $\wp_{E/S}$. This is in fact *false*, since in general if $(E; P, Q; \omega)$ is a Legendre structure, then there usually does *not* exist any Legendre structure on E with respect to any other 2-torsion basis (there are problems with square roots). We will instead show that there is a universal triple $(E; P_4, Q_2)$ over variable $\mathbf{Z}[1/2]$-scheme bases, where P_4 is a point of exact order 4 and $(2P_4, Q_2)$ is a basis of $E[2]$. By construction, the base of this universal family will be an affine scheme Y (which will be smooth of pure relative dimension 1 over $\mathbf{Z}[1/2]$ by pure throught or by construction).

Since $P_4, -P_4$ are everywhere disjoint, such structures are rigid. Following the method of proof of Lemma 4.2.3.3, it follows that given a universal Y as above, there is a universal full level 4 structure over a base Y' which is a finite étale cover of Y. Thus, to construct the desired scheme Y' it suffices to study triples $(E; P_4, Q_2)$ over a $\mathbf{Z}[1/2]$-scheme as above.

Working Zariski locally on S, we may *choose* ω generating $f_*\Omega^1_{E/S}$. Since S is a $\mathbf{Z}[1/2]$-scheme, we may write E in *unique* Weierstrass form

$$(4.2.5.3) \qquad y^2 = x^3 + ax^2 + bx + c$$

where x, y are "adapted" to ω and $x(Q_2) = 0$. In other words, $x \in \mathcal{L}(2\varepsilon)/\mathcal{L}(\varepsilon)$ and $y \in \mathcal{L}(3\varepsilon)/\mathcal{L}(2\varepsilon)$ are dual to $(\varepsilon^*\omega)^{\otimes 2}$ and $(\varepsilon^*\omega)^{\otimes 3}$, respectively. From the analysis of 2-torsion in [**35**, 2.2.8], the cubic (4.2.5.3) factors as

$$y^2 = x(x-r)(x-s)$$

with $r, s, r-s \in \mathrm{H}^0(S, \mathcal{O}_S^\times)$, $Q_2 = (0,0)$, and $2P_4 = (0, r)$.

If we rename x as rx (so that it will no longer necessarily be adapted to ω), then we have the equation

$$y^2 = ux(x-1)(x-\lambda) \tag{4.2.5.4}$$

where $u = r^3 \in \mathrm{H}^0(S, \mathcal{O}_X^\times)$, $\lambda = \frac{s}{r}$, and $\lambda, \lambda - 1 \in \mathrm{H}^0(S, \mathcal{O}_S^\times)$. This is the *unique* equation for E with y adapted to ω and with x satisfying $x(Q_2) = 0$, $x(2P_4) = 1$. We must have $\lambda = x(2P_4 + Q_2)$ and $u = r^3$, where x is dual to $r^{-1} \cdot (\varepsilon^*\omega)^{\otimes 2}$. Clearly $P_4 = (a, b)$ where $a, b, a-1 \in \mathrm{H}^0(S, \mathcal{O}_S^\times)$, as this can be easily checked on geometric fibers.

We claim that

$$u = \frac{1}{a^2}\left(\frac{-au(3a^2 - (\lambda+1)2a + \lambda)}{2b} + b\right)^2 \tag{4.2.5.5}$$

To prove this we may assume that S is local, and then local noetherian, and finally local artinian. Any local artinian ring is a quotient of a complete regular local ring, and by *completeness* we can (uniquely) lift "étale level structures" through such a surjection once we lift (E, ω) through such a surjection (first lift E by lifting the Weierstrass equation). Due to the *intrinsic* characterization of (4.2.5.4), the parameters a, b, u, λ behave well with respect to such a lifting. But once our base is a regular local ring we may replace it by an algebraically closed field!

This reduces the proof of (4.2.5.5) to the case of an algebraically closed field. The equation $[2](a,b) = (1,0)$ says that the tangent line through (a,b) meets E in $(1,0)$. Inserting the equation of the tangent line into the equation for E gives a cubic with double root a, another root 1 and leading coefficient u. Thus, the constant term must be $-ua^2$. Computing the tangent line $y = \alpha x + \beta$ at (a,b) and then the constant term of

$$-(\alpha x + \beta)^2 + ux(x-1)(x-\lambda),$$

requiring this constant term to equal $-ua^2$ gives (4.2.5.5), as desired.

Using (4.2.5.5), which says that $u = v^2$ for some v, we may replace y by $(1/v)y$ to get an equation

$$y^2 = x(x-1)(x-\lambda).$$

In this form $\omega = -\mathrm{d}x/(2y)$ is the unique differential form "adapted" to x and y.

Put another way, given any (E, P_4, Q_2), then there is a *unique* $x \in f_*\mathcal{L}(2\varepsilon)$ with $x(2P_4) = 1$, $x(Q_2) = 0$, and moreover, Zariski locally on S there is a generator ω of $f_*\Omega^1_{E/S}$ with x "adapted" to ω. Such ω is unique up to μ_2, where the μ_2-ambiguity comes from the choice of square root v above.

Let $\mathbf{E}_\lambda \to \mathcal{M}_\lambda$ denotes the universal Legendre structure and let $\mathcal{M}'_\lambda \to \mathcal{M}_\lambda$ be the finite étale cover $[2]^{-1}(e)$, where e is the point of order 2 on $\mathbf{E}_\lambda \to \mathcal{M}_\lambda$ with $x(e) = 1$. Then $\mathcal{M}'_\lambda$ represents the functor of isomorphism classes of triples of the (rigid) data $(E; P_4, Q_2, \omega)$ where P_4 has exact order 4; $(2P_4, Q_2)$ is a basis of $E[2]$; and the unique x with $x(2P_4) = 1$ and $x(Q_2) = 0$ is adapted to ω.

If we let F denote the functor of isomorphism classes of the (rigid) data $(E; P_4, Q_2)$, then F is an fppf sheaf on schemes over $\mathbf{Z}[1/2]$. The above analysis shows that the natural map of fppf sheaves

$$\mathcal{M}'_\lambda \to F \tag{4.2.5.6}$$

is equivariant for the μ_2-action defined by

$$(E; P_4, Q_2; \omega) \mapsto (E; P_4, Q_2; -\omega)$$

and (4.2.5.6) induces an isomorphism of fppf sheaves $\mathcal{M}'_\lambda/\mu_2 \xrightarrow{\simeq} F$. But the μ_2-action is free on the affine scheme $\mathcal{M}'_\lambda$ and $\mu_2 \simeq \underline{\mathbf{Z}/2}$ over $\mathbf{Z}[1/2]$. Thus, the fppf sheaf $\mathcal{M}'_\lambda/\mu_2$ is an affine scheme, visibly smooth of pure relative dimension 1 over $\mathbf{Z}[1/2]$. This scheme is the desired universal base Y. $\square$

4.2.6. Compactifications. We now focus our attention on $Y_1(N) = Y_{\Gamma_1(N)}$.

LEMMA 4.2.6.1. *The map $j : Y_1(N) \to \mathbf{A}^1_{\mathbf{Z}[1/N]}$ is finite flat.*

PROOF. Certainly j is quasi-finite. Since both $Y_1(N)$ and $\mathbf{A}^1_{\mathbf{Z}[1/N]}$ are smooth curves over $\mathbf{Z}[1/N]$, we can check flatness on fibers over $\mathbf{Z}[1/N]$; here it is clear. It thus suffices to show that j is proper.

By the valuative criterion, it is enough to show that if R is a discrete valuation ring with $N \in R^\times$ (i.e., R is a $\mathbf{Z}[1/N]$-algebra) then any $\Gamma_1(N)$-structure (E, P) over the fraction field K of R extends to such a structure over R as long as $j \in R$. Since $j \in R$ implies that E has potentially good reduction, we get the existence of a finite separable extension K'/K and a discrete valuation ring R' in K' dominating R such that $E_{/K'} \simeq \mathcal{E} \times_{R'} K'$ for an elliptic curve $\mathcal{E}$ over R'. Since $\mathcal{E}[N]$ is finite étale over R', the given level structure over K' (coming from the one we were given over K) uniquely extends by scheme-theoretic closure to an analogous such level structure on $\mathcal{E}'$ over R'. Put in more geometric terms, we have been given a K-point of $Y_1(N)$ such that the induced K'-point of $Y_1(N)$ extends to an R'-point. But since $R \cap K' = R'$, it follows that for any separated scheme X at all, we have

$$X(R) = X(K) \cap X(R')$$

where the intersection taken inside of $X(K')$ (proof: given a map $\operatorname{Spec}(R') \to X$, look at an open affine neighborhood of the image of the closed point). Applying this to $X = Y_1(N)$ finishes the proof.

$\square$

We define $X_1(N)$ to be the normalization of $\mathbf{P}^1_{\mathbf{Z}[1/N]}$ in $Y_1(N)$ under the finite flat map j. Thus, $X_1(N)$ is *finite* over $\mathbf{P}^1_{\mathbf{Z}[1/N]}$ and normal, with $X_1(N) - Y_1(N)$ finite over the section $\infty \simeq \operatorname{Spec} \mathbf{Z}[1/N]$ of $\mathbf{P}^1_{\mathbf{Z}[1/N]}$ and $X_1(N) \to \operatorname{Spec} \mathbf{Z}[1/N]$ is flat. Since $\mathbf{P}^1_{\mathbf{Z}[1/N]}$ is regular and normal 2-dimensional schemes are Cohen-Macaulay, by going up/down and [**42**, Thm 23.3] the finite map $X_1(N) \to \mathbf{P}^1_{\mathbf{Z}[1/N]}$ is *flat*. In particular, $X_1(N) \to \operatorname{Spec} \mathbf{Z}[1/N]$ has all geometric fibers of pure dimension 1, so we see that $Y_1(N) \to \operatorname{Spec} \mathbf{Z}[1/N]$ has geometrically connected fibers if and only if $X_1(N) \to \operatorname{Spec} \mathbf{Z}[1/N]$ has geometrically connected fibers.

THEOREM 4.2.6.2. *The maps*

$$Y_1(N) \to \operatorname{Spec} \mathbf{Z}[1/N], \quad X_1(N) \to \operatorname{Spec} \mathbf{Z}[1/N]$$

are smooth with geometrically connected fibers of dimension 1. *The closed subscheme $(X_1(N) - Y_1(N))_{\mathrm{red}} \hookrightarrow X_1(N)$ is a relative Cartier divisor, finite étale over $\mathbf{Z}[1/N]$. The complex analytic fiber of $E_1(N) \to Y_1(N)$ over $\mathbf{Z}[1/N]$ is the corresponding universal object in the category of complex analytic spaces.*

It follows from this theorem and Lemma 1.5.5.1 that the compactification $X_1(N)^{\mathrm{an}}$ is uniquely compatibly identified with the base of the universal family of $\Gamma_1(N)$-structures on generalized elliptic curves in the category of complex analytic spaces.

PROOF. The smoothness of $X_1(N) \to \operatorname{Spec} \mathbf{Z}[1/N]$ follows from the determination of the formal completion along ∞ using Tate curves (and normality and excellence), as is summarized in [**35**, 10.9.6]. This also gives the structure of $(X_1(N) - Y_1(N))_{\mathrm{red}} \hookrightarrow X_1(N)$ as asserted.

For geometric connectedness, we first use the *analytic* theory:

$$(Y_1(N) \times_{\mathbf{Z}[1/N]} \mathbf{C})^{\mathrm{an}}$$

is a complement of finitely many points in a finite union of compact Riemann surfaces (which we'll call an *open Riemann surface*) and the analytic map

$$(Y_1(N) \times_{\mathbf{Z}[1/N]} \mathbf{C})^{\mathrm{an}} \to \Gamma_1(N)\backslash\mathfrak{h}$$

to the *universal analytic* family (cf. §2.1.3) is a *bijection* by GAGA. But $\Gamma_1(N)\backslash\mathfrak{h}$ is *also* an open Riemann surface, and an analytic bijection of open Riemann surfaces is an analytic isomorphism. Thus, the connectedness of $\mathfrak{h}$ forces the connectedness of $Y_1(N) \times \mathbf{C}$, so $Y_1(N) \to \operatorname{Spec} \mathbf{Z}[1/N]$ has geometrically connected generic fiber. It follows that the normalization $X_1(N)_{/\mathbf{Q}}$ is the unique proper regular model of the *geometrically connected* smooth curve $Y_1(N)_{/\mathbf{Q}}$, so $X_1(N)_{/\mathbf{Q}}$ is also geometrically connected.

We conclude that $\mathrm{H}^0(X_1(N)_{/\mathbf{Q}}, \mathcal{O}_{X_1(N)}) = \mathbf{Q}$. It follows that the torsion-free $\mathrm{H}^0(X_1(N), \mathcal{O}_{X_1(N)})$ is a *finite* $\mathbf{Z}[1/N]$-algebra *inside* of $\mathbf{Q}$. Consequently, we must have

$$\mathrm{H}^0(X_1(N), \mathcal{O}_{X_1(N)}) = \mathbf{Z}[1/N],$$

so $X_1(N) \to \operatorname{Spec} \mathbf{Z}[1/N]$ is its own Stein factorization. All of its geometric fibers are therefore connected, by [**26**, 4.3.4]. □

4.2.7. $\Gamma_1(N,p)$-structures. Now choose a prime p *not* dividing N. For an elliptic curve $E \to S$ over a $\mathbf{Z}[1/N]$-scheme S, we define a $\Gamma_1(N,p)$-*structure* on $E \to S$ to be a triple (E, P, C) where P is a $\Gamma_1(N)$-structure and $C \hookrightarrow E$ is a finite locally free subgroup scheme of order p. For $N \geq 5$ and *any* p, it follows from Lemma 4.2.1.1 that such triples do *not* admit non-trivial automorphisms.

THEOREM 4.2.7.1. *Fix $N \geq 5$ and p not dividing N. There is a universal $\Gamma_1(N,p)$-structure over a base scheme $Y_1(N,p)$ which is normal and is finite flat over $Y_1(N)$ via $(E,P,C) \mapsto (E,P)$, and consequently is finite flat over $\mathbf{A}^1_{\mathbf{Z}[1/N]}$ via j. The map $Y_1(N,p) \to \mathbf{Z}[1/N]$ is smooth over $\mathbf{Z}[1/Np]$, has geometrically connected fibers, and its complex analytic fiber recovers the corresponding universal analytic family.*

REMARK 4.2.7.2. $Y_1(N,p)$ is actually regular.

PROOF. As long as we work over $\mathbf{Z}[1/Np]$, the preceding methods apply and yield the theorem since Np-torsion is étale . To work over $\mathbf{Z}[1/N]$, we note that our moduli problem is *exactly* the problem of endowing $E_1(N) \to Y_1(N)$ with a *universal* order p subgroup scheme.

For any finite locally free commutative group scheme $G \to S$ of order d and for any d' dividing d, we claim that there is a scheme $S' \to S$ of finite presentation which represents the functor that associates to an S-scheme T the set of order p subgroup schemes of G_T. Indeed, we may work Zariski locally on S, so we can assume that S and G are affine. We can then work with module bases for finite free algebras as in [**35**, 1.3.4], from which the result follows easily.

Applying this to $E_1(N)[p] \to Y_1(N)$, we deduce the existence of $Y_1(N,p)$ as a finite type $Y_1(N)$-scheme. Using the valuative criteria and the method of scheme-theoretic closure over discrete valuation rings, one sees that $Y_1(N,p) \to Y_1(N)$ is proper. For any elliptic curve E over an algebraically closed field, there are only *finitely many* order p subgroup schemes: this is clear except for supersingular E in characteristic p. In the supersingular case, there is in fact only one order p subgroup scheme since a smooth curve in characteristic p has a unique closed subscheme of a specified length supported at a specified point. Thus, the proper map $Y_1(N,p) \to Y_1(N)$ is quasi-finite and hence finite.

Once we know that $Y_1(N,p)$ is normal and finite *flat* over $Y_1(N)$, we can define $X_1(N,p)$ via normalization and use Stein factorization over $\mathbf{Z}[1/N]$ to deduce the geometric connectedness of the fibers. The verification of normality and finite flatness is quite involved, so we must refer to [**35**, 6.6.2] (with $\mathrm{M}(\mathcal{S}) = Y_1(N)$ there) and [**35**, 6.8.7]. The proof makes essential use of deformation theory and Drinfeld structures, and in the end even establishes regularity as in Remark 4.2.7.2. $\square$

Although $Y_1(N,p) \to \operatorname{Spec} \mathbf{Z}[1/N]$ has a geometrically connected fiber in characteristic p, this fiber is *not* geometrically irreducible. It turns out to be geometrically reduced, with two irreducible components, each naturally identified with the geometrically connected smooth curve $Y_1(N)_{/\mathbf{F}_p}$.

In order to make the structure in characteristic p clear, we define several maps over $\mathbf{Z}[1/N]$. Using $(E,P,C) \mapsto (E,P)$ and $(E,P,C) \mapsto (E/C, P \bmod C)$, we obtain two degeneracy maps $\pi_1, \pi_2 : Y_1(N,p) \to Y_1(N)$. By Theorem 4.2.7.1, π_1 is finite flat.

The operation

$$w(E,P,C) = \big(E/C, P \bmod C, E[p]/C\big) \tag{4.2.7.1}$$

induces a map $Y_1(N,p) \to Y_1(N,p)$. Since the degree p isogeny $\varphi : E \to E/C$ has dual $\varphi' : E/C \to E/E[p] \simeq E$ with kernel $E[p]/C$, we see that $w^2(E,P,C) = (E, p\cdot P, C)$. Since w^2 is an automorphism of $Y_1(N,p)$, we conclude that w is an automorphism as well. In particular, $\pi_2 = \pi_1 \circ w$ and thus π_2 is *also* finite flat. Over $\mathbf{Z}[1/N]$ we arrive at a diagram with cartesian squares

$$\begin{tikzcd}
& E_1(N,p) \arrow[rr] \arrow[dl] \arrow[dr, "u"] & & E_1(N,p)/\mathcal{C} \arrow[dl, "v"'] \arrow[dr] & \\
E_1(N) \arrow[dr, "f_N"'] & & Y_1(N,p) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & & E_1(N) \arrow[dl, "f_N"] \\
& Y_1(N) & & Y_1(N) &
\end{tikzcd} \tag{4.2.7.2}$$

whose complex analytic fiber is *exactly* the diagram (2.3.1.2) we used to define T_p cohomologically on $\widetilde{\mathrm{H}}^1(Y_1(N)^{\mathrm{an}}_{\mathbf{C}}, \operatorname{Sym}^k \mathrm{R}^1 f_N^{\mathrm{an}*} \underline{\mathbf{Q}}_l)$.

We likewise have maps

$$I_n : Y_1(N) \to Y_1(N) \qquad w_\zeta : Y_1(N)_{/\mathbf{Z}[1/N,\zeta_N]} \to Y_1(N)_{/\mathbf{Z}[1/N,\zeta_N]}$$

defined by

$$(E,P) \mapsto (E, n\cdot P) \qquad \big(E, P_d, P_{\frac{N}{d}}\big) \mapsto \big(E/\langle P_d\rangle, P'_d, P_{\frac{N}{d}}\big).$$

Here $n \in (\mathbf{Z}/N)^\times$, d divides N with $\gcd(d, \frac{N}{d}) = 1$, and $\langle P_d, P'_d\rangle_d = \zeta = \zeta_N^{j\frac{N}{d}}$ for some $j \in (\mathbf{Z}/d)^\times$. The complex analytic fiber of the associated diagrams with cartesian squares

$$\begin{tikzcd} E_1(N) \arrow[r, "\simeq", "I_n"'] \arrow[d, "f_N"'] & E_1(N) \arrow[d, "f_N"] \\ Y_1(N) \arrow[r, "\simeq", "I_n"'] & Y_1(N) \end{tikzcd}$$

(4.2.7.3)
$$\begin{tikzcd} E_1(N) \arrow[r] \arrow[rr, bend left, "\varphi_\zeta"] \arrow[dr] & E_1(N)/\langle \mathcal{P}_d\rangle \arrow[r] \arrow[d] & E_1(N) \arrow[d, "f_N"] \\ & Y_1(N)_{/\mathbf{Z}[1/N,\zeta_N]} \arrow[r, "w_\zeta"'] & Y_1(N)_{/\mathbf{Z}[1/N,\zeta_N]} \end{tikzcd}$$

are *exactly* the ones we constructed analytically in (2.3.1.1), (2.3.6.3), where the base change to $\mathbf{C}$ of the diagram (4.2.7.3) is defined using any desired choice of map

$$\mathbf{Z}[1/N, \zeta_N] \to \mathbf{C}.$$

4.2.8. Characteristic p fibers. Our final arithmetic task is to see what the above geometric data look like on the $\mathbf{F}_p$-fiber. The idea is that the ordinary locus $Y_1(N,p)^{\mathrm{ord}}_{/\overline{\mathbf{F}}_p}$ consists of two irreducible components, with the $\overline{\mathbf{F}}_p$-points of these components corresponding to one of the two choice of order p subgroup scheme of an ordinary elliptic curve over $\overline{\mathbf{F}}_p$: the connected μ_p and the étale $\underline{\mathbf{Z}/p}$. In $Y_1(N,p)_{/\overline{\mathbf{F}}_p}$, these two components will meet at the supersingular points, where the distinct order p group schemes in the ordinary cases "come together." Recall that if $E_{/\overline{\mathbf{F}}_p}$ is supersingular, then $E[p]$ has a *unique* subgroup scheme of order p.

The precise result is the following.

THEOREM 4.2.8.1. *The curve $Y_1(N,p)_{/\mathbf{F}_p}$ is reduced with ordinary double point singularities and two irreducible components. With their reduced structure, these components are both isomorphic to $Y_1(N)_{/\mathbf{F}_p}$. In particular, they are smooth and geometrically connected. These components intersect at exactly the (finite, non-empty) set of supersingular points.*

More precisely, the maps $Y_1(N)_{/\mathbf{F}_p} \hookrightarrow Y_1(N,p)_{/\mathbf{F}_p}$ corresponding to the irreducible components are given on moduli by

$$\Phi_1(E,P) = (E, P, \ker F) \qquad \Phi_2(E,P) = \big(E^{(p)}, P^{(p)}, \ker V\big)$$

where $E^{(p)} = E \times_{S,\mathrm{Frob}} S$ is the base change of E by the absolute Frobenius morphism on S, $F = F_{E/S} : E \to E^{(p)}$ is the relative Frobenius morphism on $E \to S$, and $V = F^\vee$ is the dual isogeny of F.

REMARK 4.2.8.2. The operator w introduced in (4.2.7.1) satisfies $\Phi_1 \circ w_{/\mathbf{F}_p} = \Phi_2$.

Proof. This is a special case of the general study of characteristic p fibers in [**35**, Ch 13], as summarized in the left column of the table in [**35**, 13.5.6] with $n = 1$. This relies heavily on Drinfeld structures and deformation theory. $\square$

Corollary 4.2.8.3. *The composition $\pi_1 \circ \Phi_1$ is the identity. Both $\pi_2 \circ \Phi_1$ and $\pi_1 \circ \Phi_2$ send (E, P) to $(E^{(p)}, P^{(p)})$ and $\pi_2 \circ \Phi_2$ sends (E, P) to $(E, p \cdot P)$. In particular, $\pi_2 \circ \Phi_2 = I_{p/\mathbf{F}_p}$.*

Proof. The only non-obvious computation is that of $\pi_2 \circ \Phi_2$. Here it comes down to showing that $F^\vee = V : E^{(p)} \to E$ takes $P^{(p)}$ to $p \cdot P$. But $p \cdot P = VF(P) = V(P^{(p)})$, so this is clear. $\square$

Combining Theorem 4.2.8.1 and Corollary 4.2.8.3, we see that (4.2.7.2) over $\mathbf{F}_p$ gives rise to *two* diagrams, one for each component $\Phi_j : Y_1(N)_{/\mathbf{F}_p} \hookrightarrow Y_1(N, p)_{/\mathbf{F}_p}$. For Φ_1, we have

$$\begin{tikzcd}
 & (E_1(N), \mathcal{P}) \arrow[rr, "F_{E_1(N)/Y_1(N)}"] \arrow[dl, equal] \arrow[dr, "f_N"] & & (E_1(N)^{(p)}, \mathcal{P}^{(p)}) \arrow[dl, "f_N^{(p)}"'] \arrow[dr] & \\
(E_1(N), \mathcal{P}) \arrow[dr, "f_N"'] & & Y_1(N)_{/\mathbf{F}_p} \arrow[dl, equal] \arrow[dr, "F"] & & (E_1(N), \mathcal{P}) \arrow[dl, "f_N"] \\
 & Y_1(N)_{/\mathbf{F}_p} & & Y_1(N)_{/\mathbf{F}_p} &
\end{tikzcd}$$

where F is the absolute Frobenius on $Y_1(N)_{/\mathbf{F}_p}$ and the squares are the canonical cartesian squares. For Φ_2, we have

$$\begin{tikzcd}
 & (E_1(N)^{(p)}, \mathcal{P}^{(p)}) \arrow[rr, "V"] \arrow[dl] \arrow[dr, "f_N^{(p)}"'] & & (E_1(N), p \cdot \mathcal{P}) \arrow[dl, "f_N"'] \arrow[dr] & \\
(E_1(N), \mathcal{P}) \arrow[dr, "f_N"'] & & Y_1(N)_{/\mathbf{F}_p} \arrow[dl, "F"] \arrow[dr, "I_p"] & & (E_1(N), \mathcal{P}) \arrow[dl] \\
 & Y_1(N)_{/\mathbf{F}_p} & & Y_1(N)_{/\mathbf{F}_p} &
\end{tikzcd}$$

where again the squares are the canonical cartesian squares. Of course, by $E_1(N)$ above we always really mean $E_1(N)_{/\mathbf{F}_p}$.

This decomposition of (4.2.7.2) over $\mathbf{F}_p$ into these two pieces governed by F, I_p, and $F^\vee_{E_1(N)/Y_1(N)} = V$ is the reason "why" the relation

$$\text{“}T_p = \pi_{1*}\varphi^*\pi_2^*\text{”}$$

in étale cohomology will decompose into something analogous to the first relation in Theorem 3.3.1.1.

The second relation in Theorem 3.3.1.1 will also have an analogue in étale cohomology, thanks to the next result. As always, $\varepsilon_{p,d} \in (\mathbf{Z}/N)^\times$ is defined to be congruent to p modulo d and 1 modulo N/d for any $d \mid N$ with $\gcd(d, N/d) = 1$ and $p \nmid d$ any prime.

THEOREM 4.2.8.4. *Let $\zeta = \zeta_N^{j\frac{N}{d}}$ with $j \in (\mathbf{Z}/d)^\times$, where d divides N and* $\gcd(d, \frac{N}{d}) = 1$. *Fix a map $\mathbf{Z}[1/N, \zeta_N] \to \overline{\mathbf{F}}_p$, so we may consider the diagram*

$$\begin{tikzcd} E_1(N) \arrow[r] \arrow[rr, bend left, "\varphi_\zeta"] \arrow[dr] & E_1(N)/\langle \mathcal{P}_d \rangle \arrow[r] \arrow[d] & E_1(N) \arrow[d, "f_N"] \\ & Y_1(N)_{/\overline{\mathbf{F}}_p} \arrow[r, "w_\zeta"'] & Y_1(N)_{/\overline{\mathbf{F}}_p} \end{tikzcd}$$

over $\overline{\mathbf{F}}_p$. Then $\varphi_\zeta \overline{F}_{E_1(N)} = \overline{F}_{E_1(N)} I_{\varepsilon_{p,d}}^{-1} \varphi_\zeta$ as maps $E_1(N)_{/\overline{\mathbf{F}}_p} \to E_1(N)_{/\overline{\mathbf{F}}_p}$ over the maps $w_\zeta \overline{F}_{Y_1(N)} = \overline{F}_{Y_1(N)} I_{\varepsilon_{p,d}}^{-1} w_\zeta$ on $Y_1(N)_{/\overline{\mathbf{F}}_p}$.

REMARK 4.2.8.5. Here $\overline{F}_{E_1(N)}$ and $\overline{F}_{Y_1(N)}$ are the base changes to $\overline{\mathbf{F}}_p$ of absolute Frobenius on $E_1(N)_{/\mathbf{F}_p}$ and $Y_1(N)_{/\mathbf{F}_p}$ respectively. These are $\overline{\mathbf{F}}_p$-maps, *not* absolute Frobenius morphisms.

PROOF. We first check that the two maps $Y_1(N)_{/\overline{\mathbf{F}}_p} \rightrightarrows Y_1(N)_{/\overline{\mathbf{F}}_p}$ coincide. Since everything is reduced, it is enough to check this on the level of $\overline{\mathbf{F}}_p$-points. Choose $(E, P) \in Y_1(N)(\overline{\mathbf{F}}_p)$ and write $P = P_d + P_{\frac{N}{d}}$ via the decomposition

$$\mathbf{Z}/N \simeq \mathbf{Z}/d \times \mathbf{Z}/(N/d)).$$

Just as in the weight 2 calculations in §3.3.3 we have

$$\begin{aligned} w_\zeta \overline{F}_{Y_1(N)}(E, P) &= w_\zeta(E^{(p)}, P^{(p)}) \\ &= \left((E/\langle P_d \rangle)^{(p)}, p^{-1}(P_d')^{(p)} + P_{\frac{N}{d}}^{(p)}\right) \end{aligned}$$

and

$$\begin{aligned} \overline{F}_{Y_1(N)} I_{\varepsilon_{p,d}}^{-1} w_\zeta(E, P) &= \overline{F}_{Y_1(N)} I_{\varepsilon_{p,d}}^{-1}\left(E/\langle P_d \rangle, P_d' + P_{\frac{N}{d}}\right) \\ &= \overline{F}_{Y_1(N)}\left(E/\langle P_d \rangle, \tfrac{1}{p} P_d' + P_{\frac{N}{d}}\right) \\ &= \left((E/\langle P_d \rangle)^{(p)}, p^{-1}(P_d')^{(p)} + P_{\frac{N}{d}}^{(p)}\right). \end{aligned}$$

Here, the commutativity of

$$\begin{tikzcd} \operatorname{Spec} \overline{\mathbf{F}}_p \arrow[r] \arrow[d, "F"'] & Y_1(N) \arrow[d, "F_{Y_1(N)}"] \\ \operatorname{Spec} \overline{\mathbf{F}}_p \arrow[r] & Y_1(N) \end{tikzcd}$$

ensures that the base change $(\cdot)^{(p)}$ above by absolute Frobenius on $\overline{\mathbf{F}}_p$ really *does* compute $\overline{F}_{Y_1(N)}$ on $\overline{\mathbf{F}}_p$-points. This proves the desired equality on $Y_1(N)_{/\overline{\mathbf{F}}_p}$.

Let $h = w_\zeta \overline{F}_{Y_1(N)} = \overline{F}_{Y_1(N)} I_{\varepsilon_{p,d}}^{-1} w_\zeta$. We can now view the two maps

$$\varphi_\zeta \overline{F}_{E_1(N)}, \quad \overline{F}_{E_1(N)} I_{\varepsilon_{p,d}}^{-1} \varphi_\zeta \tag{4.2.8.1}$$

as degree pd isogenies over the same morphism $h : Y_1(N)_{/\overline{\mathbf{F}}_p} \to Y_1(N)_{/\overline{\mathbf{F}}_p}$. (Note that for an elliptic curve $E \to S$ over an $\mathbf{F}_p$-scheme, the absolute Frobenius F_E factors as the degree p isogeny $F_{E/S}$ followed by the top row of a cartesian square.)

Since everything is reduced, to check equality we can look at $\overline{\mathbf{F}}_p$-fibers over $Y_1(N)_{/\overline{\mathbf{F}}_p}$. The $\overline{\mathbf{F}}_p$-fibers of the isogenies (4.2.8.1) over $Y_1(N)_{/\overline{\mathbf{F}}_p}$ both have as kernels *exactly* the unique connected order p subgroup scheme. By the *rigidity* in Lemma 4.2.1.1, to show the equality of the isogenies it therefore suffices to check that the point $(P'_d, P_{\frac{N}{d}})$ of exact order N goes over to the same section of

$$E_1(N)_{/\overline{\mathbf{F}}_p} \to Y_1(N)_{/\overline{\mathbf{F}}_p}$$

over h in each case.

We compute

$$\begin{aligned}\overline{F}_{E_1(N)} I_{\varepsilon_{p,d}^{-1}} \varphi_\zeta(P'_d, P_{\frac{N}{d}}) &= \overline{F}_{E_1(N)} I_{\varepsilon_{p,d}^{-1}}(P \circ w_\zeta)\\ &= \overline{F}_{E_1(N)} \circ (p \cdot P_d, P_{\frac{N}{d}}) I_{\varepsilon_{p,d}^{-1}} w_\zeta\end{aligned}$$

since $I_{\varepsilon_{p,d}^{-1}}(\varepsilon_{p,d}^{-1} P) = P \circ I_{\varepsilon_{p,d}^{-1}}$. Thus, we arrive at

$$\begin{aligned}\overline{F}_{E_1(N)} I_{\varepsilon_{p,d}^{-1}} \varphi_\zeta(P'_d, P_{\frac{N}{d}}) &= (p \cdot P_d, P_{\frac{N}{d}}) \overline{F}_{Y_1(N)} I_{\varepsilon_{p,d}^{-1}} w_\zeta\\ &= (p \cdot P_d, P_{\frac{N}{d}}) \circ h,\end{aligned}$$

where P_d, $P_{\frac{N}{d}}$ interchange $F_{E_1(N)}$ and $F_{Y_1(N)}$ since $P : Y_1(N) \to E_1(N)$ commutes with absolute Frobenii. Also, the analogous 'interchange' assertion holds with respect to $\overline{F}_{E_1(N)}$ and $\overline{F}_{Y_1(N)}$ after base change to $\overline{\mathbf{F}}_p$.

Meanwhile, we compute that

$$\begin{aligned}\varphi_\zeta \overline{F}_{E_1(N)}(P'_d, P_{\frac{N}{d}}) &= \varphi_\zeta(p \cdot P'_d, P_{\frac{N}{d}}) \circ \overline{F}_{Y_1(N)}\\ &= (p \cdot P_d, P_{\frac{N}{d}}) \circ w_\zeta \overline{F}_{Y_1(N)}\\ &= (p \cdot P_d, P_{\frac{N}{d}}) \circ h\end{aligned}$$

provided that we check that

$$\overline{F}_{E_1(N)} \circ P'_d \equiv p \cdot P'_d \circ \overline{F}_{Y_1(N)} \pmod{P_d}. \tag{4.2.8.2}$$

We have reduced the proof to checking (4.2.8.2).

Note that $\overline{F}_{E_1(N)} = (\sigma \times 1)^{-1} \circ F_{E_1(N)}$, where

$$\sigma \times 1 : \overline{\mathbf{F}}_p \times E_1(N) = \overline{E}_1(N) \to \overline{E}_1(N)$$

is the product of the absolute Frobenius σ on $\overline{\mathbf{F}}_p$ and the identity on $E_1(N)$ (and likewise for $Y_1(N)$). Also,

$$F_{\overline{E}_1(N)} P'_d = P'_d F_{\overline{Y}_1(N)} = P'_d(\sigma \times 1) \overline{F}_{Y_1(N)}.$$

Thus, it suffices to show that

$$(\sigma \times 1)^{-1} P'_d (\sigma \times 1) \equiv p \cdot P'_d \pmod{P_d}.$$

Equivalently, we need to show that for the section

$$(\sigma \times)^{-1} P'_d(\sigma \times 1) : \overline{Y}_1(N) \to \overline{E}_1(N),$$

where

$$\langle P_d, P'_d \rangle_d = \zeta \in \mu_d(\overline{\mathbf{F}}_p) \subseteq \mu_d(\overline{Y}_1(N)),$$

we also have

$$\left\langle P_d, (\sigma \times 1)^{-1} P'_d (\sigma \times 1) \right\rangle_d \stackrel{?}{=} \zeta^p. \tag{4.2.8.3}$$

Equalities in étale group schemes can be checked on fibers and $\langle \cdot, \cdot \rangle_d$ is defined over $Y_1(N)$, so to prove (4.2.8.3) we may pass to fibers over closed points of $Y_1(N)$. In other words, for finite extension fields k of $\mathbf{F}_p$ and elliptic curves $E \to \operatorname{Spec} k$

equipped with points $P \in E[d](k)$ of exact order d and points $P' \in E[d](k \otimes_{\mathbf{F}_p} \overline{\mathbf{F}}_p)$ with $\langle P, P' \rangle_d = 1 \otimes \zeta \in \mu_d(k \otimes_{\mathbf{F}_p} \overline{\mathbf{F}}_p)$, we need to check that

$$\left\langle P, (\sigma \times 1)^{-1} P'(\sigma \times 1) \right\rangle_d \stackrel{?}{=} 1 \otimes \zeta^p.$$

Since P induces a closed immersion $\underline{\mathbf{Z}/d} \hookrightarrow E[d]$ *over* k, the pairing $\langle \cdot, \cdot \rangle_d$ over k induces an isomorphism

$$E[d]/\langle P \rangle \simeq \mu_d$$

over k. We need to check that for

$$\zeta \in \mu_d(\overline{\mathbf{F}}_p) \subseteq \mu_d(k \otimes \overline{\mathbf{F}}_p) = \mu_{d/k}(k \otimes \overline{\mathbf{F}}_p) \simeq \mu_{d/k \otimes \overline{\mathbf{F}}_p}(k \otimes \overline{\mathbf{F}}_p)$$

the action of $1 \times \sigma$ on $k \otimes \overline{\mathbf{F}}_p$ induces an action on $\mu_{d/k}(k \otimes \overline{\mathbf{F}}_p) = \mu_d(k \otimes \overline{\mathbf{F}}_p)$ which restricts to the pth power on $\mu_d(\overline{\mathbf{F}}_p)$. This is *obvious*. (Note that the action on all of $\mu_d(k \otimes \overline{\mathbf{F}}_p)$ is *not* the pth power!)

□

4.3. Resolution of singularities

4.3.1. Kuga-Sato varieties. Let $E = E_1(N)_{/\mathbf{F}_p} \to Y_1(N)_{/\mathbf{F}_p}$ (for $N \geq 5$) be the universal elliptic curve. Deligne's original method for reducing the Ramanujan conjectures to the Weil conjectures required the identification of the smooth k-fold fiber product

$$E^k \stackrel{\text{def}}{=} \overbrace{E_1(N)_{/\mathbf{F}_p} \times_{Y_1(N)_{/\mathbf{F}_p}} \cdots \times_{Y_1(N)_{/\mathbf{F}_p}} E_1(N)_{/\mathbf{F}_p}}^{k}$$

with an open variety in a *projective* smooth $\mathbf{F}_p$-scheme. (Note that E^k is proper over the *affine* curve $Y_1(N)_{/\mathbf{F}_p}$, hence is *not* proper over $\mathbf{F}_p$.) This step requires a concrete resolution of singularities. Deligne's later work on generalizations of the Weil conjectures make this step unnecessary, but due to the interest in E^k and the sketchy discussion of the resolution in [**10**], we explain the proof of resolution in this case. Of course, E^k is not very interesting when $k = 0$ (corresponding to the weight 2 case), and indeed Deligne's arguments for the $k = 0$ case do not involve non-constant étale sheaves. Thus, the present discussion is not needed in that case.

The main result here is the following theorem of Deligne.

THEOREM 4.3.1.1. *For each $k \geq 1$, there exists an open immersion of E^k into a projective smooth $\mathbf{F}_p$-scheme.*

REMARK 4.3.1.2. The projective smooth varieties arising from this proof are called *Kuga-Sato varieties*. Before Deligne, others had the idea to use the Weil conjecture for such a variety to reduce the Ramanujan conjecture to the Weil conjecture. However, Deligne was the one who (among other things) figured out how to find the required projective smooth variety as in Theorem 4.3.1.1.

PROOF. Consider the compactification of $Y_1(N)_{/\mathbf{F}_p}$, a smooth geometrically connected curve. For the purposes of this proof it is irrelevant that this compactification is $X_1(N)_{/\mathbf{F}_p}$. We will write $X_1(N)_{/\mathbf{F}_p}$ for this curve, but we will *not* use that it is the characteristic p fiber of $X_1(N)_{\mathbf{Z}[1/N]}$. We define $\overline{E} \to X_1(N)_{/\mathbf{F}_p}$ to be the minimal regular proper model of E over the Dedekind base $X_1(N)_{/\mathbf{F}_p}$; clearly $\overline{E}$ is a finite type $\mathbf{F}_p$-scheme.

At this point, it seems essential to invoke the theory of generalized elliptic curves in [**14**] to most naturally understand $\overline{E}$ over the cusps. To be precise, the proof of [**14**, IV, 1.6] implies that $\overline{E} \to X_1(N)$ has at worst ordinary double point singularities over the cusps if and only if it is a DM semistable curve of genus 1. This in turn is true if and only if the smooth locus $\overline{E}^{\mathrm{sm}}$ (which is the Néron model of the elliptic curve $E \to Y_1(N)_{/\mathbf{F}_p}$, by [**2**, 7.5/1]) has semistable reduction in the usual sense that the connected components of the cuspidal fibers are either elliptic curves or tori. (If this is true, then $\overline{E}$ has a unique structure of generalized elliptic curve extending the group law on E.)

The first option of good reduction over a cusp cannot occur, by the description of the cusps via Tate curves (or because $j : Y_1(N)_{/\mathbf{F}_p} \to \mathbf{A}^1_{/\mathbf{F}_p}$ extends to $\overline{\jmath} : X_1(N) \to \mathbf{P}^1_{\mathbf{F}_p}$ with $\overline{\jmath}^{-1}(\infty) = X_1(N) - Y_1(N)$). We claim that $\overline{E}^{\mathrm{sm}} \to X_1(N)$ *does* have toric (i.e., semistable) reduction over the cusps. To prove this, it suffices to show that there is *some* semi-abelian model for E at the discrete valuation ring for each cusp (as semi-abelian schemes with proper generic fibers over discrete valuation rings are *always* the connected component of the Néron model of their generic fiber [**2**, 7.4/3]). Since j is non-integral at the discrete valuation ring for each cusp, the Néron models there for E have potentially multiplicative reduction. Thus, it suffices to show the following: let R be a discrete valuation ring with fraction field K and let $E_{/K}$ be an elliptic curve with potentially multiplicative reduction (i.e., $j \notin R$). If $E(K)$ has a point of exact order $N \geq 5$ with $N \in R^\times$, then E has semistable (and hence multiplicative) reduction.

Let l be an odd prime dividing N or let $l = 4$, so the representation of the inertia group of K on the l-adic Tate module is congruent to $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$ modulo l. By the theory of Tate models, if E isn't semistable then a quadratic twist of E is, so that this l-adic representation is the quadratic twist of a unipotent representation. But looking modulo l shows that the auxiliary quadratic character is trivial, so indeed we do have semistable reduction.

We have now shown that $\overline{E} \to X_1(N)_{/\mathbf{F}_p}$ is a DM semistable curve of genus 1 and thus, as we mentioned above, it has only ordinary double point singularities. By the étale local description of ordinary double point singularities [**17**, Ch III, Prop 2.8] and the *generic smoothness* of $\overline{E}$ over $X_1(N)_{/\mathbf{F}_p}$, for each non-smooth point $\xi \in \overline{E}$ over $X_1(N)_{/\mathbf{F}_p}$, the strict Henselization $\mathcal{O}^{\mathrm{sh}}_{\overline{E},\xi}$ is isomorphic to the strict Henselization of $\mathbf{F}_p[q,x,y]/(xy - q^{m_\xi})$ at (q,x,y), for some $m_\xi \geq 1$. Since $\overline{E}$ is *regular*, we must have $m_\xi = 1$ for all ξ, so $\overline{E} \to \operatorname{Spec} \mathbf{F}_p$ is *smooth* and proper. As $\overline{E}$ is also a regular curve over the Dedekind scheme $X_1(N)_{/\mathbf{F}_p}$, it follows from the method of proof of [**40**, Thm 2.8] that there exists an invertible sheaf $\mathcal{L}$ on $\overline{E}$ such that $\mathcal{L}$ is relatively ample over $X_1(N)_{/\mathbf{F}_p}$. Since $X_1(N)_{/\mathbf{F}_p}$ is itself projective over $\mathbf{F}_p$, it follows that $\overline{E}$ is *projective* (and smooth) over $\mathbf{F}_p$. This settles the case $k = 1$.

Now assume that $k \geq 2$. We will study the non-smooth projective $\mathbf{F}_p$-scheme

$$\overline{E}^k \stackrel{\mathrm{def}}{=} \overbrace{\overline{E}_{/\mathbf{F}_p} \times_{X_1(N)_{/\mathbf{F}_p}} \cdots \times_{X_1(N)_{/\mathbf{F}_p}} \overline{E}_{/\mathbf{F}_p}}^{k}$$

whose smooth locus contains E^k. Explicitly, consider an $\overline{\mathbf{F}}_p$-point

$$\xi = (\xi_1, \dots, \xi_k) \in \overline{E}^k$$

with $\xi_1, \ldots, \xi_r$ non-smooth in $\overline{E}^k$, $\xi_{r+1}, \ldots, \xi_k$ smooth in $\overline{E}^k$, and $r \geq 1$. Working étale locally at this point, $\overline{E}^k$ is isomorphic to

$$\text{(4.3.1.1)} \quad \begin{aligned} &\mathbf{F}_p[q, X_1, Y_1, \ldots, X_r, Y_r, T_{r+1}, \ldots, T_k]/\big(X_1Y_1 = \cdots = X_rY_r = q\big) \simeq \\ &\mathbf{F}_p[X_1, Y_1, \ldots, X_r, Y_r, T_{r+1}, \ldots, T_k]/\big(X_1Y_1 = \cdots = X_rY_r\big). \end{aligned}$$

(In fact, there is no relation for $r = 1$.) Of course, we have an analogous ring for any permutation of the ξ_i.

Our task is to find an $\mathbf{F}_p$-smooth tower of blow-ups of $\overline{E}^k$ over the non-smooth locus. This will settle the theorem for $k \geq 2$. The hard part is to give an *intrinsic* description of what to blow-up; once we have defined an intrinsic algorithm, we will be able to *compute* it étale locally (using (4.3.1.1)) to see that the process really leads to a desingularization. The following clear description is due to Tony Scholl [**52**], and is inspired by [**10**, Lemma 5.5].

Let $\{x_1, \ldots\}$ be the finitely many points in $X_1(N)_{/\mathbf{F}_p} - Y_1(N)_{/\mathbf{F}_p}$. Let $\overline{F} \hookrightarrow \overline{E}$ be the finite disjoint union of the closed reduced fiber subschemes $\overline{E}_{\mathbf{F}_p(x_i)} \stackrel{\text{def}}{=} \overline{F}_i$ over the x_i's. We write $\xi_{1,i}, \ldots$ for the finite non-empty set of ordinary double points on $\overline{F}_i$. We view each $\xi_{j,i} \in \overline{F}_i$ as a closed (reduced) subscheme concentrated at a point; of course, $\xi_{j,i} \in \overline{F}_i = \overline{E}_{\mathbf{F}_p(x_i)}$ is typically *not* $\mathbf{F}_p(x_i)$-rational.

Let $\overline{F}^k$ denote the k-fold fiber product of $\overline{F}$ over $X_1(N)_{/\mathbf{F}_p}$. We define a stratification of

$$\overline{F}^k \hookrightarrow \overline{E}^k$$

by closed subschemes

$$\overline{F}^k = F_k^k \supsetneq F_{k-1}^k \supsetneq \cdots \supsetneq F_0^k \supseteq F_{-1}^k = \emptyset$$

where, for $0 \leq r \leq k$, $F_r^k \subseteq \overline{F}^k$ is the closed subscheme whose x_i-fiber is the scheme-theoretic union of the closed subschemes

$$\{\xi_{j_1,i}\} \times_{\mathbf{F}_p(x_i)} \cdots \times_{\mathbf{F}_p(x_i)} \{\xi_{j_{k-r},i}\} \times_{\mathbf{F}_p(x_i)} \overline{F}_{x_i}^r,$$

(of course, by $\overline{F}_{x_i}^r$ we mean the r-fold fiber product of the fiber $\overline{F}_{x_i}$ over $\mathbf{F}_p(x_i)$) together with the other permutations of this arising from the k "coordinates" of $\overline{F}_{x_i}^k$. For example, working étale locally over $\overline{E}$, we see that

$$F_{k-2}^k = (\overline{E}^k)^{\text{non-regular}}.$$

Define

$$E^k\langle 0\rangle = \overline{E}^k, \qquad F_i^k\langle 0\rangle = F_i^k$$

for $0 \leq i \leq k$. For $1 \leq r \leq k-1$, we recursively define

$$E^k\langle r\rangle = \mathrm{Bl}_{F_{r-1}^k\langle r-1\rangle}\big(E^k\langle r-1\rangle\big).$$

and we let $F_i^k\langle r\rangle$ be the proper transform of $F_i^k\langle r-1\rangle$ for $r \leq i \leq k-1$.

We claim that

$$F_r^k\langle r\rangle \subseteq E^k\langle r\rangle^{\text{non-regular}}$$

for all $0 \leq r \leq k-2$. In particular, this will imply that $E^k\langle k-1\rangle \to E^k\langle 0\rangle = \overline{E}^k$ is an isomorphism over $(\overline{E}^k)^{\text{reg}} \supseteq \overline{E}^k$ and we claim also that $E^k\langle k-1\rangle$ is regular (and hence $\mathbf{F}_p$-smooth). This yields the desired desingularization.

To verify both claims we may work étale locally over a non-regular point of $\overline{E}^k$, which amounts to replacing $\overline{E}^k$ with

$$\widetilde{E}^m\langle 0\rangle = \mathbf{F}_p[X_0, Y_0, \ldots, X_m, Y_m, T_{m+1}, \ldots, T_k]/\big(X_0Y_0 = \cdots = X_{m+1}Y_{m+1}\big),$$

where $2 \leq m \leq k$.

We define $\widetilde{F}_i^m = \widetilde{F}_i^m\langle 0\rangle$ to be the locus where $m-i$ pairs (X_j, Y_j) vanish. Using inductive definitions analogous to those above, we define $\widetilde{E}^m\langle r\rangle$ and $\widetilde{F}_i^m\langle r\rangle$ (with $r \leq i \leq m-1$) for $0 \leq r \leq m-1$. We want

- $\widetilde{F}_r^m\langle r\rangle \subseteq \widetilde{E}^m\langle r\rangle^{\mathrm{reg}}$ for all $0 \leq r \leq m-2$;
- $\widetilde{E}^m\langle m-1\rangle$ is regular.

This will clearly prove both claims about $\overline{E}^k$. The $T_m, \ldots, T_k$ just get "carried along", so they can (and will) now be dropped.

It is easy to see that $\widetilde{E}^m\langle 1\rangle$ has an open cover by $2m$-copies U_j of $\mathbf{A}^1 \times \widetilde{E}^{m-1}\langle 0\rangle$ such that $U_j \cap \widetilde{F}_r^m\langle 1\rangle = \mathbf{A}^1 \times \widetilde{F}_{r-1}^{m-1}\langle 0\rangle$ for $1 \leq r \leq m-1$. Here, we define $\widetilde{E}^1 = \widetilde{E}^1\langle 0\rangle = \operatorname{Spec}\mathbf{F}_p[X_0, Y_0]/(X_0Y_0)$ and $\widetilde{F}_0^1 = (0,0)$.

By induction on m, we see that for $0 \leq r \leq m-2$ there exists an open cover of $\widetilde{E}^m\langle r\rangle$ by copies V_j of $\mathbf{A}^r \times \widetilde{E}^{m-r}\langle 0\rangle$ with $V_j \cap \widetilde{F}_i^m\langle r\rangle = \mathbf{A}^r \times \widetilde{F}_{i-r}^{m-r}\langle 0\rangle$ for all $r \leq i \leq m-1$. Thus, $\widetilde{F}_r^m\langle r\rangle \subseteq \widetilde{F}_0^{m-r} \subseteq \widetilde{E}^{m-r}\langle 0\rangle^{\mathrm{reg}}$ since $m-r \geq 2$.

It remains to note that when $m=2$, so

$$\widetilde{E}^2 = \operatorname{Spec}\mathbf{F}_p[X_0, Y_0, X_1, Y_1]/(X_0Y_0 - X_1Y_1),$$

the scheme $\mathrm{Bl}_{(0)}(\widetilde{E}^2)$ is regular, covered by copies of $\mathbf{A}^3$. $\square$

This completes the geometric preparations for the arguments using étale cohomology.

CHAPTER 5

Étale cohomology

In this chapter, we give an overview of the foundational results in étale cohomology, including some important examples and remarks on the proofs, and then we apply these results to prove Deligne's theorems. We discuss in detail the various manifestation of "Frobenius morphism" in étale cohomology, since one of the key problems we have to solve is the determination of characteristic polynomials for Frobenius actions in terms of Hecke eigenvalues on spaces of modular forms. The comparison theorems between étale cohomology and topological cohomology over $\mathbf{C}$, together with various delicate base change theorems, will provide the link (via the Eichler-Shimura isomorphism) between the analytic theory of cusp forms over $\mathbf{C}$ and the purely algebraic theory of l-adic cohomology over finite fields. This is what makes it possible to use the Weil conjectures over finite fields to deduce estimates of eigenvalues of Hecke operators. For reasons of space, we only give an ad hoc treatment of l-adic sheaves which is barely adequate for applications to relative smooth curves (and is woefully insufficient for essentially any other application, such as the proof of the Weil conjectures).

It is assumed that the reader has some previous exposure to the *idea* of a Grothendieck topology and sheaves in such a context, and the definition of an étale morphism, but we will review some of these concepts with a special emphasis on the case of the étale site. We recommend [**2**, Ch 1, §6.1, §6.2, §8.1] for a polished overview of the basic general facts and examples related to étale morphisms, Grothendieck topologies, descent, and sheaves on a site.

5.1. Étale sheaves

5.1.1. Étale sheaves and Galois modules. Let S be a scheme. The (*small*) *étale site* $S_{\text{ét}}$ has as objects the étale S-schemes, and a *cover* of an object S' is defined to be a set of étale maps $S'_i \to S'$ whose (open) images cover S'. A *presheaf* (of sets) on $S_{\text{ét}}$ (called an *étale sheaf* on S) is a contravariant functor from $S_{\text{ét}}$ to the category of sets; a *sheaf* (of sets) on $S_{\text{ét}}$ is a presheaf satisfying the usual sheaf axioms. We can *sheafify* presheaves since there is a cofinal *set* of objects in $S_{\text{ét}}$.

Before we delve into the general theory of étale sheaves and their cohomology, it is intructive to begin with a detailed explication of the dictionary between étale sheaves over $\mathrm{Spec}(k)$ and discrete Galois modules over $\mathrm{Gal}(k_{\text{sep}}/k)$, where k is a field. After all, we are interested in constructing certain Galois modules related to modular forms, and these will in fact be realized as the $\mathbf{Q}$-fibers of certain étale sheaves on $\mathrm{Spec}(\mathbf{Z}[1/N])$ constructed from the arithmetic models of modular curves that we analyzed in the previous chapter. It is important to thoroughly understand the situation over a field before exploring geometrically complicated situations.

Let $S = \operatorname{Spec} k$ with k a field and let $\mathcal{F}$ be an étale sheaf on $\operatorname{Spec} k$. Fix a separable closure k_s of k. An étale k-scheme is just a disjoint union $\coprod \operatorname{Spec} k_i$ where the k_i are finite separable field extensions of k. By the sheaf axioms we must have

$$\mathcal{F}\big(\coprod \operatorname{Spec} k_i\big) = \prod \mathcal{F}(\operatorname{Spec} k_i).$$

Also, if $g : k' \xrightarrow{\simeq} k''$ is an isomorphism over k, then there is an induced isomorphism of sets

$$\mathcal{F}(\operatorname{Spec} g) : \mathcal{F}(\operatorname{Spec} k') \xrightarrow{\simeq} \mathcal{F}(\operatorname{Spec} k'').$$

If k' is a finite *Galois* extension of k, then $\operatorname{Gal}(k'/k) = \operatorname{Aut}(\operatorname{Spec} k'/\operatorname{Spec} k)^{\mathrm{opp}}$ thereby has a natural *left* action on $\mathcal{F}(k')$. The diagram

$$\operatorname{Spec} k' \times_{\operatorname{Spec} k} \operatorname{Spec} k' \rightrightarrows \operatorname{Spec} k' \to \operatorname{Spec} k$$

is naturally identified with the diagram

$$\coprod_{g \in \operatorname{Gal}(k'/k)} \operatorname{Spec} k' \rightrightarrows \operatorname{Spec} k' \to \operatorname{Spec} k \tag{5.1.1.1}$$

via the isomorphism

$$k' \otimes_k k' \xrightarrow{\simeq} \prod_{g \in \operatorname{Gal}(k'/k)} k'$$
$$x \otimes y \mapsto \big(xg(y)\big)$$

(In particular, the top projection q_1 in (5.1.1.1) is the "identity" and the bottom projection q_2 is induced by $\operatorname{Spec} g$ on the gth coordinate.)

The sheaf axiom yields an exact sequence of sets

$$\prod_g \mathcal{F}(\operatorname{Spec} k') \leftleftarrows \mathcal{F}(\operatorname{Spec} k') \leftarrow \mathcal{F}(\operatorname{Spec} k) \tag{5.1.1.2}$$

where the top projection q_1 is the diagonal map $q_1(s) = (s, \dots, s)$ and where the bottom projection q_2 is the "action map" $q_2(s) = (gs)_g$. Thus exactness of (5.1.1.2) says that $\mathcal{F}(\operatorname{Spec} k) \to \mathcal{F}(\operatorname{Spec} k')$ is injective and $\mathcal{F}(\operatorname{Spec} k) = \mathcal{F}(\operatorname{Spec} k')^{\operatorname{Gal}(k'/k)}$.

The same argument shows that for any finite Galois extension k''/k' with $\operatorname{Spec} k' \in S_{\text{ét}}$, we have $\mathcal{F}(\operatorname{Spec} k') = \mathcal{F}(\operatorname{Spec} k'')^{\operatorname{Gal}(k''/k')}$; here $\mathcal{F}(\operatorname{Spec} k') \to \mathcal{F}(\operatorname{Spec} k'')$ is induced by functoriality and $\operatorname{Gal}(k''/k')$ acts on $\mathcal{F}(\operatorname{Spec} k'')$ by functoriality as well.

Let Σ be the *set* of finite Galois extensions of k inside of our fixed separable algebraic closure k_s. For each $k' \in \Sigma$, we have a set $\mathcal{F}(\operatorname{Spec} k')$ on which $\operatorname{Gal}(k'/k)$ acts on the left, and for $k'' \in \Sigma$ *containing* k' (inside of k_s) we have an injection $\mathcal{F}(\operatorname{Spec} k') \hookrightarrow \mathcal{F}(\operatorname{Spec} k'')$ identifying $\mathcal{F}(\operatorname{Spec} k')$ with $\mathcal{F}(\operatorname{Spec} k'')^{\operatorname{Gal}(k''/k')}$. Moreover, the map $\mathcal{F}(\operatorname{Spec} k') \to \mathcal{F}(\operatorname{Spec} k'')$ is compatible with Galois actions via the natural surjection $\operatorname{Gal}(k''/k) \twoheadrightarrow \operatorname{Gal}(k'/k)$. Thus,

$$M_{\mathcal{F}} = \varinjlim_{k' \in \Sigma} \mathcal{F}(\operatorname{Spec} k')$$

has a natural structure of discrete left $\operatorname{Gal}(k_s/k)$-set (i.e., it is a set with a left action of $\operatorname{Gal}(k_s/k)$ which is continuous with respect to the Krull topology on $\operatorname{Gal}(k_s/k)$ and the discrete topology on $M_{\mathcal{F}}$). Furthermore, the natural map $\mathcal{F}(k') \to M_{\mathcal{F}}$ is an isomorphism onto $M_{\mathcal{F}}^{\operatorname{Gal}(k_s/k')}$ as $\operatorname{Gal}(k'/k)$-sets. This leads us to the next theorem.

THEOREM 5.1.1.1. *The functor* $\mathcal{F} \to M_{\mathcal{F}}$ *from sheaves on* $(\operatorname{Spec} k)_{\text{ét}}$ *to discrete left* $\operatorname{Gal}(k_s/k)$*-sets is fully faithful. There is a natural quasi-inverse functor, so that these categories are equivalent. This equivalence is compatible with formation of finite products, and thus identifies the category of abelian sheaves with the category of discrete* $\operatorname{Gal}(k_s/k)$*-modules.*

REMARK 5.1.1.2. The category of sheaves on $(\operatorname{Spec} k)_{\text{ét}}$ is independent of any *choice* of k_s, but many important calculations require this choice of "base point." Of course, when k is separably closed and $k_s = k$, then sheaves on $(\operatorname{Spec} k)_{\text{ét}}$ are *canonically* the same as sets.

Because of Theorem 5.1.1.1, we will see that étale cohomology over $\operatorname{Spec} k$ is the "same" as Galois cohomology over k, except that the former does not mention any choice of k_s!

PROOF. By the very definition of the sheaf axioms, and since every finite separable extension k'/k is covered by a Galois extension of k, we see that if $\mathcal{F}$ and $\mathcal{G}$ are two étale sheaves on $\operatorname{Spec} k$, then to give a map $\mathcal{F} \to \mathcal{G}$ is the same as to specify a map $\mathcal{F}(\operatorname{Spec} k') \to \mathcal{G}(\operatorname{Spec} k')$ for all finite Galois extensions k' of k, functorially in k'. (In particular, these maps are compatible with the actions of $\operatorname{Gal}(k'/k)$.) Equivalently, we must specify maps $\mathcal{F}(\operatorname{Spec} k') \to \mathcal{G}(\operatorname{Spec} k')$ of $\operatorname{Gal}(k'/k)$-sets for each $k' \in \Sigma$, functorial with respect to inclusions $k' \hookrightarrow k''$ inside of k_s; this is sufficient because an *arbitrary* map $k' \to k''$ over k factors as an inclusion $k' \hookrightarrow k''$ inside of k_s, followed by the action of some $g \in \operatorname{Gal}(k''/k') \subseteq \operatorname{Gal}(k''/k)$. This is exactly the data of a map $M_{\mathcal{F}} \to M_{\mathcal{G}}$ as discrete $\operatorname{Gal}(k_s/k)$-sets, so the fact that our functor is fully faithful is clear.

Now let M be a discrete $\operatorname{Gal}(k_s/k)$-set. For any "abstract" finite separable extension k' over k, define

$$\mathcal{F}_M(\operatorname{Spec} k') = \left\{ (m_i) \in \prod_{i:k' \hookrightarrow k_s} M; m_{g(i)} = g(m_i) \text{ for all } g \in \operatorname{Gal}(k_s/k) \right\}.$$

Here i runs through the finitely many k-embeddings of k' into k_s. Note that m_i lies in $M^{\operatorname{Gal}(k_s/i(k'))}$. For any $k' \in \Sigma$, projection to the coordinate labelled by the *inclusion* $k' \hookrightarrow k_s$ defines a bijection

$$\mathcal{F}_M(\operatorname{Spec} k') \xrightarrow{\simeq} M^{\operatorname{Gal}(k_s/k')}.$$

For any k-embedding $j : k' \hookrightarrow k''$ of finite separable extensions of k, any $i' : k' \hookrightarrow k_s$ over k extends to $i'' : k'' \hookrightarrow k_s$ over k. Thus, there is a natural map

$$\mathcal{F}_M(j) : \mathcal{F}_M(\operatorname{Spec} k') \to \mathcal{F}_M(\operatorname{Spec} k'') \tag{5.1.1.3}$$

defined by $\mathcal{F}_M(j)\big((m_{i'})_{i''}\big) = m_{i'' \circ j}$ and this is *injective*. When $k' = k''$ is a finite Galois extension of k, then the action of $\operatorname{Gal}(k'/k)$ on $\mathcal{F}_M(\operatorname{Spec} k')$ makes $\mathcal{F}_M(\operatorname{Spec} k') \xrightarrow{\simeq} M^{\operatorname{Gal}(k_s/k')}$ an isomorphism of $\operatorname{Gal}(k'/k)$-sets. Defining

$$\mathcal{F}_M\big(\coprod \operatorname{Spec} k_i\big) = \prod \mathcal{F}_M(\operatorname{Spec} k_i),$$

we have defined a *presheaf* of sets on $(\operatorname{Spec} k)_{\text{ét}}$. In fact, this presheaf is separated (i.e., $\mathcal{F}_M(X) \to \prod \mathcal{F}_M(X_i)$ is injective for every cover $\coprod X_i \to X$) since the maps (5.1.1.3) are injective.

Thus, to check the gluing axiom we can pass to a refinement covering, so we are reduced (by connectedness considerations) to the case of a covering $\operatorname{Spec} k'' \to \operatorname{Spec} k'$ with k''/k' Galois. We just need to show that the injection $\mathcal{F}_M(\operatorname{Spec} k') \hookrightarrow$

$\mathcal{F}_M(\operatorname{Spec} k'')$ is an isomorphism onto the $\operatorname{Gal}(k''/k')$-invariants. This is immediate from the way in which we defined $\mathcal{F}_M(\operatorname{Spec} k') \to \mathcal{F}_M(\operatorname{Spec} k'')$ and the fact that two k-embeddings $k'' \rightrightarrows k_s$ coincide on k' if and only if they are related by the action of some $g \in \operatorname{Gal}(k''/k')$.

We conclude that $\mathcal{F}_M$ is a sheaf on $(\operatorname{Spec} k)_{\text{ét}}$, and we have isomorphisms

$$\mathcal{F}_M(k') \simeq M^{\operatorname{Gal}(k_s/k')}$$

for all $k' \in \Sigma$, compatible with the action of $\operatorname{Gal}(k'/k)$ and the inclusions $k' \hookrightarrow k_s$ as subfields of k_s. We therefore have a canonical injection

$$M_{\mathcal{F}_M} \hookrightarrow M$$

compatible with with $\operatorname{Gal}(k_s/k)$-actions. In fact, this is an isomorphism since every element of M is fixed by some $\operatorname{Gal}(k_s/k')$ for some finite extension k' of k.

When $M = M_{\mathcal{F}}$, the above construction yields a $\operatorname{Gal}(k'/k)$-compatible bijection

$$\mathcal{F}_{M_{\mathcal{F}}}(\operatorname{Spec} k') \simeq M_{\mathcal{F}}(\operatorname{Spec} k') = \mathcal{F}(\operatorname{Spec} k') \tag{5.1.1.4}$$

for all $k' \in \Sigma$. These bijections are compatible with inclusions inside k_s by the sheaf axioms, so (5.1.1.4) uniquely extends to an isomorphism of sheaves $\mathcal{F}_{M_{\mathcal{F}}} \xrightarrow{\simeq} \mathcal{F}$. Thus, our constructions are indeed quasi-inverses. The compatibility with products is clear. □

5.1.2. Locally constant sheaves. If $X \to S$ is any map of schemes, it is a basic theorem of Grothendieck that the functor $\underline{X}(T) = \operatorname{Hom}_S(T, X)$ on S-schemes is an fpqc sheaf and hence is an étale sheaf. (Note that it may happen that $\underline{X}|_{S_{\text{ét}}} = \emptyset$ if X is not étale over S, such as $S = \operatorname{Spec} k$, $X = \operatorname{Spec} k'$ for k'/k a purely inseparable extension with degree > 1.) In particular, if $X \to S$ is finite étale of constant non-zero degree, then there is some étale cover $S' \to S$ such that $X' = X \times_S S' \to S'$ is *constant* (i.e., a disjoint union of copies of S' as an S'-scheme). Put differently, the étale sheaf $\underline{X}$ becomes a constant sheaf over a cover of S. The next theorem gives a more precise statement of this fact.

THEOREM 5.1.2.1. *The functor $X \rightsquigarrow \underline{X}$ from finite étale S-schemes of degree $n \geq 1$ to étale sheaves is fully faithful, with essential image consisting of étale sheaves $\mathcal{F}$ for which $\mathcal{F}|_{S'} \simeq \coprod_{1 \leq i \leq n} \underline{S'}$ for some étale surjection $S' \to S$.*

REMARK 5.1.2.2. The sheaves $\mathcal{F}$ arising in this way are called the *locally constant constructible sheaves* (of order n). Note that if $S = \operatorname{Spec} k$, then the discrete $\operatorname{Gal}(k_s/k)$-set associated to $\underline{X}$ is *exactly* $\underline{X}(k_s)$ with its *usual* $\operatorname{Gal}(k_s/k)$-action.

PROOF. The full faithfulness follows from Yoneda's lemma and the fact that X is an object in $S_{\text{ét}}$. Recall that any section to a separated étale map $Y \to Z$ splits off as a closed and open piece (since the diagonal $\Delta_{Y/Z}$ is an open and closed immersion). Since $X \times_S X \xrightarrow{p_2} X$ has a section $\Delta_{X/S}$, we can therefore write

$$X \times_S X = X \coprod X'$$

where $X' \to X$ has degree $n-1$. By induction we conclude that $X|_{S'} \simeq \coprod_{1 \leq i \leq n} \underline{S'}$ for some finite étale cover $S' \to S$.

Conversely, suppose that $\mathcal{F}$ is an étale sheaf with $\mathcal{F}|_{S'} \simeq \prod_{1 \leq i \leq n} \underline{S'}$ for some étale surjection $S' \to S$. To show that $\mathcal{F} \simeq \underline{X}$ for some finite étale $X \to S$ of degree n, we may work Zariski locally over S (by the full faithfulness which we have already proved). We thus may assume that S and S' are both affine, so $S' \to S$ is faithfully flat and quasi-compact. Since fpqc descent is effective for affine maps, we get an isomorphism $\mathcal{F} \simeq \underline{X}$ for some affine S-scheme $X \to S$ with $X \times_S S' \to S'$ finite étale of degree n. Since $S' \to S$ is an fpqc cover, it follows by descent that $X \to S$ is finite étale of degree n. $\square$

5.1.3. Images of étale sheaves. Let $f : S' \to S$ be a map of schemes. We define f_* as usual as a map from étale sheaves on S' to étale sheaves on S. The functor f_* in general has a left adjoint f^* with the usual properties, although the construction is more delicate than in the ordinary topology since objects can now have non-trivial automorphisms. More precisely, for any sheaf $\mathcal{F}$ on $S_{\text{ét}}$ we define the presheaf $f^{-1}\mathcal{F}$ on $S'_{\text{ét}}$ by

$$(f^{-1}\mathcal{F})(X') = \varinjlim \mathcal{F}(X)$$

where the direct limit is taken over *commutative* diagrams

(5.1.3.1)
$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

as $X \to S$ runs through a "cofinal" *set* of objects in $S_{\text{ét}}$ and the transition maps are by pullback. The sheafification of this presheaf is denoted be $f^*\mathcal{F}$, and there are obvious maps $f^*f_*\mathcal{F} \to \mathcal{F}$ and $\mathcal{F} \to f_*f^*\mathcal{F}$ making f^* and f_* adjoints in the usual way:

$$\mathrm{Hom}_{S'}(f^*\mathcal{G}, \mathcal{F}) \simeq \mathrm{Hom}_S(\mathcal{G}, f_*\mathcal{F})$$

(for sheaves of sets or for sheaves of abelian groups). The point is that any diagram (5.1.3.1) has $X' \to X$ factoring uniquely through $X \times_S S' \to X$, and everything follows easily from this.

For example, $f^*(\underline{X}) = \underline{X \times_S S'}$ naturally in f and X; this follows from the equality

$$\begin{aligned} \mathrm{Hom}_{S'}(f^*\underline{X}, \mathcal{F}) &= \mathrm{Hom}_S(\underline{X}, f_*\mathcal{F}) \\ &= (f_*\mathcal{F})(X) \\ &= \mathcal{F}(X \times_S S') \\ &= \mathrm{Hom}_{S'}(\underline{X \times_S S'}, \mathcal{F}). \end{aligned}$$

It is straightforward to directly construct maps $f^*g^* \to (g \circ f)^*$ compatible with triple composites and compatible with the identification $(fg)_* \xrightarrow{\simeq} f_*g_*$ via adjointness. In particular, this map $f^*g^* \to (g \circ f)^*$ is an isomorphism.

Let $k \to k'$ be an extension of fields with compatible choices of separable closures. Let $f : \mathrm{Spec}\, k' \to \mathrm{Spec}\, k$ be the natural map. It is clear that under Theorem 5.1.1.1 the operation f^* associates to a $\mathrm{Gal}(k_s/k)$-module the underlying discrete $\mathrm{Gal}(k'_s/k')$-module via the continuous map $\mathrm{Gal}(k'_s/k') \to \mathrm{Gal}(k_s/k)$.

COROLLARY 5.1.3.1. *Let R be a Dedekind domain and let $i : \mathrm{Spec}\, K \to \mathrm{Spec}\, R$ be the inclusion of the generic point. The functor $\mathcal{F} \rightsquigarrow i^*\mathcal{F}$ is an equivalence of categories from locally constant constructible sheaves of degree $n \geq 1$ over* $\mathrm{Spec}\, R$ *to discrete* $\mathrm{Gal}(K_s/K)$*-sets of order n which are unramified over all closed points of* $\mathrm{Spec}\, R$.

Proof. By Theorem 5.1.2.1 this amounts to showing that if $X_K \to \operatorname{Spec} K$ is finite étale with $X(K_s)$ unramified over $\operatorname{Spec} R$, then there is a finite étale R-scheme X with $X \times_{\operatorname{Spec} R} \operatorname{Spec} K \simeq X_K$ and such an X is unique up to unique isomorphism. Clearly X is normal, so it must be the normalization $\widetilde{X}_K$ of $\operatorname{Spec} R$ in X_K. Thus, the only question is whether or not $\widetilde{X}_k$ is étale over $\operatorname{Spec} R$.

For this we may assume that R is a discrete valuation ring and that X_K is connected. We're reduced to checking that for any finite separable extension L/K with $\operatorname{Hom}_K(L, K_s)$ unramified over R, the integral closure R_L of R in L (which is certainly finite flat over R) is étale over R. By hypothesis, R_L is unramified at all closed points over R, so the closed fiber of $\operatorname{Spec} R_L \to \operatorname{Spec} R$ is a finite union of finite separable extensions of the residue field of R. Thus $R \to R_L$ is étale , as desired. $\square$

As an example of crucial importance for the construction of $G_{\mathbf{Q}}$-actions as in Theorem 4.1.3.1, a locally constant constructible sheaf on $(\operatorname{Spec} \mathbf{Z}[1/N])_{\text{ét}}$ of degree n is the "same" as a discrete $G_{\mathbf{Q}}$-set of order n, unramified at all p not dividing N.

There is a more general concept of a *constructible sheaf* which is crucial for the *proofs* of many theorems in étale cohomology, as this often permits one to argue by localizing on the source of a map so as to exploit stratification by smooth schemes or induction on dimension. We will not discuss constructible sheaves here.

It is worth noting that "topological invariance" of the étale site with respect to nilpotent thickenings can be generalized. If $f : S' \to S$ is radiciel, integral and surjective (i.e., if it is a universal homeomorphism [**28**, 18.12.11]), then in fact the natural transformations $\mathrm{id} \to f_* f^*$ and $f^* f_* \to \mathrm{id}$ are isomorphisms, so f_* and f^* give *equivalences* between the categories $\mathrm{Et}(S')$ and $\mathrm{Et}(S)$ of étale sheaves of sets on S' and S. For example, if k' is a purely inseparable field extension of k and if X is a k-scheme, then $\mathrm{Et}(X)$ and $\mathrm{Et}(X \times_{\operatorname{Spec} k} \operatorname{Spec} k')$ are the "same" via the maps π_* and π^*, with $\pi : X \times_{\operatorname{Spec} k} \operatorname{Spec} k' \to X$ the projection (more specifically, the canonical adjointness maps $1 \to \pi_* \circ \pi^*$ and $\pi_* \circ \pi_* \to 1$ are isomorphisms which give the same identification of π_* with the "inverse" of π^*).

5.1.4. Stalks of étale sheaves.

Definition 5.1.4.1. A *geometric point* of S is a map $\overline{s} : \operatorname{Spec} k \to S$ with k a separably closed field. If $f : S \to S'$ is a map of schemes, we denote by $f(\overline{s})$ the geometric point $f \circ \overline{s}$ of S'.

Analogously to affine schemes for quasi-coherent sheaves, it is the *geometric points* which play the role of cohomologically trivial objects with respect to the cohomology of abelian étale sheaves (see Theorem 5.1.1.1).

For any map $x : \operatorname{Spec} k \to S$ and for any étale sheaf $\mathcal{F}$ on S, denote $x^*\mathcal{F}$ by $\mathcal{F}_x$. By Theorem 5.1.1.1 this can be interpreted as a discrete $\operatorname{Gal}(k_s/k)$-set, or even just as a set if x is a geometric point. We call $\mathcal{F}_x$ a *stalk* of $\mathcal{F}$ if x is a geometric point.

Due to the concrete sheaf-theoretic construction of the "image of a map", "quotient by an equivalence relation", and "extension by $\emptyset$ or 0" (for sheaves of sets or abelian groups, respectively), it follows formally that in both the category of sheaves of sets and the category of sheaves of abelian groups, monic maps are "subsheaf" inclusions, epic maps are "local surjections" and epic monomorphisms are isomorphisms, just as in classical sheaf theory.

Given a geometric point $\overline{s} : \operatorname{Spec} k \to S$, we define an *étale neighborhood* of $\overline{s}$ to be an étale map $U \to S$ together with a section $u : \operatorname{Spec} k \to U$ whose composite with $U \to S$ is $\overline{s}$:

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r, "u"] \arrow[dr, "\overline{s}"'] & U \arrow[d] \\ & S \end{tikzcd}$$

THEOREM 5.1.4.2.

1. *For $s \in S$ and $\overline{s} : \operatorname{Spec} \overline{k} \to S$ with $\overline{k}$ a separable closure of $k(s)$, the natural map*
$$\varinjlim_{(U,u)\to(S,\overline{s})} \mathcal{F}(U) \to \mathcal{F}_{\overline{s}} = \overline{s}^*\mathcal{F}(\operatorname{Spec} \overline{k})$$
is a bijection. Here the direct limit is taken over the category of étale neighborhoods of $\overline{s}$.
2. *Let Σ be a set of maps $s : \operatorname{Spec} k \to S$ whose images cover S. A map $\varphi : \mathcal{F} \to \mathcal{G}$ is epic (resp. monic) if and only if φ_s is for all $s \in \Sigma$, and the functor $\mathcal{F} \rightsquigarrow \mathcal{F}_s$ is of formation compatible with equalizer diagrams for $\varphi_1, \varphi_2 : \mathcal{F} \rightrightarrows \mathcal{G}$. That is, $\varphi_1 = \varphi_2$ if and only if $\varphi_{1,s} = \varphi_{2,s}$ for all $s \in \Sigma$. In particular, $\mathcal{F} = \emptyset$ (resp. $\mathcal{F} = \underline{0}$) if and only if this is true on stalks at all $s \in \Sigma$.*

PROOF. The special case of (2) where S is the spectrum of a field and s is a geometric point follows from Theorem 5.1.1.1 and the remarks preceding Corollary 5.1.3.1. For (2) in general we may suppose that Σ is a set of geometric points of the type in (1), in which case (2) is a formal consequence of (1) and the local description of monic and epic maps, because (1) clearly implies that $a, b \in \mathcal{F}(S)$ are equal in an étale neighborhood of s if and only if $a_{\overline{s}} = b_{\overline{s}}$ in $\mathcal{F}_{\overline{s}}$. Thus, we just have to prove (1).

As a special case of (1), when $S = \operatorname{Spec} R$ with R strictly Henselian with residue field $\overline{k}$, by [**28**, 18.5.11(c)] and [**28**, 18.5.15] the *only* separated étale R-schemes through which $\overline{s}$ factors are those of the form $\operatorname{Spec} R \coprod X'$ with $X' \to \operatorname{Spec} R$ étale separated. Thus, the presheaf $\overline{s}^{-1}\mathcal{F}$ satisfies $(\overline{s}^{-1}\mathcal{F})(\overline{k}) = \mathcal{F}(R)$. But $\overline{k}$ is separably closed, so that

$$(\overline{s}^*\mathcal{F})(\overline{k}) = (\overline{s}^{-1}\mathcal{F})(\overline{k}) = \mathcal{F}(R).$$

This makes (1) clear in this case.

In general, if $\overline{f} : \operatorname{Spec} \mathcal{O}^{\mathrm{sh}}_{S,\overline{s}} \to S$ is a strict Henselization of $\mathcal{O}_{S,\overline{s}}$ with respect to the separable closure $\overline{k}$ of $k(s)$, then $(\overline{f}, \overline{s})$ factors uniquely through any étale neighborhood (U, u) of $\overline{s}$. The reason is that for any local ring $(A, \mathfrak{m})$ with strict Henselization $(A^{\mathrm{sh}}, \mathfrak{m}^{\mathrm{sh}})$, for any étale A-algebra B with (necessarily closed) point $\mathfrak{n}$ over $\mathfrak{m}$, and for any map $h : B/\mathfrak{n} \to A^{\mathrm{sh}}/\mathfrak{m}^{\mathrm{sh}}$ over $A/\mathfrak{m}$, there is a *unique* local A-algebra map $B \to A^{\mathrm{sh}}$ lifting h [**28**, 18.8.4]. Thus, we can just as well consider the *canonical* map

$$\varinjlim_{(U,u)} \mathcal{F}(U) \to \overline{f}^*\mathcal{F}\big(\operatorname{Spec} \mathcal{O}^{\mathrm{sh}}_{S,\overline{s}}\big).$$

In view of the previous remarks about $(\overline{f}, \overline{s})$ uniquely factoring through étale neighborhoods, we see that the left side is *exactly* the group $(\overline{f}^{-1}\mathcal{F})(\mathcal{O}^{\mathrm{sh}}_{S,\overline{s}})$ for the *presheaf* $f^{-1}\mathcal{F}$.

We are thus reduced to showing that if $\mathcal{G}$ is a *presheaf* (of sets) on Spec R for a strictly Henselian ring R, and if $\widetilde{\mathcal{G}}$ is the sheafification of $\mathcal{G}$, then the map $\mathcal{G}(R) \to \widetilde{\mathcal{G}}$ is bijective. Since every separated (e.g., affine) étale neighborhood of a closed point contains Spec R as a closed and open piece, this assertion is clear in view of the construction of $\widetilde{\mathcal{G}}$. □

COROLLARY 5.1.4.3. *The functor f^* is exact on abelian sheaves. In particular, its right adjoint f_* takes injectives sheaves to injective sheaves.*

PROOF. For a geometric point $\overline{s} : \operatorname{Spec} \overline{k} \to X$, we have $(f^*\mathcal{F})_{\overline{s}} \simeq \mathcal{F}_{f\circ\overline{s}}$. The corollary follows immediately from this. □

Theorem 5.1.4.2 allows us to carry over many classical "stalk" arguments, such as the construction of enough injectives, the identification of higher direct images with "sheafified" cohomology, the Leray spectral sequence for higher direct images, and pullback maps in cohomology.

5.1.5. Finite maps. A very special case of proper base change theorem for étale cohomology is the following result.

LEMMA 5.1.5.1. *Let $f : X \to Y$ be a finite map, let $\mathcal{F}$ be an abelian sheaf on $X_{\text{ét}}$ and let $\overline{y}$ be a geometric point of Y such that $f^{-1}(\overline{y})$ has all residue fields equal to $k(\overline{y})$. Then the natural map*

$$(f_*\mathcal{F})_{\overline{y}} \to \prod_{\overline{x}\in f^{-1}(\overline{y})} \mathcal{F}_{\overline{x}}$$

is an isomorphism. In particular, the functor f_* is exact for finite maps f.

PROOF. See [**17**, I, 3.3]. □

This result ensures that finite maps are essentially "harmless" for the analysis of étale cohomology in general situations. Here is an interesting example of a finite map. Consider the map

$$f : \overbrace{\mathbf{P}^1_S \times_S \cdots \times_S \mathbf{P}^1_S}^{n} \to \mathbf{P}^n_S$$

which sends $[a_1, a_1'] \times \cdots \times [a_n, a_n']$ to the coefficients of the polynomial

$$\sum_{i=0}^{n} p_i(a, a')X^iY^{n-i} = (a_1X - a_1'Y)\cdots(a_nX - a_nY).$$

(This description works Zariski locally on S and then clearly globalizes.) This map (over any base) is proper and is visibly quasi-finite (as can be checked in the case of an algebraically closed base field, by unique factorization). Thus it is *finite*. By using the map $\mathcal{F} \to f_*f^*\mathcal{F}$ for a sheaf $\mathcal{F}$ on $\mathbf{P}^n_S$, general cohomological questions on $\mathbf{P}^n_S$ can sometimes be reduced to general cohomological questions on $\mathbf{P}^1_S \times_S \cdots \times_S \mathbf{P}^1_S$. This last scheme has the pleasant property that it factors as a composite of maps of relative dimension 1. This kind of trick (and variants on it) allows one to reduce some general problems about higher direct images to the case of relative dimension 1 (c.f. Theorem 5.2.5.1). It should be noted that there are much more powerful geometric methods for carrying out reduction arguments of this type.

5.2. Cohomology

We will mostly just state the main results and content ourselves with general remarks on some of the proofs. The proofs are quite difficult and are best formulated via derived categories. They also require deep results on formal schemes and excellent rings from EGA.

5.2.1. The étale fundamental group. We first give the basic cohomological property of the étale fundamental group. Let S be a connected scheme and let $\overline{s}$ be a geometric point of S. Let $S' \to S$ be a finite étale map with S' connected, and let $\overline{s}'$ be a geometric point of S' over $\overline{s}$. For simplicity we assume that $\overline{s}$ and $\overline{s}'$ have the same residue field. Note that the map $S' \to S$ has constant degree.

Any $f \in \operatorname{Aut}(S'/S)$ is entirely determined by $f(\overline{s}') = f \circ \overline{s}'$. Indeed, let $f, g : S' \rightrightarrows S$ be two maps and assume that $f(\overline{s}') = g(\overline{s}')$, The diagonal $\Delta_{S'/S} : S' \to S' \times_S S'$ is an open and closed immersion, so $S' \times_S S'$ can be written as $\Delta \coprod S''$ with $\Delta \simeq S'$ the diagonal. Since S' is connected and the map $f \times g : S' \to S' \times_S S'$ has image hitting Δ by assumption, the entire image lies in Δ by connectedness. Thus, $f = g$, as claimed.

If we denote the degree of the map $S' \to S$ by n, it follows that $\operatorname{Aut}(S'/S) \leq n$ since there are exactly n geometric points in the fiber over $\overline{s}$. A generalization of the classical method of constructing Galois closures [**23**, Exp V, §2–4] shows that we can always find a finite étale map $S'' \to S'$ with S'' connected and $\#\operatorname{Aut}(S''/S) = \deg(S''/S)$. We call such an $S'' \to S$ *Galois*, and $\operatorname{Aut}(S''/S)^{\mathrm{op}}$ is called the *Galois group*; it acts simply transitively on the fiber over $\overline{s}$.

By Theorem 5.1.1.1 and Theorem 5.1.2.1, this is all compatible with Galois theory. If $(S', \overline{s}') \to (S, \overline{s})$ and $(S'', \overline{s}'') \to (S, \overline{s})$ are two such "pointed" connected Galois covers, then by the same reasoning as above there is at most one S-map

$$(S'', \overline{s}'') \to (S', \overline{s}')$$

taking $\overline{s}''$ to $\overline{s}'$. When such a map exists, for *any* $f'' \in \operatorname{Aut}(S''/S)$ there exists a unique $f' \in \operatorname{Aut}(S'/S)$ fitting into a commutative diagram

$$\begin{tikzcd} S'' \arrow[r, "f''"] \arrow[d, "\pi"'] & S'' \arrow[d, "\pi"] \\ S' \arrow[r, "f'"'] & S' \end{tikzcd}$$

Indeed, $\pi \circ f''(\overline{s}'') \in S'_{\overline{s}'}$ has the form $f'(\overline{s}')$ for a unique $f' \in \operatorname{Aut}(S'/S)$ by the transitivity of Galois group actions on fibers, so $(f \circ \pi)(\overline{s}'') = (\pi \circ f'')(\overline{s}'')$. Thus, $f\pi = \pi f''$.

Given this, we can now introduce the fundamental

Definition 5.2.1.1. The *étale fundamental group* of the connected scheme S (with respect to the geometric base point $\overline{s}$) is the profinite group

$$\pi_1(S, \overline{s}) = \varprojlim_{(S', \overline{s}')} \operatorname{Aut}(S'/S)^{\mathrm{op}}.$$

This is visibly *covariant* in the pair $(S, \overline{s})$.

Note that if $S = \operatorname{Spec} k$, then a connected étale cover of S is just a finite separable extension. Thus, there is a canonical isomorphism of profinite groups

$\pi_1(S, \overline{s}) \simeq \mathrm{Gal}(k_s/k)$ via the choice of a geometric point $\overline{s} : \operatorname{Spec} k_s \to \operatorname{Spec} k$. Moreover, if

$$\begin{tikzcd} k_s \arrow[r] & k'_s \\ k \arrow[u] \arrow[r] & k' \arrow[u] \end{tikzcd}$$

is a commutative diagram, the diagram

$$\begin{tikzcd} \pi_1(S, \overline{s}) \arrow[d, equal] & \pi_1(S', \overline{s}') \arrow[l] \arrow[d, equal] \\ \mathrm{Gal}(k_s/k) & \mathrm{Gal}(k'_s/k') \arrow[l] \end{tikzcd}$$

commutes, where the top row is induced by covariant functoriality and the bottom row is the classical map.

The starting point of the theory of the étale fundamental group is the following theorem of Grothendieck.

THEOREM 5.2.1.2. *There is a canonical equivalence of categories between (possibly disconnected) isomorphism classes of finite étale $S' \to S$ and finite discrete left $\pi_1(S, \overline{s})$-sets, given by $S' \rightsquigarrow S'_{\overline{s}}$. This equivalence is functorial in $(S, \overline{s})$.*

PROOF. One first checks that S' is the union of finitely many connected components, so we can reduce to the case of a connected S'. Consider a connected Galois $S'' \to S$ with factorization

$$S'' \xrightarrow{\pi} S' \to S,$$

so $S'' \to S'$ is Galois as well. Choose a geometric point $\overline{s}'' \in S''$ over $\overline{s}$ in order to *uniquely* define a continuous surjection $\pi_1(S, \overline{s}) \twoheadrightarrow \mathrm{Aut}(S''/S)^{\mathrm{op}}$. Define a *map* $\mathrm{Aut}(S''/S)^{\mathrm{op}} \to S'_{\overline{s}}$ by sending g to $\pi \circ g(\overline{s}'')$. If $\pi \circ h \circ g(\overline{s}'') = \pi \circ g(\overline{s}'')$, then $\pi, \pi \circ h : S'' \rightrightarrows S'$ are S-maps which coincide on $g(\overline{s}'')$, so they must be equal.

That is, we have a bijection of sets

$$\mathrm{Aut}(S''/S)^{\mathrm{op}} / \mathrm{Aut}(S''/S')^{\mathrm{op}} \xrightarrow{\simeq} S'_{\overline{s}}.$$

One then checks that the induced $\pi_1(S, \overline{s})$-set structure on $S'_{\overline{s}}$ is independent of the choice of $(S'', \overline{s}'')$ and that any $\pi_1(S, \overline{s})$-set with transitive action arises in this way. See [**23**, Exp V] for details. The functoriality in $(S, \overline{s})$ is easy. ☐

5.2.2. Étale cohomology groups. Let X be a scheme. Analogously to cohomology on ordinary ringed spaces, we define the *étale cohomology functors* $\mathrm{H}^{\bullet}_{\text{ét}}(X, \cdot)$ of X to be the right derived functors of the left exact global sections functor

$$\mathcal{F} \rightsquigarrow \mathcal{F}(X)$$

from étale abelian sheaves on X to abelian groups. If $f : X \to Y$ is a map of schemes, we define the *higher direct images* $\mathrm{R}^{\bullet} f_*$ of f to be the right derived functors of the left exact pushforward functor

$$\mathcal{F} \rightsquigarrow f_* \mathcal{F}$$

from étale abelian sheaves on X to étale abelian sheaves on Y.

As a first interpretation of these functors, one can use Theorem 5.1.2.1 and Čech techniques to show the next result result.

THEOREM 5.2.2.1. [**17**, I, 2.11] *Let G be a finite abelian group and let X be a connected scheme with geometric point $\overline{x}$. There is a canonical bijection between $\mathrm{H}^1_{\text{ét}}(X, \underline{G})$ and the set $\mathrm{Hom}_{\text{cont}}(\pi_1(X, \overline{x}), G)$ of isomorphism classes of (possibly disconnected) finite étale covers $X' \to X$ which are G-torsors. This correspondence is functorial in X.*

Let X be a connected scheme, $x \in X$ is a point, and $\overline{x} : \operatorname{Spec} k(x)_s \to X$ a geometric point over x. For any étale sheaf $\mathcal{F}$ on X, the "stalk" $\mathcal{F}_x = i_x^*\mathcal{F}$ at x is an étale sheaf on $(\operatorname{Spec} k(x))_{\text{ét}}$, and thus is a $\mathrm{Gal}(k(\overline{x})/k(x))$-module via the choice of $\overline{x}$. If $\mathcal{F}$ is a locally constant constructible sheaf on X, then the pointed map

$$(x, \overline{x}) \to (X, \overline{x})$$

induces a continuous map of profinite groups

$$\mathrm{Gal}\big(k(\overline{x})/k(x)\big) = \pi_1(x, \overline{x}) \to \pi_1(X, \overline{x}).$$

Thus, the finite discrete $\pi_1(X, \overline{x})$-set $\mathcal{F}_{\overline{x}}$ is canonically identified with the finite discrete $\pi_1(x, \overline{x})$-set $(\mathcal{F}_x)_{\overline{x}}$. This makes explicit the functoriality in Theorem 5.2.1.2.

For example, suppose that R is a Dedekind ring with fraction field K, and let $x \in X = \operatorname{Spec} R$ be a closed point. Choose a geometric point $\overline{x}$ over x and let R_x^{h} denote the Henselization of the local ring of R at x. By [**28**, 18.5.15], every finite étale cover of $\operatorname{Spec} k(x)$ uniquely lifts to a finite étale cover of the henselian local ring R_x^{h}. Equivalently, the map $\operatorname{Spec} k(x) \to \operatorname{Spec} R_x^{\mathrm{h}}$ induces an isomorphism $\pi_1(X, \overline{x}) \xrightarrow{\simeq} \pi_1(\operatorname{Spec} R_x^{\mathrm{h}}, \overline{x})$. The map

$$\pi_1(\operatorname{Spec} R_x^h, \overline{x}) \to \pi_1(\operatorname{Spec} R_x, \overline{x})$$

is just a choice of "decomposition group" at x in the maximal quotient of $\mathrm{Gal}(K_s/K)$ which is unramified over x (here we are using Corollary 5.1.3.1 and Theorem 5.2.1.2).

In particular, let M be a finite discrete $\mathrm{Gal}(K_s/K)$-set and let $\mathcal{F}_K$ be the associated étale sheaf on $(\operatorname{Spec} K)_{\text{ét}}$. Assume that the Galois action is unramified, with $\mathcal{F}$ the unique associated locally constant constructible sheaf over $(\operatorname{Spec} R)_{\text{ét}}$ via Corollary 5.1.3.1. Then for any closed point $x \in X = \operatorname{Spec} R$, the $\pi_1(X, \overline{x})$-module $\mathcal{F}_x$ is *canonically* identified with the $\mathrm{Gal}(K_s/K)$-set M (which is unramified at x) once we *choose* a strict Henselization R_x^{sh} inside of K_s, or equivalently once we choose a Henselization R_x^h inside of K_s and a separable closure $k(\overline{x})$ of $k(x)$; this data gives us a compatibility of geometric points on R and K, as in the commutative diagram

$$\begin{tikzcd}
 & \mathrm{Spec}(K_s) \arrow[r] \arrow[d] & \mathrm{Spec}(K) \arrow[d] \\
\mathrm{Spec}(k(\overline{x})) \arrow[r] & \mathrm{Spec}(R_x^{\mathrm{h}}) \arrow[r] & \mathrm{Spec}(R)
\end{tikzcd}$$

For example, given a finite discrete $G_{\mathbf{Q}}$-set unramified at p and a choice of decomposition group $D_p \hookrightarrow G_{\mathbf{Q}}$ at p, we can identify the action of $\mathrm{Frob}_p \in D_p/I_p \simeq \mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p)$ on a $G_{\mathbf{Q}}$-module M unramified at p with the Frob_p-action on a locally constant constructible sheaf on $\mathrm{Spec}(\mathbf{Z}_{(p)}^{\mathrm{h}})$ (or equivalently, on $\mathrm{Spec}(\mathbf{F}_p)$).

One consequence of this type of reasoning with closed fibers over henselian local rings is the following rather deep fact which underlies the powerful base change theorems in étale cohomology.

COROLLARY 5.2.2.2. *Let $X \to \operatorname{Spec} R$ be a proper map, with R a Henselian noetherian local ring with closed point s. Then the map*

$$\mathrm{H}^i_{\text{ét}}(X, \underline{\mathbf{Z}/n}) \to \mathrm{H}^i_{\text{ét}}(X_s, \underline{\mathbf{Z}/n})$$

is an isomorphism for all n and for $i = 0, 1$.

PROOF. Since an element of

$$\mathrm{H}^0_{\text{ét}}(X, \underline{\mathbf{Z}/n}) = \mathrm{Hom}_{\operatorname{Spec} R}\big(X, \textstyle\coprod_{j \in \mathbf{Z}/n} \operatorname{Spec} R\big)$$

amounts to an assignment of $j \in \mathbf{Z}/n$ to each (closed and open) connected component of X, for $i = 0$ this just says that X and X_s have the same number of connected components. Thus, we need to show that the primitive idempotents of $\mathrm{H}^0_{\text{ét}}(X, \mathcal{O}_X) \simeq \mathrm{H}^0_{\text{ét}}(\operatorname{Spec} R, f_*\mathcal{O}_X)$ and $\mathrm{H}^0_{\text{ét}}(X_s, \mathcal{O}_{X_s})$ correspond under "reduction". Since $\mathrm{H}^0_{\text{ét}}(R, f_*\mathcal{O}_X)$ is *finite* over the Henselian ring R, it is a finite product of local R-algebras, so its primitive idempotents and maximal ideals are in natural bijection. By [**26**, 4.3.3], these maximal ideals are in natural bijection with the connected components of X_s. This settles the case $i = 0$. In particular, to check the case $i = 1$ we may assume that X and X_s are connected.

For $i = 1$, by Theorem 5.2.2.1 (including the functoriality) we want to show that every $\mathbf{Z}/n$-Galois cover of X_s is induced by a unique $\mathbf{Z}/n$-Galois cover of X. Since finite X-schemes can be identified with coherent $\mathcal{O}_X$-algebras, Grothendieck's existence theorem for proper formal schemes [**26**, 5.1.4] implies that any $\mathbf{Z}/n$-Galois cover of X_s lifts uniquely to a $\mathbf{Z}/n$-Galois cover of $X \times_{\operatorname{Spec} R} \operatorname{Spec} \widehat{R}$, where $\widehat{R}$ is the completion of R. (The existence of the unique such liftings for each $X \times_{\operatorname{Spec} R} \operatorname{Spec} R/\mathfrak{m}^n$ follow from the topological invariance of the étale site [**28**, 18.1.2].) See [**28**, 18.3.4] for details.

This already ensures uniqueness, since we claim that any two $\mathbf{Z}/n$-Galois covers of X which are isomorphic as such over $X \times_{\operatorname{Spec} R} \operatorname{Spec} \widehat{R}$ must be isomorphic in the first place. Equivalently, by Theorem 5.2.2.1, we are asserting that the map

$$\mathrm{Hom}_{\mathrm{cont}}(\pi_1(X), \mathbf{Z}/n) \to \mathrm{Hom}_{\mathrm{cont}}(\pi_1(X \times_{\operatorname{Spec} R} \operatorname{Spec} \widehat{R}), \mathbf{Z}/n) \tag{5.2.2.1}$$

is *injective* (for compatible choices of base points). Here we are implicitly using the fact that $X \times_{\operatorname{Spec} R} \operatorname{Spec} \widehat{R}$ is connected, which follows from the already treated case $i = 0$ for the Henselian ring $\widehat{R}$ (and the fact that $X \times_{\operatorname{Spec} R} \operatorname{Spec} \widehat{R}$ has closed fiber X_s, which is connected by assumption).

To see the injectivity of (5.2.2.1), it suffices to show that the map

$$\pi_1(X \times_{\operatorname{Spec} R} \operatorname{Spec} \widehat{R}) \to \pi_1(X)$$

is *surjective*. By Theorem 5.2.1.2, this amounts to checking that if $Y \to X$ is a *connected* finite étale cover, then $Y \times_X (X \times_{\operatorname{Spec} R} \operatorname{Spec} \widehat{R}) = Y \times_{\operatorname{Spec} R} \operatorname{Spec} \widehat{R}$, which is a finite étale cover of $X \times_{\operatorname{Spec} R} \operatorname{Spec} \widehat{R}$, is still connected. For this we may apply the $i = 0$ case to $Y \to \operatorname{Spec} R$ to deduce that Y_s is connected, and then applying the $i = 0$ case to $Y \times_{\operatorname{Spec} R} \operatorname{Spec} \widehat{R}$ (as with $X \times_{\operatorname{Spec} R} \operatorname{Spec} \widehat{R}$ above) we conclude that $Y \times_{\operatorname{Spec} R} \operatorname{Spec} \widehat{R}$ is connected.

Now that we have uniqueness, we need to check the *existence* of liftings of Galois covers. Since R is a direct limit of Henselizations of $\mathbf{Z}$-algebras essentially of finite type (i.e., localizations of finite type $\mathbf{Z}$-algebras), it suffices to assume that R is the Henselization of a $\mathbf{Z}$-algebra essentially of finite type. That is, we have a proper $X \to \operatorname{Spec} R$ over such an R and a $\mathbf{Z}/n$-Galois cover $Y_s \to X_s$, and we want to lift this to a $\mathbf{Z}/n$-Galois cover of X. The argument with formal schemes above gives us a $\mathbf{Z}/n$-Galois cover $\widehat{Y}$ over $X \times_{\operatorname{Spec} R} \operatorname{Spec} \widehat{R}$. Thus, given such a cover we wish to find a $\mathbf{Z}/n$-Galois cover $Y \to X$ with the same closed fiber over R.

By direct limit arguments, there exists a finite type R-algebra A, an R-algebra map $g : A \to \widehat{R}$, and a $\mathbf{Z}/n$-Galois cover $Y_A \to X \times_{\operatorname{Spec} R} \operatorname{Spec} A$ whose base change to $\operatorname{Spec} \widehat{R}$ is $\widehat{Y}$. Writing

$$A = R[T_1, \dots, T_n]/(f_1, \dots, f_m),$$

the map $g : A \to \widehat{R}$ corresponds to a set of solutions of the polynomial equations $f_1 = \cdots = f_m = 0$ in $\widehat{R}$. If we can find a solution to this system in R which induces the same solution in the residue field, then we get a map $s : A \to R$ over R with $A \to R \to \widehat{R}$ and $g : A \to \widehat{R}$ induced the *same* composite on the residue field. Thus $Y_A \times_{\operatorname{Spec} A} \operatorname{Spec} R$ would be the desired $\mathbf{Z}/n$-cover of X, realizing the closed fiber cover Y_s.

Thus, we want to show that for a ring R which is the Henselization of an algebra essentially of finite type over a field or an excellent Dedekind domain (e.g, $\mathbf{Z}$) and for a fixed $n \geq 1$, any solution in $\widehat{R}$ to a finite system of polynomials over R admits a solution in R with the same image in $R/\mathfrak{m}^n \simeq \widehat{R}/\mathfrak{m}^n$. This is *exactly* the Artin approximation theorem [**2**, 3.6/16]; the above argument of Artin is a massive simplification of the original proof of Corollary 5.2.2.2, although the Artin approximation theorem itself is far from trivial. $\square$

5.2.3. The Kummer and Artin-Schreier sequences. Note that by fpqc descent of quasi-coherent sheaves, for any quasi-coherent sheaf $\mathcal{F}$ on S, the presheaf $\mathcal{F}_{\text{ét}}$ on $S_{\text{ét}}$ defined on $\pi : S' \to S$ by

$$\mathcal{F}_{\text{ét}}(S') = \mathrm{H}^0_{\text{zar}}(S', \pi^*\mathcal{F})$$

is a *sheaf*. In fact, using Čech methods one can show that there is a natural isomorphism

$$\mathrm{H}^i_{\text{zar}}(S, \mathcal{F}) \xrightarrow{\simeq} \mathrm{H}^i_{\text{ét}}(S, \mathcal{F}_{\text{ét}});$$

[**17**, Ch I, 1.13(2), 2.9]. For $\mathcal{F} = \mathcal{O}_X$, the étale sheaf $\mathcal{O}_{X,\text{ét}}$ is the sheaf of groups $\mathbf{G}_a$ "represented" by the smooth (but not étale) group scheme $\mathbf{G}_{a/S}$ in the sense discussed at the beginning of §5.1.2, and the multiplicative subsheaf of units $\mathbf{G}_m$ ("represented" by the smooth group scheme $\mathbf{G}_{m/S}$) is likewise defined by

$$\mathbf{G}_m(S') = \mathrm{H}^0_{\text{zar}}(S', \mathcal{O}^\times_{S'}).$$

If $n \in \mathrm{H}^0(S, \mathcal{O}^\times_S)$, then for any S-scheme $\operatorname{Spec} A$ and for any $f \in A^\times$, the finite faithfully flat A-algebra $A[X]/(X^n - f)$ is *etale* over A. It follows that the *Kummer sequence*

$$0 \to \mu_n \to \mathbf{G}_m \xrightarrow{n} \mathbf{G}_m \to 0$$

is not only left exact (as is obvious on points), but it even exact for the *étale* topology. Indeed, for any affine étale S-scheme $\operatorname{Spec}(A)$ and any $f \in \mathbf{G}_m(A)$, f is an nth power in $\mathbf{G}_m(A')$ for the étale cover

$$\operatorname{Spec}(A') = \operatorname{Spec}(A[T]/(T^n - f) \to \operatorname{Spec}(A).$$

Similarly, if $p = 0$ on S, then for any S-scheme $\operatorname{Spec} A$ and for any $f \in A$, the finite faithfully flat A-algebra $A[X]/(X^p - X - f)$ is *étale* over A, so the

Artin-Schreier sequence

$$0 \to \underline{\mathbf{Z}/p} \to \mathbf{G}_a \xrightarrow{t^p - t} \mathbf{G}_a \to 0$$

is an exact sequence of étale sheaves.

Note that if $n \notin \mathrm{H}^0(S, \mathcal{O}_S^\times)$, then the map $n : \mathbf{G}_m \to \mathbf{G}_m$ of étale sheaves is *not* surjective in general. Beware that since $\mathbf{G}_m$ and $\mathbf{G}_a$ are *not* represented by objects in $S_{\text{ét}}$, the pullback of these sheaves under a map $f : S' \to S$ need *not* be $\mathbf{G}_{m/S'}$ and $\mathbf{G}_{a/S'}$ respectively. That is, the canonical maps $f^*\mathbf{G}_{m/S} \to \mathbf{G}_{m/S'}$ and $f^*\mathbf{G}_{a/S} \to \mathbf{G}_{a/S'}$ need *not* be isomorphisms. (Of course, if f is étale then these are isomorphisms.) For example, if $S = \operatorname{Spec} k$ with k separably closed, then $f^*\mathbf{G}_{m/S}$ is the *constant* sheaf attached to the group $k^\times$.

5.2.4. Cohomology of curves. The Kummer and Artin-Schreier sequences play a crucial role in the basic cohomological calculations for curves, upon which the whole theory rests. For example, we have the next result.

THEOREM 5.2.4.1. *Let X be a proper scheme of dimension ≤ 1 over a separably closed field k. For any torsion sheaf $\mathcal{F}$ on X, the groups $\mathrm{H}^i_{\text{ét}}(X, \mathcal{F})$ vanish for $i > 2$. If $\mathcal{F}$ is a locally constant constructible sheaf, then $\mathrm{H}^i_{\text{ét}}(X, \mathcal{F})$ is finite for $i \leq 2$.*

A *torsion* (abelian) sheaf $\mathcal{F}$ is one whose sections are locally killed by a non-zero integer.

PROOF. See [**17**, I, §5] for details, although this reference assumes that the torsion orders are relatively prime to the characteristic of k (which is all we will need). One reduces to the case k is algebraically closed (via the remarks at the end of §5.1.3), X is smooth and $\mathcal{F} = \underline{\mathbf{Z}/l}$ for a prime l. Since $\mu_l \simeq \underline{\mathbf{Z}/l}$ when l does not divide characteristic of k, the Kummer and Artin-Schreier sequences reduce us to statements about $\mathrm{H}^i_{\text{ét}}(X, \mathbf{G}_m)$ for $l \neq \operatorname{char}(k)$ and $\mathrm{H}^i_{\text{ét}}(X, \mathcal{O}_X) = \mathrm{H}^i_{\text{zar}}(X, \mathcal{O}_X)$ for $l = \operatorname{char}(k)$.

The vanishing of $\mathrm{H}^i_{\text{ét}}(X, \mathbf{G}_m)$ for $i \geq 2$ rests upon the Leray spectral sequence and Tsen's Theorem that the C_1 property holds for function fields K of transcendence degree 1 over algebraically closed fields k. Basic Galois cohomology results about C_1 fields in [**54**, Ch I, Prop 13] and [**54**, Ch II, Prop 5 and 8] then imply that *all* such fields K/k have vanishing Galois cohomology in degrees greater than 2 and that $\mathrm{H}^2(K, \mathbf{G}_m)$ vanishes for such fields as well.

By the effectivity of étale descent for line bundles, we have a canonical isomorphism

$$\mathrm{H}^1_{\text{ét}}(X, \mathbf{G}_m) \simeq \check{\mathrm{H}}^1(X, \mathbf{G}_m) \simeq \operatorname{Pic}(X)$$

whose n-torsion coincides with $\operatorname{Pic}^0_{X/k}[n]$. This is *finite* by theory of abelian varieties!

The vanishing of $\mathrm{H}^i_{\text{ét}}(X, \mathcal{O}_X)$ for $i \geq 2$ is just the usual vanishing of higher coherent sheaf cohomology on a curve. Also, $\mathrm{H}^1_{\text{ét}}(X, \underline{\mathbf{Z}/p})$ for $p = \operatorname{char}(k)$ is the kernel of $\phi - 1$ on $\mathrm{H}^1_{\text{ét}}(X, \mathcal{O}_X)$, where ϕ is the Frob_k-semilinear endomorphism of $\mathrm{H}^1_{\text{ét}}(X, \mathcal{O}_X)$ induced by the map $t \mapsto t^p$ on $\mathcal{O}_X$. By [**45**, p. 143], $\phi - 1$ has *finite* kernel with $\mathbf{F}_p$-dimension at most $\dim_k \mathrm{H}^1_{\text{ét}}(X, \mathcal{O}_X)$. (In fact, this kernel can be identified with the dual abelian group to $\operatorname{Jac}_{X/k}[p](\overline{\mathbf{F}}_p) \simeq \operatorname{Pic}^0_{X/k}[p](\overline{\mathbf{F}}_p)$.) $\square$

5.2.5. Proper base change. The preceding results, together with the rather precise description of the cohomology of proper smooth curves, provide some of the essential ingredients in the proof of the following algebraic analogue of the relatively straightforward Theorem 1.2.1.1. The proof is extremely difficult; see [**17**, Ch I, 6.1].

THEOREM 5.2.5.1 (Proper base change). *Let S be a locally noetherian scheme, let $f : X \to S$ be a proper map, and let $\mathcal{F}$ be a torsion étale abelian sheaf on X. For any cartesian square*

$$\begin{tikzcd} X_T \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ T \arrow[r, "g"'] & S \end{tikzcd}$$

the natural base change map

$$g^*\mathrm{R}^i f_*\mathcal{F} \to \mathrm{R}^i f'_*(g'^*\mathcal{F})$$

is an isomorphism.

PROOF. We will just make some minor remarks about the proof. Using direct limit techniques one can reduce to the case where $S = \mathrm{Spec}\, R$ for a strictly Henselian ring R, $g : T \to S$ is the closed point $\{s\}$ and $\mathcal{F}$ is "almost" $\underline{\mathbf{Z}/n}$. (More precisely, $\mathcal{F}$ is constructible.) One needs to show that the map

$$\mathrm{H}^i_{\text{ét}}(X, \mathcal{F}) \to \mathrm{H}^i_{\text{ét}}(X_s, \mathcal{F}_s)$$

is an isomorphism in this case.

When $\mathcal{F} = \underline{\mathbf{Z}/n}$ and $i \leq 1$, this assertion is *exactly* Corollary 5.2.2.2. This is the "hard part" of the proof. The general case builds up from this by general arguments, exploiting the compatibility of the *general* case with respect to composites of cartesian squares. For example, using Chow's lemma, Lemma 5.1.5.1 and the example following Lemma 5.1.5.1, one can reduce the problem to the case where the fibers of f have dimension at most 1. The details are rather technical. □

5.2.6. Smooth base change. Theorem 5.2.5.1 is a crucial ingredient in the even more difficult smooth base change theorem.

THEOREM 5.2.6.1 (Smooth base change). *Let S be a locally noetherian scheme, let $f : X \to S$ be quasi-compact quasi-separated, and let $\mathcal{F}$ be a torsion étale sheaf on X with torsion orders invertible on S. Let $g : T \to S$ be a smooth map. Then for the cartesian square*

$$\begin{tikzcd} X_T \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ T \arrow[r, "g"'] & S \end{tikzcd}$$

the natural base change map

$$g^*\mathrm{R}^i f_*\mathcal{F} \to \mathrm{R}^i f'_*(g'^*\mathcal{F})$$

is an isomorphism.

REMARK 5.2.6.2. 1. This is the analogue of the (comparatively trivial) homotopy invariance of singular cohomology in classical topology, corresponding to the case $T = S \times [0,1]$ as in the discussion in the latter part of 1.7.3.

2. Using direct limit methods, the same result holds when g is a limit of smooth maps with affine transition maps. In particular, this applies to a separable algebraic extension or a purely transcendental extension $K \to K'$ of fields. Combining this with the remarks preceding the definition of a geometric point, we conclude that the étale cohomology *groups* of a torsion sheaf on a quasi-compact quasi-separated scheme X over a separably closed field k are *canonically* unaffected by extension of separably closed base field, at least when the torsion orders of the sheaf are relatively prime to the characteristic. The most notable example for our purposes is the extension $\overline{\mathbf{Q}} \to \mathbf{C}$ which we will need later on.
3. The hypothesis on torsion orders in Theorem 5.2.6.1 is crucial. For example, if $X = \operatorname{Spec} A$ is an *affine* scheme over $\mathbf{F}_p$, then by the Artin-Schreier sequence we have

$$\mathrm{H}^1_{\text{ét}}(\operatorname{Spec} A, \underline{\mathbf{Z}/p}) \simeq A/\{a^p - a \mid a \in A\},$$

naturally in A. If $\operatorname{Spec} A$ is a scheme over a separably closed field k, then smooth base change for $\operatorname{Spec} A \to \operatorname{Spec} k$ would say that for any separably closed extension k'/k and $A' = k' \otimes_k A$, the map

$$A/\{a^p - a \mid a \in A\} \to A'/\{a'^p - a' \mid a' \in A'\}$$

is bijective. This is very false in general.

Note that this does not contradict *proper* base change since the map $\operatorname{Spec} A \to \operatorname{Spec} k$ is rarely proper. The role of the torsion condition in the proof is that when $\mathcal{F} = \underline{\mathbf{Z}/n}$ and $S = \operatorname{Spec} R$ for a strictly Henselian ring R *with* $n \in R^\times$, then $\mathcal{F} \simeq \mu_n$. Thus, one can use the Kummer sequence to reformulate things in terms of line bundles and nth powers in rings.

PROOF. We again make only some minor comments on the proof. Using spectral sequences and direct limit methods, one reduces to the case of an affine S of finite type over $\operatorname{Spec} \mathbf{Z}$ and an affine finite type map f. The theorem is compatible with "composites" of cartesian squares, so by factoring through projective space and using *proper base change*, one reduces to the crucial case where f is an open immersion. The proof of this case is quite technical and requires deep facts from commutative algebra, such as Grothendieck's theory of excellent rings (e.g., to know that finitely generated $\mathbf{Z}$-algebras A have *open* regular locus).

□

Let $f : X \to \operatorname{Spec} K$ be a quasi-compact quasi-separated scheme over a field K, let K_s/K be a separable closure, and let $\mathcal{F}$ be a torsion étale sheaf on X with torsion orders prime to $\operatorname{char}(K)$. Using (2) above, Theorem 5.1.1.1, and the comments preceding Corollary 5.1.3.1, we see that the "underlying" abelian group of the étale sheaf

$$\mathrm{R}^i f_*(\mathcal{F})_{K_s}$$

on $(\operatorname{Spec} K_s)_{\text{ét}}$ is *canonically* identified with

$$\mathrm{H}^i_{\text{ét}}\big(X \times_{\operatorname{Spec} K} \operatorname{Spec} K_s, \mathcal{F}_{K_s}\big). \tag{5.2.6.1}$$

Note also that for $g \in \mathrm{Gal}(K_s/K)$, we have a canonical *left* action on (5.2.6.1) via

(5.2.6.2)

$$\mathrm{H}^i_{\text{ét}}\big(X \times_{\operatorname{Spec} K} \operatorname{Spec} K_s, \mathcal{F}_{K_s}\big) \overset{(1\times g)^*}{\longrightarrow} \mathrm{H}^i_{\text{ét}}\big(X \times_{\operatorname{Spec} K} \operatorname{Spec} K_s, (1 \times g)^*\mathcal{F}_{K_s}\big)$$
$$\simeq \mathrm{H}^i_{\text{ét}}\big(X \times_{\operatorname{Spec} K} \operatorname{Spec} K_s, \mathcal{F}_{K_s}\big).$$

This is because the diagram

$$\begin{tikzcd} X \times_{\operatorname{Spec} K} \operatorname{Spec} K_s \arrow[rr, "1\otimes g"] \arrow[dr] & & X \times_{\operatorname{Spec} K} \operatorname{Spec} K_s \arrow[dl] \\ & X & \end{tikzcd}$$

commutes.

We claim that the $\mathrm{Gal}(K_s/K)$-action on the group $\mathrm{H}^i_{\text{ét}}(X \times_{\operatorname{Spec} K} \operatorname{Spec} K_s, \mathcal{F}_{K_s})$ as in (5.2.6.2) is *exactly* the same as that coming from the structure of $\mathrm{R}^i f_*\mathcal{F}$ as an étale sheaf on $\operatorname{Spec} K$. In particular, the description (5.2.6.2) really is *continuous* with respect to the discrete topology. The ability to "package" this $\mathrm{Gal}(K_s/K)$-action intrinsically in terms of the étale sheaf $\mathrm{R}^i f_*\mathcal{F}$ (without having to mention K_s or work with K_s-schemes) will be *very* useful later on.

To verify this compatibility we can use Theorem 5.1.4.2(1) and the identification of higher direct images with "sheafified" cohomology to deduce that the canonical map

$$\varinjlim_{K'} \mathrm{H}^i_{\text{ét}}\big(X \times_{\operatorname{Spec} K} \operatorname{Spec} K', \mathcal{F}_{K'}\big) \to \mathrm{H}^i_{\text{ét}}\big(X \times_{\operatorname{Spec} K} \operatorname{Spec} K_s, \mathcal{F}_{K_s}\big) = \mathrm{R}^i f_*(\mathcal{F})_{K_s}$$

is an isomorphism of discrete $\pi_1(K, K_s)$-modules, where K' runs through finite Galois extensions of K inside of K_s. Consequently, the "abstract" $\mathrm{Gal}(K_s/K)$-action on $\mathrm{H}^i_{\text{ét}}(X \times_{\operatorname{Spec} K} \operatorname{Spec} K_s, \mathcal{F}_{K_s})$ is exactly that induced by the (compatible) actions on $\mathrm{H}^i_{\text{ét}}(X \times_{\operatorname{Spec} K} \operatorname{Spec} K', \mathcal{F}_{K'})$ for all K'. This is precisely the desired compatibility.

5.2.7. Cohomology with proper support. We now turn to the crucial issue of "cohomology with compact supports", or rather its relativization using "proper" supports. We begin with the topological analogue for motivation. Let $f : X \to S$ be a map of topological spaces, and let $j : X \hookrightarrow \overline{X}$ be an open immersion with $f = \overline{f} \circ j$, where $\overline{f} : \overline{X} \to S$ is *proper*. Let $\mathcal{F}$ be an sheaf on X. Define the sheaf $f_!\mathcal{F}$ on $\overline{X}$ by

$$(f_!\mathcal{F})(U) \overset{\text{def}}{=} \{s \in \Gamma(U, \mathcal{F}) \mid \operatorname{supp}(s) \text{ proper over } S\}.$$

When $f = j$, we recover the classical "extension by zero" and it is easy to see that $f_! = \overline{f}_* \circ j_!$ since for any closed set $Z \hookrightarrow X$ with Z proper over S, the immersion $Z \hookrightarrow \overline{X}$ is proper and hence a closed embedding. If X is paracompact Hausdorff, then by [**18**, Ch II, Thm 3.5.5(c)] the functor $j_!$ takes soft (e.g., injective or flabby) abelian sheaves to soft sheaves, which are acyclic for cohomology. If S has a base of paracompact Hausdorff opens (e.g., S an analytic space), then over these opens the proper $\overline{X}$ is paracompact Hausdorff. Thus, universal δ-functor arguments (and the exactness of $j_!$) yield an isomorphism

$$\mathrm{R}^\bullet\overline{f}_* \circ j_! \simeq \mathrm{R}^\bullet(\overline{f}_* \circ j_!) = \mathrm{R}^\bullet f_!$$

of δ-functors.

In the étale world, the acyclicity property for $j_!$ on injectives does *not* hold (even for smooth affine curves over algebraically closed fields). This motivates the next definition.

DEFINITION 5.2.7.1. Let $f : X \to S$ be separated, finite type with S noetherian. Let $j : X \to \overline{X}$ be an open immersion into a proper S-scheme $\overline{f} : \overline{X} \to S$. (One can find such an $\overline{X}$ either by the Nagata compactification theorem [**41**] or by assuming that f is quasi-projective.) Define the higher direct images with proper support to be the δ-functor

$$\mathrm{R}^\bullet f_! \stackrel{\mathrm{def}}{=} \mathrm{R}^\bullet \overline{f}_* \circ j_!$$

on the category of *torsion* étale sheaves.

Note that this is *not* a derived functor. The natural "independence" of the choice of j requires the proper base change theorem, which is the reason for the torsion condition. As an example of the significance of the proper base change theorem, note that if we want a Leray type spectral sequence, then for a commutative diagram

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[d, "f"'] & \overline{X} \arrow[d, "\overline{f}"] \\ Y \arrow[r, hook, "j"] & \overline{Y} \end{tikzcd}$$

with i, j open immersions and $f, \overline{f}$ proper, we want

$$j_! \mathrm{R}^\bullet f_* \mathcal{F} \simeq \mathrm{R}^\bullet \overline{f}_*(i_! \mathcal{F}).$$

In particular, the sheaf $\mathrm{R}^\bullet \overline{f}_*(i_! \mathcal{F})$ should vanish over $\overline{Y} - Y$. This follows from *proper base change* and Theorem 5.1.4.2(2).

There is a base change theorem for $\mathrm{R}^\bullet f_!$, as well as an analogue of the Leray spectral sequence and excision sequence. This is all compatible with the natural maps $\mathrm{R}^\bullet f_! \to \mathrm{R}^\bullet f_*$. When $S = \operatorname{Spec} k$ for a separably closed field k, one writes $\mathrm{H}^\bullet_{\mathrm{c,\acute{e}t}}$ for the corresponding groups. The theory of $\mathrm{R}^\bullet f_!$ provides the ability to work locally on the *source*. For example, by reduction to the case of curves one can show that $\mathrm{R}^i f_!$ vanishes on torsion sheaves for i larger than twice the maximum dimension of the fibers of f; see Theorem 5.2.4.1.

A very special case of the general finiteness theorems is the following result, whose proof (as with many others) makes essential use of the theory of constructible sheaves:

LEMMA 5.2.7.2. [**17**, Ch I, 8.9] *Let X be a finite type separated scheme over a separably closed field k and let $\mathcal{F}$ be a locally constant constructible sheaf on X. Then $\mathrm{H}^i_{\mathrm{c,\acute{e}t}}(X, \mathcal{F})$ is finite for all i (and vanishes for $i > 2\dim(X)$).*

By using the *smooth* base change theorem and more general considerations with $\mathrm{R}^\bullet f_!$ and the theory of constructible sheaves, one obtains the following crucial fact which is analogous to Theorem 1.2.1.6 and is closely related to Lemma 5.2.7.2.

THEOREM 5.2.7.3. [**17**, Ch I, 8.13] *Let $f : X \to S$ be proper and smooth with S locally noetherian and let $\mathcal{F}$ be a locally constant constructible sheaf on X with torsion orders invertible on S. Then $\mathrm{R}^i f_* \mathcal{F}$ is locally constant constructible for all i.*

For an example, let $f : X \to \operatorname{Spec} R$ be proper smooth over a discrete valuation ring R with fraction field K and residue field k. Theorem 5.2.7.3 asserts that $\mathrm{R}^i F_*(\underline{\mathbf{Z}/l^n})$ is locally constant constructible on R, so by proper base change the geometric generic and closed fibers of X have the "same" cohomology!

Note that for a scheme X over a non-separably closed field K, $\mathrm{H}^i_{\text{ét}}(X, \underline{\mathbf{Z}/l^n})$ is typically *much* bigger than $\mathrm{H}^i_{\text{ét}}(X \times_K K_s, \underline{\mathbf{Z}/l^n})$, and is heavily influenced by the Galois cohomology groups

$$\mathrm{H}^j\big(K, \mathrm{R}^{i-j} f_*(\underline{\mathbf{Z}/l^n})\big)$$

through a Leray spectral sequence.

5.2.8. Cohomology of the universal elliptic curve. The geometric context in which we want to study cohomology is the universal algebraic family over a modular curve

$$E_1(N) \xrightarrow{f_N} Y_1(N) \xrightarrow{a} \mathbf{Z}[\tfrac{1}{N}].$$

Pick a prime l not dividing N, so $\mathrm{R}^1 f_{N*}(\underline{\mathbf{Z}/l^n})$ is locally constant constructible on $(Y_1(N)[\frac{1}{l}])_{\text{ét}}$ by Theorem 5.2.7.3. Let $\mathcal{F}_k = \operatorname{Sym}^k \mathrm{R}^1 f_{N*}(\underline{\mathbf{Z}/l^n})$, which is locally constant constructible on $(Y_1(N)[\frac{1}{l}])_{\text{ét}}$. We want to study the image $\mathrm{R}^1\widetilde{a}(\mathcal{F}_l) \subseteq \mathrm{R}^1 a_*(\mathcal{F}_k)$ of the map of sheaves

$$\mathrm{R}^1 a_!(\mathcal{F}_k) \to \mathrm{R}^1 a_*(\mathcal{F}_k)$$

on $(\operatorname{Spec} \mathbf{Z}[1/Nl])_{\text{ét}}$. These are sheaves of $\mathbf{Z}/l^n$-modules, and the higher direct images with proper supports $\mathrm{R}^\bullet a_!$ can be computed using the open immersion $Y_1(N) \hookrightarrow X_1(N)$ into a proper $\mathbf{Z}[1/N]$-scheme.

Recall from Theorem 4.2.6.2 that $X_1(N) \to \mathbf{Z}[1/N]$ is smooth and

$$X_1(N)_\infty \stackrel{\text{def}}{=} (X_1(N) - Y_1(N))_{\text{red}}$$

is finite étale over $\mathbf{Z}[1/N]$, defined Zariski locally by a single equation in $X_1(N)$. For any locally constant constructible sheaf $\mathcal{E}$ on $Y_1(N)[\frac{1}{l}]$, with torsion orders only divisible by primes dividing Nl (e.g, the $\mathcal{F}_k$'s), we may use Theorem 5.1.2.1 to identify $\mathcal{E}$ with $\underline{Y}$ for a finite étale $Y \to Y_1(N)[\frac{1}{l}]$. Let X be the normalization of $X_1(N)[\frac{1}{l}]$ in Y, so X is a normal 2-dimensional scheme and thus is Cohen-Macaulay. The regularity of $X_1(N)$ then implies via [**42**, Thm 23.3] that $X \to X_1(N)[\frac{1}{l}]$ is finite *flat*. In particular, there is a Cartier divisor $X_\infty \hookrightarrow X$ supported exactly over $X_1(N)_\infty[\frac{1}{l}]$. Since $X_1(N)_\infty[\frac{1}{l}]$ has generic points with characteristic 0, the map $X \to X_1(N)[\frac{1}{l}]$ is *tamely ramified* over all of these codimension 1 points of $X_1(N)[1/l]$. Thus, by the relative version of Abhyankar's lemma [**23**, XIII, 5.5], étale locally over $X_1(N)[\frac{1}{l}]$ we have that X looks like $U[t]/(T^n - f)$ for an étale $U \to X$ and f defining $U_\infty = U \times_X X_\infty$, with n *invertible* on U.

The above geometric conclusions about $X_1(N)_\infty$ are precisely the hypotheses required to use the difficult [**23**, XIII, 2.7] to conclude that if $\mathcal{E}$ is a sheaf of $\mathbf{Z}[1/Nl]$-modules, then $\mathrm{R}^1 a_* \mathcal{E}$ is locally constant constructible and of formation compatible with base change to fibers over $\operatorname{Spec} \mathbf{Z}[1/N]$. This result is in the spirit of Theorem 5.2.7.3, but due to the non-properness of a this result is actually quite deep. Since $X_1(N)_\infty \to \mathbf{Z}[1/N]$ is finite étale, similar methods show that $\mathrm{R}^1 a_! \mathcal{E}$ is locally constant constructible (and we know that this is of formation compatible with base change by the general base change theory for $\mathrm{R}^\bullet f_!$ for finite type separated maps f to any noetherian base).

We emphasize that the proofs of these "locally constant constructible" results make essential use of the concept of constructible sheaves (more general than locally constant constructible sheaves) and a specialization criterion for a constructible sheaf to be locally constant constructible. Thus, the framework that we have described is completely inadequate to prove the results we have cited. Summarizing and using Theorem 5.1.4.2 as well as smooth base change (e.g., for $\mathrm{Spec}(k_s) \to \mathrm{Spec}(k)$, as in Remark 5.2.6.2(2)), we have the following fundamental result.

THEOREM 5.2.8.1. *The sheaves*

$$\mathrm{R}^1\widetilde{a}\big(\mathrm{Sym}^k\,\mathrm{R}^1 f_{N*}\underline{\mathbf{Z}/l^n}\big)$$

on $(\mathrm{Spec}\,\mathbf{Z}[1/Nl])_{\text{ét}}$ *are locally constant constructible and of formation compatible with base change to (geometric) fibers over* $\mathrm{Spec}\,\mathbf{Z}[1/Nl]$.

5.2.9. Comparison with singular cohomology. By Corollary 5.1.3.1, we can identify the sheaves

$$ {}^{k}_{N}W_{\mathbf{Z}/l^n} = \mathrm{R}^1\widetilde{a}\big(\mathrm{Sym}^k\,\mathrm{R}^1 f_{N*}\underline{\mathbf{Z}/l^n}\big) \tag{5.2.9.1}$$

on $(\mathrm{Spec}\,\mathbf{Z}[1/Nl])_{\text{ét}}$ with finite discrete $(\mathbf{Z}/l^n)[G_{\mathbf{Q}}]$-modules, unramified at p not dividing Nl. Upon choosing an injection $\overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$ (which is the direct limit of *smooth* maps), these are in turn canonically identified (by a variant of smooth base change as in Remark 5.2.6.2(2)). with the underlying groups of the $\mathbf{C}$-fibers, which are just

$$\widetilde{\mathrm{H}}^1_{\text{ét}}\big(Y_1(N)_{/\mathbf{C}}, \mathrm{Sym}^k\,\mathrm{R}^1(f_{N/\mathbf{C}})_*\underline{\mathbf{Z}/l^n}\big);$$

here we are also using proper base change for $\mathrm{R}^1 f_{N*}$.

It remains to add the crucial comparison theorem with the analytic topology over $\mathbf{C}$ [**24**, Exp XVI, §4]; see also [**17**, Ch I, 11.6]. We define the *analytic site* for an analytic space X to be the site on which the "opens" are maps $U \to X$ which are local isomorphisms. It is straightfoward to check that sheaf theory and cohomology theory for the analytic site is "the same" as usual sheaf theory and cohomology theory on analytic spaces.

THEOREM 5.2.9.1. *Let* $f : X \to S$ *be a separated finite type map of finite type* $\mathbf{C}$*-schemes. For any étale sheaf* $\mathcal{F}$ *on* X *(resp.* S*), let* $\mathcal{F}^{\mathrm{an}}$ *denote the canonical pullback to the analytic site on* X^{an} *(resp.* S^{an}*). For any torsion abelian étale sheaf* $\mathcal{F}$ *on* X*, there is a canonical isomorphism of* δ*-functors*

$$\mathrm{R}^\bullet f_!(\cdot)^{\mathrm{an}} \xrightarrow{\simeq} \mathrm{R}^\bullet f_!^{\mathrm{an}}((\cdot)^{\mathrm{an}})$$

compatible with natural base change maps. When $\mathcal{F}$ *is locally constant constructible, there are* δ*-functorial isomorphisms*

$$\mathrm{R}^\bullet f_*(\cdot)^{\mathrm{an}} \xrightarrow{\simeq} \mathrm{R}^\bullet f_*^{\mathrm{an}}(\cdot)^{\mathrm{an}}$$

compatible with base change maps.

REMARK 5.2.9.2. The case of direct image sheaves is somewhat deeper. The argument proceeds by induction on the fiber dimension of X, and the original proof requires resolution of singularities for varieties of dimension at most X. We only care about the case of dimension 1, namely $Y_1(N)_{/\mathbf{C}} \to \mathrm{Spec}(\mathbf{C})$ (as $f_N : E_1(N)_{/\mathbf{C}} \to Y_1(N)_{/\mathbf{C}}$ is proper, so $\mathrm{R}^\bullet f_{N!} = \mathrm{R}^\bullet f_{N*}$ takes care of the 2-dimensional $E_1(N)_{/\mathbf{C}}$ without needing resolution of singularities for surfaces). An elegant proof that avoid resolution is given by the [**1**, 7.5.1], which carries over essentially verbatim to the complex analytic case (if one interprets "elementary fibration of pure dimension 1" to mean "relative unit disc").

Using direct limits and the Kummer sequence, the key case in the proof of the comparison theorem is that of proper smooth curves. Here the case $\mathcal{F} = \underline{\mathbf{Z}/n}$ reduces us to the claim that the *natural* map of groups

$$\mathrm{Pic}(X)[n] \to \mathrm{Pic}(X^{\mathrm{an}})[n]$$

is an isomorphism. In fact, the map $\mathrm{Pic}(X) \to \mathrm{Pic}(X^{\mathrm{an}})$ itself is an isomorphism by GAGA!

By exactness of inverse limits on inverse systems of finite groups, the group

$$\varprojlim_n \widetilde{\mathrm{H}}^1\big(Y_1(N)^{\mathrm{an}}_{/\mathbf{C}}, \mathrm{Sym}^k \mathrm{R}^1 f^{\mathrm{an}}_{N*}\underline{\mathbf{Z}/l^n}\big) = \varprojlim_n {}^k_N W_{\mathbf{Z}/l^n}$$

is canonically identified with the image of the map

(5.2.9.2)
$$\varprojlim_n \mathrm{H}^1_{\mathrm{c}}\big(Y_1(N)^{\mathrm{an}}_{/\mathbf{C}}, \mathrm{Sym}^k \mathrm{R}^1 f^{\mathrm{an}}_{N*}\underline{\mathbf{Z}/l^n}\big) \to \varprojlim_n \mathrm{H}^1\big(Y_1(N)^{\mathrm{an}}_{/\mathbf{C}}, \mathrm{Sym}^k \mathrm{R}^1 f^{\mathrm{an}}_{N*}\underline{\mathbf{Z}/l^n}\big).$$

Note that $\mathrm{H}^\bullet_{\mathrm{c}}(Y_1(N)^{\mathrm{an}}_{/\mathbf{C}}, (\cdot))$ is isomorphic to $\mathrm{H}^\bullet(X_1(N)^{\mathrm{an}}_{/\mathbf{C}}, (\cdot)) \circ j_!$ for the open immersion $j : Y_1(N)^{\mathrm{an}}_{/\mathbf{C}} \hookrightarrow X_1(N)^{\mathrm{an}}_{/\mathbf{C}}$, due to the discussion at the start of §5.2.7. Considerations with [**26**, Ch 0, 13.3.1] and constructible sheaves imply that the map (5.2.9.2) is canonically identified with the map

$$\mathrm{H}^1_{\mathrm{c}}\big(Y_1(N)^{\mathrm{an}}_{/\mathbf{C}}, \mathrm{Sym}^k \mathrm{R}^1 f^{\mathrm{an}}_{N*}\underline{\mathbf{Z}_l}\big) \to \mathrm{H}^1\big(Y_1(N)^{\mathrm{an}}_{/\mathbf{C}}, \mathrm{Sym}^k \mathrm{R}^1 f^{\mathrm{an}}_{N*}\underline{\mathbf{Z}_l}\big).$$

We therefore have an isomorphism

$$\varprojlim_n {}^k_N W_{\mathbf{Z}/l^n} \simeq \widetilde{\mathrm{H}}^1\big(Y_1(N)^{\mathrm{an}}_{/\mathbf{C}}, \mathrm{Sym}^k \mathrm{R}^1 f^{\mathrm{an}}_{N*}\underline{\mathbf{Z}_l}\big) \simeq {}^k_N W \otimes_{\mathbf{Z}} \mathbf{Z}_l \tag{5.2.9.3}$$

of $\mathbf{Z}_l$-modules. Note that this is even a topological isomorphism since finite $\mathbf{Z}_l$-modules have a *unique* profinite topological group structure (namely, the l-adic one).

Recalling that ${}^k_N W_{\mathbf{Z}/l^n}$ admits an action of $G_{\mathbf{Q}}$ which is continuous for the discrete topology, using (5.2.9.1) and (5.2.9.3) and admitting the many unproven facts which we have discussed above finally achieves the goal of defining a *continuous* action of $G_{\mathbf{Q}}$ on

$${}^k_N W \otimes_{\mathbf{Z}} \mathbf{Z}_l$$

which is unramified at p not dividing Nl and compatible with change in $N \geq 5$. We thus have the same structure on ${}^k_N W_l = {}^k_N W \otimes_{\mathbf{Z}} \mathbf{Q}_l$. Our next task is to give an algebraic definition of Hecke operators and l-adic pairings in terms of étale cohomology. Recalling the role of trace maps in the definitions of Hecke operators on topological cohomology, the first order of business is to discuss a theory of trace maps in étale cohomology.

5.2.10. Trace maps. One can define cup products in étale cohomology (which are compatible with those in Galois cohomology via Theorem 5.1.1.1). There are also δ-functorial relative cup products among $\mathrm{R}^\bullet f_*$ and $\mathrm{R}^\bullet f_!$ compatible with the maps $\mathrm{R}^\bullet f_! \to \mathrm{R}^\bullet f_*$ and base change maps, as well as with the classical constructions on the analytic side.

For quasi-finite separated flat maps $f : X \to Y$ of finite presentation, there is a fundamental *trace map* for abelian sheaves $\mathcal{F}$ on $Y_{\text{ét}}$:

$$\operatorname{tr}_f : f_! f^* \mathcal{F} \to \mathcal{F}. \tag{5.2.10.1}$$

It is uniquely determined on *geometric points* by the formula

$$\begin{aligned} \bigoplus_{x \in f^{-1}(y)} \mathcal{F}_y &\to \mathcal{F}_y \\ (s_x) &\mapsto \sum n_x s_x \end{aligned} \tag{5.2.10.2}$$

on stalks; here n_x is the length of the artinian local ring $\mathcal{O}_{f^{-1}(g),x}$. The existence of the map (5.2.10.1) is *not* obvious; see [**24**, Exp XVII, §6.2]. This is the first step towards the more general theory of relative trace maps. We now explain the motivation in the case of curves.

Let Z be a compact Riemann surface and recall the isomorphism $\operatorname{Pic}(Z) \simeq \mathrm{H}^1(Z, \mathcal{O}_Z^\times)$ given by our Čech convention as is made explicit in §0.1. Consider the composite map

$$\operatorname{Pic}(Z) \simeq \mathrm{H}^1(Z, \mathcal{O}_Z^\times) \xrightarrow{\partial} \mathrm{H}^2(Z, \underline{\mathbf{Z}}(1)) \xrightarrow{\simeq} \mathbf{Z}, \tag{5.2.10.3}$$

where the last isomorphism is defined via Poincaré duality as in Theorem 1.2.2.2 and where ∂ is the coboundary map obtained from the exact sequence

$$0 \to \underline{\mathbf{Z}}(1) \to \mathcal{O}_Z \xrightarrow{\exp} \mathcal{O}_Z^\times \to 1.$$

If we choose $i = \sqrt{-1} \in \mathbf{C}$, (5.2.10.3) can also be expressed as

$$\operatorname{Pic}(Z) \simeq \mathrm{H}^1(Z, \mathcal{O}_Z^\times) \xrightarrow{\partial_i} \mathrm{H}^2(Z, \underline{\mathbf{Z}}) \simeq \mathrm{H}^2_{\text{top}}(Z, \underline{\mathbf{Z}}) \xrightarrow{\simeq} \underline{\mathbf{Z}},$$

where ∂_i is the coboundary map coming from

$$0 \to \underline{\mathbf{Z}} \to \mathcal{O}_Z \xrightarrow{e^{2\pi i(\cdot)}} \mathcal{O}_Z^\times \to 1$$

and the Poincaré duality isomorphism at the end is defined with respect to the i-orientation of Z.

The fundamental fact is that the composite map (5.2.10.3) is *exactly* the usual notion of degree of a line bundle. (Note that the choice of sign in the definition of the isomorphism $\operatorname{Pic}(Z) \simeq \mathrm{H}^1(Z, \mathcal{O}_Z^\times)$ matters here.) The verification of this claim is a somewhat long exercise in unwinding definitions and studying the coherent ideal sheaf of a point on a compact Riemann surface; since we are merely trying to give some motivation, we omit the tedious details of this exercise. Using this reformulation of the definition of the degree of a line bundle, together with the commutativity of the diagram

$$\begin{tikzcd} 0 \arrow[r] & \underline{\mathbf{Z}}(1) \arrow[r] \arrow[d, "\exp(\frac{1}{n}\cdot)"] & \mathcal{O}_Z \arrow[r, "\exp"] \arrow[d, "\exp(\frac{1}{n}\cdot)"] & \mathcal{O}_Z^\times \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & \mu_n \arrow[r] & \mathcal{O}_Z^\times \arrow[r, "n"'] & \mathcal{O}_Z^\times \arrow[r] & 1 \end{tikzcd}$$

(where the bottom row is the "analytic" Kummer sequence), we obtain the commutativity of

$$\begin{tikzcd} \mathrm{H}^2(Z,\mu_n) \arrow[r, "\simeq"] \arrow[d, "\simeq"] & \mathrm{coker}\big(\mathrm{Pic}(Z) \xrightarrow{n} \mathrm{Pic}(Z)\big) \arrow[d, "\deg"', "\simeq"] \\ \mathrm{H}^2(Z,\underline{\mathbf{Z}/n}) \arrow[r] & \mathbf{Z}/n \end{tikzcd} \tag{5.2.10.4}$$

where the left column is defined by the isomorphism $\mathbf{Z}/n \simeq \mu_n$ using $e^{\frac{2\pi i}{n}(\cdot)}$ and the bottom row is Poincaré duality with the i-orientation. This commutative diagram is the motivation for how to define the étale cohomology trace map for curves (and in the complex analytic case is ultimately the reason why the étale cohomology trace in arbitrary dimension agrees via the comparison isomorphism with the trace map coming from topology).

More generally, for *any* d-dimensional Hausdorff complex manifold Z, there is a canonical isomorphism

$$\begin{aligned} \mathrm{H}^{2d}_{\mathrm{c}}\big(Z,\mu_n^{\otimes d}\big) &\simeq \mathbf{Z}(d)\otimes \mathrm{H}^{2d}_{\mathrm{c}}\big(Z,\underline{\mathbf{Z}/n}\big) \\ &\simeq \mathbf{Z}(d)\otimes(\mathbf{Z}/n)(-d) \\ &\simeq \mathbf{Z}/n \end{aligned}$$

via topological Poincaré duality

In étale cohomology, the theory of the trace map aims to construct, for any smooth separated finite type $f : Y \to S$ with pure relative dimension d over a noetherian base S, a canonical map

$$\mathrm{R}^{2d}f_!(\mu_n^{\otimes d}) \to \underline{\mathbf{Z}/n} \tag{5.2.10.5}$$

for n invertible on S. The map (5.2.10.5) should be compatible with base change and with change in n, should be transitive in f in a suitable sense, and should be an *isomorphism* when f has *geometrically connected* fibers.

In the case $d = 0$, (5.2.10.5) is defined to be the trace map (5.2.10.1). When $d = 1$ and $f : Y \to S$ is a proper smooth connected curve over a separably closed field, (5.2.10.2) suggests that (5.2.10.5) should be

$$\mathrm{H}^2(Y,\mu_n) \simeq \mathrm{coker}\big(\mathrm{Pic}(Y) \xrightarrow{n} \mathrm{Pic}(Y)\big) \xrightarrow{\deg} \mathbf{Z}/n \tag{5.2.10.6}$$

where the first map comes from the Kummer sequence and the isomorphism

$$\mathrm{H}^1_{\text{ét}}(Y,\mathbf{G}_m) \simeq \mathrm{Pic}(Y).$$

Note that our Čech sign convention for defining the isomorphism of $\mathrm{Pic}(Y)$ and $\mathrm{H}^1_{\text{ét}}(Y,\mathbf{G}_m)$ arises here.

The general case is built up from the étale case ($d = 0$) and the fact that any smooth map is Zariski locally an étale map over an affine space $\mathbf{A}^n_S$. The case of $\mathbf{A}^n_S \to S$ can be handled directly. There are a number of delicate matters to be handled, such as well-definedness and obtaining the expected answer for proper smooth curves. See [**17**, Ch 2, §1] for details.

In order to show that the relative trace map is an isomorphism when the fibers are *geometrically* connected (this is crucial for Poincaré duality), one first verifies the transitivity of relative trace maps with respect to composites in f. Then one does a direct calculation for curves over separably closed fields, where (5.2.10.6) *is* recovered in the proper smooth connected case.

By base change for $\mathrm{R}^\bullet f_!$ and induction on fiber dimension, the remaining problem is to show that any smooth finite type separated X over an *algebraically closed field* k of dimension $d > 1$ contains a Zariski dense open set $U \subseteq X$ which is "fibered by curves". More precisely, there should exist such a U admitting a smooth map $U \to V$ where V is a smooth connected finite type separated k-scheme of dimension $d-1$ and all fibers of $U \to V$ are *geometrically connected* of dimension 1. The construction of such a U can be done by means of the Bertini theorems:

LEMMA 5.2.10.1. *Let X be a reduced and irreducible scheme of finite type and dimension $d > 1$ over an algebraically closed field k. Then there exists a dense open $U \subseteq X$ and a smooth map $U \to Y$ with Y open in $\mathbf{A}_k^{d-1}$ and all fibers of $U \to Y$ geometrically connected of dimension* 1.

PROOF. Without loss of generality we can assume that X is quasi-projective (e.g., affine), so we may choose an immersion $X \hookrightarrow \mathbf{P}_k^N$ with $N \geq 2d$. Let $\overline{X}$ denote the closure of X in this projective space, so that $\overline{X}$ is a projective variety of dimension d. Let $\mathrm{Gr}(\delta, N)$ denote the Grassmannian of codimension δ planes in $\mathbf{P}_k^N$. By the Bertini theorems [**34**, Ch I, Cor 6.11], there exists a Zariski dense open set $V \subseteq \mathrm{Gr}(d-1, N)$ such that for all $H \in V(k)$, $\overline{X} \cap H$ is irreducible and reduced of dimension 1. On the other hand, since all irreducible components of $\overline{X} - X$ have dimension at most $d-1 > 0$, $(\overline{X} - X) \cap H$ is empty or 0-dimensional for each such H. For such an H, note also that $X \cap H$ is a dense open in $\overline{X} \cap H$ and so is irreducible and reduced of dimension 1.

Pick an $H_0 \in V(k)$ and choose a linear subspace $L \subseteq H_0$ (an $(N-d)$-plane) with L not containing the curve $\overline{X} \cap H_0$. (That is, $L \cap \overline{X}$ has dimension 0.) Since $\dim H_0 = N - (d-1) \geq 2d - (d-1) > d > 1$, H_0 is not a line and thus such an L can be found.

More importantly, since $N - (d-1) > d$, by the same Bertini theorems there exists a $(d-1)$-plane $H_0' \subseteq \mathbf{P}_k^N$ disjoint from the d-dimensional variety $\overline{X}$, as $(d-1)$-planes have codimension greater than d in $\mathbf{P}_k^N$.

Choose projective coordinates so that

$$\begin{aligned} H_0 &= \{[x_0, \dots, x_{N-d+1}, 0, \dots, 0]\} \\ L &= \{[x_0, \dots, x_{N-d}, 0, \dots, 0]\} \\ H_0' &= \{[0, \dots, 0, x_{N-d+1}, \dots, x_n]\} \end{aligned}$$

Consider the projection $\pi : \mathbf{P}_k^N - L \to H_0'$ defined by

$$[x_0, \dots, x_N] \to [x_{N-d+1}, \dots, x_N].$$

Since X has dimension $d > 1$ and $X \cap H_0$ has dimension 1, certainly X is not contained in H_0. Thus, $U_0 = X \cap (\mathbf{P}_k^N - L)$ is a non-empty open set in X. We are interested in the map

$$\pi|_{U_0} : U_0 \to H_0'.$$

The scheme-theoretic fiber of $\pi|_{U_0}$ over $[1, 0, \dots, 0] \in H_0'$ is $X \cap H_0$, which is reduced and irreducible of dimension 1. Since L and H_0' are disjoint, there is a morphism

$$H_0' \to \mathrm{Gr}(d-1, N) \tag{5.2.10.7}$$

determined on k-rational points by

$$p \mapsto \mathrm{span}(p, L).$$

Since $L \cap H_0'$ is empty, we see that $\mathrm{span}(p, L) \cap H_0' = p$ for any p, so (5.2.10.7) is injective on k-rational points. By the properness of H_0' and $\mathrm{Gr}(d-1, N)$, the (reduced) image $Z \subseteq \mathrm{Gr}(d-1, N)$ of H_0' is closed and $H_0' \to Z$ is proper and bijective on k-rational points. Thus $H_0' \to Z$ is a homeomorphism. Note that $H_0 \in Z(k)$ since H_0 is the span of L and $[0, \dots, 0, 1, 0, \dots, 0]$.

Now recall that for the dense open $V \subseteq \mathrm{Gr}(d-1, N)$ defined at the start of the proof, all $H \in V(k)$ have the property that $X \cap H$ is irreducible and reduced of dimension 1. Since $H_0 \in V(k)$, the open set $Z \cap V$ in Z contains a k-rational point; in particular, it is non-empty. But $H_0' \to Z$ is a homeomorphism and H_0' is a projective space, so there exists a Zariski dense open set in $H_0'(k)$ whose points p have the property that $\mathrm{span}(p, L) \cap X$ is reduced and irreducible of dimension 1.

Since $L \cap \overline{X}$ has dimension 0, for all such p the open subset

$$\mathrm{span}(p, L) \cap (X \cap (\mathbf{P}_k^N - L)) \subseteq \mathrm{span}(p, L) \cap X$$

is non-empty and therefore is reduced and irreducible of dimension 1. But this is just the scheme $(\pi|_{U_0})^{-1}(p)$!

We can now use generic flatness [**28**, 6.9.1] and replace H_0' by a dense open so that we get an open $U \subseteq X$, a non-empty open $Y \subseteq \mathbf{A}_k^{d-1}$ and a *flat* map $U \to Y$ such that all fibers of $U \to Y$ over $Y(k)$ are reduced and irreducible of dimension 1. Thus the open smooth locus of $U \to Y$ is non-empty (it meets all k-rational fibers). Upon shrinking U we may assume that $U \to Y$ is smooth, necessarily of pure dimension 1 since this is true on the dense set $U(k) \subseteq U$. Now [**28**, 9.7.8] ensures that *all* fibers of $U \to Y$ are geometrically connected, as this holds over $Y(k)$ and k is algebraically closed. □

5.2.11. Poincaré duality. A special case of Poincaré duality is given in Theorem 5.4.2.2 below. As usual, a crucial step in the proof is the case of a proper smooth connected curve X over an algebraically closed field k. Here the essential calculation is that the cup product pairing

$$\mathrm{H}^1_{\text{ét}}(X, \mu_n) \otimes \mathrm{H}^1_{\text{ét}}(X, \underline{\mathbf{Z}/n}) \to \mathrm{H}^2_{\text{ét}}(X, \mu_n) \simeq \mathbf{Z}/n \tag{5.2.11.1}$$

is perfect when n is prime to the characteristic of k. Note that $\mathrm{H}^1_{\text{ét}}(X, \mu_n)$ is canonically identified with $\mathrm{Pic}_{X/k}(k)[n] = \mathrm{Pic}^0_{X/k}[n](k)$ by the Kummer sequence, whereas by Theorem 5.2.2.1 the group $\mathrm{H}^1_{\text{ét}}(X, \underline{\mathbf{Z}/n})$ is canonically identified with $\mathrm{Hom}_{\mathrm{cont}}(\pi_1(X, \overline{x}), \mathbf{Z}/n)$, which classifies all (possibly disconnected) $\mathbf{Z}/n$-Galois covers of X. If X has genus g, then by the theory of abelian varieties and the Jacobian, $\mathrm{Pic}^0_{X/k}[n]$ is a finite free $\mathbf{Z}/n$-module of rank g. Meanwhile, by Grothendieck's determination of the "prime-to-characteristic" part of the étale fundamental groups of curves, $\mathrm{Hom}_{\mathrm{cont}}(\pi_1(X, \overline{x}), \mathbf{Z}/n)$ is a finite free $\mathbf{Z}/n$-module of rank g (essentially because one knows this when $k = \mathbf{C}$ by topological arguments!). Thus, (5.2.11.1) is a pairing between two finite free $\mathbf{Z}/n$-modules of the same rank. When $k = \mathbf{C}$ it is perfect because of comparison with the known topological analogue. The general case can be reduced to this case by means of the proper and smooth base change theorems in étale cohomology.

Here is a more concrete way to think about the "meaning" of this pairing (5.2.11.1) without invoking lifts to characteristic 0. Recall the interpretation of $\mathrm{Pic}^0_{X/k}$ as classifying abelian Galois covers of X. Upon choosing the canonical map $X \to \mathrm{Pic}^0_{X/k}$ given by $x \mapsto (x) - (\overline{x})$ which takes the base point $\overline{x}$ to 0, the classical theory of the Jacobian shows that every *connected* abelian étale cover of X arises as the pullback of a unique connected (abelian) étale cover of $\mathrm{Pic}^0_{X/k}$, and conversely all such covers of $\mathrm{Pic}^0_{X/k}$ have *connected* pullback to X. In this way, we get a canonical isomorphism ("independent" of the choice of base point)

$$\pi_1(X,\overline{x})^{\mathrm{ab}} \simeq \pi_1(\mathrm{Pic}^0_{X/k},0).$$

Meanwhile, since $\mathrm{Pic}^0_{X/k}$ is an abelian variety we have

$$\mathrm{Pic}^0_{X/k}[n](k) \simeq \pi_1(\mathrm{Pic}^0_{X/k},0)/n;$$

see the proof of Lemma 5.3.3.2. Depending upon ones choice of signs, the pairing (5.2.11.1) is, up to sign, nothing other than the *canonical* evaluation map

$$(\pi_1(\mathrm{Pic}^0_{X/k},0)/n) \otimes \mathrm{Hom}_{\mathrm{cont}}\big(\pi_1(\mathrm{Pic}^0_{X/k},0),\mathbf{Z}/n\big) \to \mathbf{Z}/n, \tag{5.2.11.2}$$

which is visibly perfect since $\pi_1(\mathrm{Pic}^0_{X/k},0)/n$ is a finite free $\mathbf{Z}/n$-module (with the discrete topology)! However, this is a somewhat misleading argument because it seems rather tricky to unwind the general construction of the trace map for X and the computation of the two H^1's in terms of Kummer theory and fundamental groups in order to *directly* prove the crucial ingredient that the pairing (5.2.11.2) really *is* the expected natural pairing (up to sign). The only proof I can see for this latter fact is to use reduction to the case $k = \mathbf{C}$ and the following facts about a compact Riemann surface X and its "abstract" Pic^0_X:

- the analytic model $\mathrm{H}^1(X,\mathcal{O}_X)/\mathrm{H}^1(X,\underline{\mathbf{Z}}(1))$ for Pic^0_X as Picard variety explicates the "algebraic" isomorphism between $\mathrm{H}^1(X,\mu_n)$ and $\mathrm{Pic}^0_X[n]$ as well as the "algebraic" isomorphism between $\mathrm{H}^1(X,\mathcal{O}_X)$ and $T_0(\mathrm{Pic}^0_X)$,
- the analytic model $\mathrm{H}^0(X,\Omega^1_X)^\vee/\mathrm{H}_1(X,\mathbf{Z})$ for $J \stackrel{\mathrm{def}}{=} \mathrm{Pic}^0_X$ as Jacobian variety explicates the Galois-theoretic intepretation of
$$\mathrm{H}^1(X,\underline{\mathbf{Z}/n}) = \mathrm{Hom}(\mathrm{H}_1(X,\mathbf{Z}),\mathbf{Z}/n)$$
in terms of coverings of the J and the "algebraic" isomorphism between $\mathrm{H}^0(X,\Omega^1_X)^\vee$ and $T_0(J) = \mathrm{H}^0(J,\Omega^1_J)^\vee$ via a standard inclusion $X \hookrightarrow J$,
- the analytic isomorphism of complex tori
$$\mathrm{H}^1(X,\mathcal{O}_X)/\mathrm{H}^1(X,\underline{\mathbf{Z}}(1)) \simeq \mathrm{H}^0(X,\Omega^1_X)^\vee/\mathrm{H}_1(X,\mathbf{Z}) = \mathrm{H}^0(X,\Omega^1_X)^\vee/\mathrm{H}^1(X,\underline{\mathbf{Z}})^\vee$$
which respects the tangent space identifications with $T_0(\mathrm{Pic}^0_X)$ (and hence which respects the abstract identifications of each analytic model with Pic^0_X!) is – up to sign – induced by the Hodge theory isomorphism
$$\mathrm{H}^1(X,\mathcal{O}_X) \simeq \mathrm{H}^0(X,\overline{\Omega}^1_X) \simeq \mathrm{H}^0(X,\Omega^1_X)$$
arising from $(\overline{\omega},\eta) \mapsto \frac{1}{2\pi i}\int_X \overline{\omega}\wedge\eta$, and this isomorphism carries the lattice $\mathrm{H}^1(X,\underline{\mathbf{Z}}(1))$ over to the lattice $\mathrm{H}^1(X,\underline{\mathbf{Z}})^\vee$ via the cup product pairing
$$\mathrm{H}^1(X,\underline{\mathbf{Z}}(1)) \times \mathrm{H}^1(X,\underline{Z}) \to \mathrm{H}^2(X,\underline{\mathbf{Z}}(1)) \simeq \mathbf{Z},$$
as one proves by computations with differential forms.

By proper base change and Theorem 5.2.7.3, the perfectness of (5.2.11.1) implies that the cup product pairing of locally constant constructible sheaves on $Y_1(N)[\frac{1}{l}]_{\mathrm{\acute{e}t}}$

$$\mathrm{R}^1 f_{N*}\underline{\mathbf{Z}/l^n} \otimes \mathrm{R}^1 f_{N*}\underline{\mathbf{Z}/l^n} \to \mathrm{R}^2 f_{N*}\underline{\mathbf{Z}/l^n} \simeq \mu_{l^n}^{-1}$$

is a perfect pairing. Consequently, the induced pairing

$$\mathrm{Sym}^k\,\mathrm{R}^1 f_{N*}\underline{\mathbf{Z}/l^n} \otimes \mathrm{Sym}^k\,\mathrm{R}^1 f_{N*}\underline{\mathbf{Z}/l^n} \to \mu_{l^n}^{\otimes -k}$$

is perfect as well.

It follows from another application of Theorem 5.4.2.2 (using Theorem 5.2.8.1 to compute on fibers) that the cup product pairing on $(\operatorname{Spec}\mathbf{Z}[1/Nl])_{\text{ét}}$

$$\mathrm{R}^1\widetilde{a}\big(\operatorname{Sym}^k \mathrm{R}^1 f_{N*}\underline{\mathbf{Z}/l^n}\big)\otimes \mathrm{R}^1\widetilde{a}\big(\operatorname{Sym}^k \mathrm{R}^1 f_{N*}\underline{\mathbf{Z}/l^n}\big)\to \mathrm{R}^2 a_! \mu_{l^n}^{\otimes -k}\simeq \mu_{l^n}^{\otimes(-k-1)}$$

is a perfect pairing. On the $\overline{\mathbf{Q}}$-fiber, this yields a perfect pairing of $G_{\mathbf{Q}}$-modules

$${}^k_N(\cdot,\cdot)_{\mathbf{Z}/l^n} : {}^k_N W_{\mathbf{Z}/l^n}\otimes {}^k_N W_{\mathbf{Z}/l^n}\to \mu_{l^n}^{\otimes(-k-1)}$$

Furthermore, by compatibility with the analytic theory of cup product and trace map on the $\mathbf{C}$-fiber, upon choosing an embedding $\overline{\mathbf{Q}}\hookrightarrow\mathbf{C}$ the resulting pairing

$${}^k_N(\cdot,\cdot)_{\mathbf{Z}_l} : \varprojlim_n {}^k_N W_{\mathbf{Z}/l^n}\otimes \varprojlim_n {}^k_N W_{\mathbf{Z}/l^n}\to \mathbf{Z}_l(-k-1)$$

is *exactly* the $\mathbf{Z}_l$-extension of scalars of ${}^k_N(\cdot,\cdot)$ on ${}^k_N W$ over $\mathbf{Z}$. Thus, we have a *purely algebraic* reconstruction of the l-adic pairing ${}^k_N(\cdot,\cdot)_l$; in particular, it is $G_{\mathbf{Q}}$-equivariant.

Beware that although ${}^k_N(\cdot,\cdot)_{\mathbf{Z}/l^n}$ is a perfect pairing, ${}^k_N(\cdot,\cdot)_{\mathbf{Z}_l}$ need not be a perfect pairing. The problem is that the ${}^k_N W_{\mathbf{Z}/l^n}$'s might not be *free* $\mathbf{Z}/l^n$-modules of formation compatible with change in n. The theory of cohomology for l-adic sheaves nevertheless yields a purely *algebraic* proof that ${}^k_N(\cdot,\cdot)_{\mathbf{Z}_l}\otimes_{\mathbf{Z}_l}\mathbf{Q}_l$ is a perfect pairing (which we can see in our situation by comparison with the pairing ${}^k_N(\cdot,\cdot)_l$ from the analytic theory).

Now choose a prime p not dividing Nl. Consider the action F of geometric Frobenius at p on ${}^k_N W_{\mathbf{Z}/l^n}$. By the discussion after Theorem 5.2.2.1, as well as proper base change with respect to $\operatorname{Spec}\overline{\mathbf{F}}_p\to\operatorname{Spec}\mathbf{Z}[1/N]$, we may study this action of F using only étale cohomology on the $\mathbf{F}_p$-fiber! Likewise, by base change compatibility of trace for finite flat maps (as in §5.2.10), we may imitate the cohomological definitions in §2.3.1, §2.3.6 to directly *define* Hecke and w_ζ actions in étale cohomology over $\mathbf{Z}[1/N]$ and $\mathbf{Z}[1/N,\zeta_N]$, respectively. On the complex analytic fiber, this recovers *exactly* the earlier defined analytic operations on

$${}^k_N W_l\simeq \mathbf{Q}_l\otimes_{\mathbf{Z}_l}\varprojlim {}^k_N W_{\mathbf{Z}/l^n},$$

via the comparison isomorphisms (5.2.9.3). On the $\mathbf{F}_p$ and $\overline{\mathbf{F}}_p$ fibers, we can study ${}^k_N W_{\mathbf{Z}/l^n}$ for fixed n. by using *characteristic* p diagrams such as those in Corollary 4.2.8.3. In particular, we have finally achieved our goal of giving a purely *algebraic* definition for *all* of the structures which arise in Theorem 4.1.3.1 and Corollary 4.1.3.3. We now need to verify the properties in the hypotheses of these two results.

5.3. Frobenius calculations

5.3.1. Frobenius actions on sheaves. We want to describe the action of the "geometric Frobenius" on ${}^k_N W_{\mathbf{Z}/l^n}$ *without* mentioning a choice of $\overline{\mathbf{F}}_p$. The classical context is the following. Let X be a proper smooth scheme over $\mathbf{F}_p$ and let $\phi: X\to X$ be the absolute Frobenius morphism. Let $\overline{\mathbf{F}}_p$ be an algebraic closure of $\mathbf{F}_p$ and $\overline{X}=X\times_{\mathbf{F}_p}\overline{\mathbf{F}}_p$. Let $\overline{\phi}=\phi\times 1$. This is an $\overline{\mathbf{F}}_p$-endomorphism of $\overline{X}$ and is *not* the intrinsic absolute Frobenius endomorphism of $\overline{X}$ as an $\mathbf{F}_p$-scheme. This map is "geometric", since it corresponds to raising coordinates to the pth power if we view $\overline{X}$ sitting inside of some projective space over $\overline{\mathbf{F}}_p$, defined by equations with coefficients in $\mathbf{F}_p$. Meanwhile, if Frob denotes the arithmetic Frobenius endomorphism of $\overline{\mathbf{F}}_p$ over $\mathbf{F}_p$, then $1 \times \mathrm{Frob}$ and $\overline{\phi}$ commute as endomorphisms of the scheme $\overline{X}$, with composite (in either order) equal to the absolute Frobenius endomorphism of $\overline{X}$.

Let M be an abelian group. The endomorphism ring of $\overline{X}$ acts on the cohomology group $\mathrm{H}^i_{\text{ét}}(\overline{X}, \underline{M})$, so the operators of pullback by $\overline{\phi} = \phi \times 1$ and by $1 \times \mathrm{Frob}$ commute. The composite of these commuting endomorphisms is pullback along the absolute Frobenius endomorphism of the scheme $\overline{X}$. But as we shall see below, for an arbitrary $\mathbf{F}_p$-scheme Z and an arbitrary constant sheaf $\underline{M}$ on Z, the action of absolute Frobenius on $\mathrm{H}^i_{\text{ét}}(Z, \underline{M})$ is the *identity*. Thus, back in the situation over $\overline{X}$, we see that the "geometric" endomorphism $\overline{\phi}$ of $\overline{X}$ which acts by "raising coordinates to pth power" induces a pullback action on $\mathrm{H}^i_{\text{ét}}(\overline{X}, \underline{M})$ which is equal to the pullback action of $1 \times \mathrm{Frob}^{-1}$. For this reason, we call $\mathrm{Frob}^{-1} \in \mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p)$ the *geometric* Frobenius element: this is the algebraic element in the Galois group whose pullback action on étale cohomology (of a constant sheaf) coincides with that induced by the honest geometric Frobenius map on the $\overline{\mathbf{F}}_p$-scheme $\overline{X}$.

For technical flexibility in our later considerations, we now generalize these constructions to the case of étale sheaves that are not necessarily constant. For any $\mathbf{F}_p$-scheme X, we define the absolute Frobenius

$$F_X : X \to X$$

to be given by the identity map on the underlying space and the pth power map on the structure sheaf. For any morphism $f : X \to Y$ of $\mathbf{F}_p$-schemes one sees that the ringed space maps $F_Y \circ f$ and $f \circ F_X$ coincide. In more concrete terms, over an open affine $\mathrm{Spec}(A)$ in X the map F_X is induced by the pth power map on A and all maps between $\mathbf{F}_p$-algebras commute with the pth power maps on source and target.

For any étale sheaf of sets $\mathcal{F}$ on an $\mathbf{F}_p$-scheme X, there is a natural isomorphism

(5.3.1.1) $$\mathcal{F} \simeq (F_X)_* \mathcal{F}$$

defined as follows: the (relative Frobenius) map over X

(5.3.1.2) $$U \to F_X^{-1}(U)$$

sitting in the diagram

$$\begin{tikzcd} U \arrow[r] \arrow[dr] \arrow[rr, bend left, "F_U"] & F_X^{-1}(U) \arrow[r] \arrow[d] & U \arrow[d] \\ & X \arrow[r, "F_X"'] & X \end{tikzcd}$$

is a radiciel surjection (since F_U and F_X both are). This map is also a map of étale X-schemes, and thus it is an étale radiciel surjection; by [**28**, 17.9.1], it is therefore an isomorphism. This is a vast generalization of the fact that if K/k is a finite separable extension of fields, then $k \cdot K^p = K$. The isomorphism (5.3.1.2) yields an isomorphism $\mathcal{F}(F_X^{-1}(U)) \xrightarrow{\simeq} \mathcal{F}(U)$, and this defines (5.3.1.1).

The adjoint sheaf map to (5.3.1.1) is denoted

(5.3.1.3) $$\mathrm{Fr}_{\mathcal{F}} : F_X^* \mathcal{F} \to \mathcal{F};$$

for sheaves of modules this is visibly compatible with tensor constructions (e.g., symmetric powers). Note also that when $\mathcal{F} = \underline{M}$ is a constant sheaf for a finite set M, then the composite of $\mathrm{Fr}_{\underline{M}}$ and the *canonical* isomorphism $\underline{M} \simeq F_X^*(\underline{M})$ is the identity. (To prove this, use that $\underline{M}$ is representable and chase definitions.) It is also clear that $\mathrm{Fr}_{\mathcal{F}}$ is compatible with Zariski localization on X.

This construction is compatible with higher direct images in the sense that if $f : X \to S$ is a map of $\mathbf{F}_p$-schemes, so the diagram

$$\begin{tikzcd} X \arrow[r, "F_X"] \arrow[d, "f"'] & X \arrow[d, "f"] \\ S \arrow[r, "F_S"'] & S \end{tikzcd}$$

commutes (although it is *not* cartesian in general), then the composite

$$F_S^* \mathrm{R}^i f_* \mathcal{F} \to \mathrm{R}^i f_*(F_X^* \mathcal{F}) \xrightarrow{\mathrm{R}^i f_*(\mathrm{Fr}_{\mathcal{F}})} \mathrm{R}^i f_* \mathcal{F} \tag{5.3.1.4}$$

is equal to $\mathrm{Fr}_{\mathrm{R}^i f_* \mathcal{F}}$. Briefly, the method of proof is to use universal δ-functor arguments to reduce to the case $i = 0$; this case is handled by adjointness and the commutativity of

$$\begin{tikzcd} f_* \arrow[r, "\simeq"] \arrow[dr, "\simeq"'] & \mathrm{Fr}_{X*}\, f_* \arrow[d, equal] \\ & f_*\, \mathrm{Fr}_{S*} \end{tikzcd}$$

This all takes a particularly nice form when $S = \operatorname{Spec} \mathbf{F}_p$, for then F_S is the *identity*. However, the more general picture (over any S) keeps clear what is going on. Note also that even when $S = \operatorname{Spec} \mathbf{F}_p$, the operation (for $\mathcal{F}$ an étale sheaf on S)

$$\mathrm{Fr}_{\mathcal{F}} : F_S^* \mathcal{F} \to \mathcal{F}$$

is *not* generally the identity. Since F_S is the identity map, $\mathrm{Fr}_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}$ is exactly the *inverse* of the map $\mathcal{F}(U) \to \mathcal{F}(U)$ induced by pullback along F_U. That is, in terms of discrete left $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p)$-sets, $\mathrm{Fr}_{\mathcal{F}}$ is the action of *geometric Frobenius* (under the dictionary of Theorem 5.1.1.1). This will be vastly generalized in Lemma 5.3.2.1 below.

5.3.2. Frobenius actions on the universal elliptic curve. Applying the above considerations to $f_N : E_1(N) \to Y_1(N)_{/\mathbf{F}_p}$, the *canonical* map

$$F_{Y_1(N)}^* \mathrm{R}^1 F_{N*}(\underline{\mathbf{Z}/l^n}) \to \mathrm{R}^1 f_{N*}(F_{E_1(N)}^* \underline{\mathbf{Z}/l^n}) = \mathrm{R}^1 f_{N*}(\underline{\mathbf{Z}/l^n}) \tag{5.3.2.1}$$

is $\mathrm{Fr}_{\mathrm{R}^1 f_{N*}(\underline{\mathbf{Z}/l^n})}$. (Here we are using the fact that $\mathrm{Fr}_{\underline{M}}$ *is* the canonical isomorphism $F^* \underline{M} \simeq \underline{M}$, as was noted after (5.3.1.3).) The analogous assertion follows for $\operatorname{Sym}^k \mathrm{R}^1 f_{N*} \underline{\mathbf{Z}/l^n}$, by tensor compatibility.

Using the map (5.3.2.1) and the structure map $a : Y_1(N)_{/\mathbf{F}_p} \to \operatorname{Spec} \mathbf{F}_p$, we get an étale sheaf map

$$\begin{aligned} F_{\mathbf{F}_p}^* \mathrm{R}^1 a_!(\operatorname{Sym}^k \mathrm{R}^1 f_{N*} \underline{\mathbf{Z}/l^n}) \to \mathrm{R}^1 a_!(F_{Y_1(N)}^* \operatorname{Sym}^k \mathrm{R}^1 F_{N*} \underline{\mathbf{Z}/l^n}) \to \\ \mathrm{R}^1 a_!(\operatorname{Sym}^k \mathrm{R}^1 f_{N*} \underline{\mathbf{Z}/l^n}) \end{aligned}$$

which we claim *is*

$$\mathrm{Fr}_{\mathrm{R}^1 a_!(\mathrm{Sym}^k \mathrm{R}^1 f_{N*}\underline{\mathbf{Z}/l^n})}\,.$$

More generally, we have the next lemma.

LEMMA 5.3.2.1. *For any finite-type separated $\mathbf{F}_p$-scheme $f : X \to \operatorname{Spec}\mathbf{F}_p$ with absolute Frobenius F_X, and for any torsion sheaf $\mathcal{F}$ on X, the endomorphism*

$$\mathrm{R}^i f_! \mathcal{F} = F^*_{\mathbf{F}_p}\mathrm{R}^i f_!\mathcal{F} \to \mathrm{R}^i f_!(F_X^*\mathcal{F}) \xrightarrow{\mathrm{Fr}_{\mathcal{F}}} \mathrm{R}^i f_!\mathcal{F}$$

of the étale sheaf on $(\operatorname{Spec}\mathbf{F}_p)_{\text{ét}}$ *is equal to* $\mathrm{Fr}_{\mathrm{R}^i f_!\mathcal{F}}$.

PROOF. Let $j : X \to \overline{X}$ be an open immersion into a proper $\mathbf{F}_p$-scheme with structure map $\overline{f} : \overline{X} \to \operatorname{Spec}\mathbf{F}_p$. Then the diagram

$$\begin{tikzcd} j_! F_X^*\mathcal{F} \arrow[r, "j_!\,\mathrm{Fr}_{\mathcal{F}}"] \arrow[d, "\simeq"'] & j_!\mathcal{F} \\ F^*_{\overline{X}}(j_!\mathcal{F}) \arrow[ur, "\mathrm{Fr}_{j_!\mathcal{F}}"'] & \end{tikzcd}$$

commutes; indeed, since the stalks are 0 away from X, this can be checked on the open $X \hookrightarrow \overline{X}$, and here it is just the fact that our Frobenius construction (5.3.1.3) respects Zariski localization.

Replacing f by $\overline{f}$ and $\mathcal{F}$ by $j_!\mathcal{F}$ reduces us to the case of higher direct images. This case is treated by universal δ-functor arguments as mentioned in our consideration of (5.3.1.4). □

By the compatibility checked after Theorem 5.2.6.1 (and the *definition* $\mathrm{R}^\bullet f_! = \mathrm{R}^\bullet \overline{f}_* \circ j_!$), we see that in the setup of Lemma 5.3.2.1 on a geometric fiber $\operatorname{Spec}\overline{\mathbf{F}}_p \to \operatorname{Spec}\mathbf{F}_p$ the endomorphism of the étale sheaf $\mathrm{R}^i f_!\mathcal{F}$ corresponds to the endomorphism of $\mathrm{H}^i_{\mathrm{c,ét}}(X_{/\overline{\mathbf{F}}_p}, \mathcal{F}_{/\overline{\mathbf{F}}_p})$ induced by pullback along geometric Frobenius $\phi^{-1} \in \operatorname{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p)$. Of course, this *is* a $\operatorname{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p)$-module endomorphism (as it *must* be, coming from an étale sheaf map on $\operatorname{Spec}\mathbf{F}_p$), since $\operatorname{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p)$ is commutative!

To summarize, the action of a *geometric* Frobenius in $G_{\mathbf{Q}}$ on the *generic* fiber of the locally constant constructible sheaf

$$\mathrm{R}^1 a_!\big(\mathrm{Sym}^k \mathrm{R}^1 f_{N*}\underline{\mathbf{Z}/l^n}\big)$$

over $(\operatorname{Spec}\mathbf{Z}[1/Nl])_{\text{ét}}$ is identified (via base change to the $\mathbf{F}_p$-fiber) with the pullback action induced by the *characteristic p* diagram

$$\begin{tikzcd} E_1(N) \arrow[r, "F_{E_1(N)}"] \arrow[d, "f_N"'] & E_1(N) \arrow[d, "f_N"] \\ Y_1(N)_{/\mathbf{F}_p} \arrow[r, "F_{Y_1(N)}"'] & Y_1(N)_{/\mathbf{F}_p} \end{tikzcd}$$

Note that this action makes perfectly good sense on the $\mathbf{F}_p$-fiber of the locally constant constructible sheaf $\mathrm{R}^1 a_*(\mathrm{Sym}^k \mathrm{R}^1 f_{N*}\underline{\mathbf{Z}/l^n})$ as well. Since

$$\mathrm{R}^1 a_! \to \mathrm{R}^1 a_*$$

respects base change and pullback maps, as does $\mathrm{R}^1 f_{N*}$, we arrive at the following crucial "geometrization" of Galois actions in our situation of interest.

LEMMA 5.3.2.2. *Under the canonical identification of ${}^k_N W_{\mathbf{Z}/l^n}$ with the characteristic p geometric fiber of the locally constant constructible sheaf*

$$\mathrm{R}^1\widetilde{a}(\mathrm{Sym}^k\,\mathrm{R}^1 f_{N*}(\underline{\mathbf{Z}/l^n}))$$

on $(\mathrm{Spec}\,\mathbf{Z}[1/Nl])_{\text{ét}}$*, the action on ${}^k_N W_{\mathbf{Z}/l^n}$ by geometric Frobenius at p in $G_{\mathbf{Q}}$ is identified with the endomorphism of sheaves on* $(\mathrm{Spec}\,\mathbf{F}_p)_{\text{ét}}$

$$\begin{aligned} F : \mathrm{R}^1\widetilde{a}\big(\mathrm{Sym}^k\,\mathrm{R}^1 f_{N*}\underline{\mathbf{Z}/l^n}\big) &\longrightarrow \mathrm{R}^1\widetilde{a}\big(F^*_{Y_1(N)}\,\mathrm{Sym}^k\,\mathrm{R}^1 f_{N*}\underline{\mathbf{Z}/l^n}\big) \longrightarrow \\ &\mathrm{R}^1\widetilde{a}\big(\mathrm{Sym}^k\,\mathrm{R}^1 f_{N*}F^*_{E_1(N)}\underline{\mathbf{Z}/l^n}\big) = \mathrm{R}^1\widetilde{a}\big(\mathrm{Sym}^k\,\mathrm{R}^1 f_{N*}\underline{\mathbf{Z}/l^n}\big) \end{aligned}$$

5.3.3. Hecke identities. We are now ready to prove the identities

$$T_p = F + \langle p\rangle\, p^{k+1}F^{-1} \qquad w_\zeta^{-1}Fw_\zeta = \langle\varepsilon_{d,p}\rangle^{-1}\,F \tag{5.3.3.1}$$

from Theorem 4.1.3.1; this will complete the proof of Main Theorem 2. The point is that, thanks to Theorem 5.2.8.1 and Lemma 5.3.2.2, we have a canonical identification between the $\overline{\mathbf{Q}}$-fiber

$$\widetilde{\mathrm{H}}^1_{\text{ét}}(Y_1(N)_{/\overline{\mathbf{Q}}}, \mathrm{Sym}^k\,\mathrm{R}^1\overline{f}_{N*}\underline{\mathbf{Z}/l^n})$$

and the $\overline{\mathbf{F}}_p$-fiber

$$\widetilde{\mathrm{H}}^1_{\text{ét}}(Y_1(N)_{/\overline{\mathbf{F}}_p}, \mathrm{Sym}^k\,\mathrm{R}^1\overline{f}_{N*}\underline{\mathbf{Z}/l^n})$$

of the locally constant constructible sheaf $\mathrm{R}^1\widetilde{a}(\mathrm{Sym}^k\,\mathrm{R}^1 f_{N*}\underline{\mathbf{Z}/l^n})$ on the etale site over $\mathbf{Z}[1/N]$, with the the action of a geometric Frobenius element at p in $G_{\mathbf{Q}}$ on the former going over to the action of F on the latter which can be interpreted via pullback by *geometric* maps on cohomology (rather than as just pullback by the "algebraic" action of geometric Frobenius from $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p)$). Thus, in order to get the identities (5.3.3.1) we are reduced to prove the following theorem about geometric maps on cohomology:

THEOREM 5.3.3.1. *Let $f_N : E_1(N)_{/\mathbf{F}_p} \to Y_1(N)_{/\mathbf{F}_p}$ and $a : Y_1(N)_{/\mathbf{F}_p} \to \mathrm{Spec}\,\mathbf{F}_p$ denote the structure maps on characteristic p fibers. We have the identities*

$$T_p = F + \langle p\rangle\, p^{k+1}F^{-1} \qquad w_\zeta^{-1}Fw_\zeta = \langle\varepsilon_{d,p}\rangle^{-1}\,F$$

on $\mathrm{R}^1\widetilde{a}(\mathrm{Sym}^k\,\mathrm{R}^1 f_{N*}\underline{\mathbf{Z}/l^n})$ *and its $\overline{\mathbf{F}}_p$-fiber respectively; here $\zeta \in \overline{\mathbf{F}}_p$ is a primitive dth root of unity with d dividing N and* $\gcd(d, \frac{N}{d}) = 1$.

PROOF. We begin with the second relation. By Lemma 5.3.2.2, the action of F on $\widetilde{\mathrm{H}}^1_{\text{ét}}(Y_1(N)_{/\overline{\mathbf{F}}_p}, \mathrm{Sym}^k\,\mathrm{R}^1\overline{f}_{N*}\underline{\mathbf{Z}/l^n})$ is induced by the $\overline{\mathbf{F}}_p$-base changes of $F_{E_1(N)}$ and $F_{Y_1(N)}$ over $\mathbf{F}_p$.

Recall from Theorem 4.2.8.4 the identity

$$\varphi_\zeta\overline{F}_{E_1(N)} = \overline{F}_{E_1(N)}I^{-1}_{\varepsilon_{p,d}}\varphi_\zeta \tag{5.3.3.2}$$

on $E_1(N)_{/\overline{\mathbf{F}}_p}$, lying over the identity $w_\zeta\overline{F}_{Y_1(N)} = \overline{F}_{Y_1(N)}I^{-1}_{\varepsilon_{p,d}}w_\zeta$ on $Y_1(N)_{/\overline{\mathbf{F}}_p}$. By definition (and Theorem 5.2.8.1), the action of w_ζ on

$$\widetilde{\mathrm{H}}^1_{\text{ét}}(Y_1(N)_{/\overline{\mathbf{F}}_p}, \mathrm{Sym}^k\,\mathrm{R}^1 f_{N*}\underline{\mathbf{Z}/l^n})$$

is induced by the commutative diagram

$$\begin{tikzcd} E_1(N)_{/\overline{\mathbf{F}}_p} \arrow[r, "\varphi_\zeta"] \arrow[d, "f_N"'] & E_1(N)_{/\overline{\mathbf{F}}_p} \arrow[d, "f_N"] \\ Y_1(N)_{/\overline{\mathbf{F}}_p} \arrow[r, "w_\zeta"'] & Y_1(N)_{/\overline{\mathbf{F}}_p} \end{tikzcd}$$

while F and $\langle \varepsilon_{p,d} \rangle^{-1}$ are induced by the base change to $\overline{\mathbf{F}}_p$ of the commutative diagrams

$$\begin{tikzcd} E_1(N) \arrow[r, "F_{E_1(N)}"] \arrow[d] & E_1(N) \arrow[d] \\ Y_1(N)_{/\mathbf{F}_p} \arrow[r, "F_{Y_1(N)}"'] & Y_1(N)_{/\mathbf{F}_p} \end{tikzcd}$$

$$\begin{tikzcd} E_1(N) \arrow[r, "I_{\varepsilon_{p,d}^{-1}}"] \arrow[d] & E_1(N) \arrow[d] \\ Y_1(N)_{/\mathbf{F}_p} \arrow[r, "I_{\varepsilon_{p,d}^{-1}}"'] & Y_1(N)_{/\mathbf{F}_p} \end{tikzcd}$$

respectively. The (contravariant) *functoriality* of pullback (on $\mathrm{H}^1_{\text{ét}}$ and $\mathrm{H}^1_{\text{c,ét}}$, and thus on $\widetilde{\mathrm{H}}^1_{\text{ét}}$ as well) with respect to composite commutative squares now implies that the desired identity $w_\zeta^{-1} F w_\zeta = \langle \varepsilon_{p,d} \rangle^{-1} F$ follows from (5.3.3.2).

We now turn to showing that $T_p = F + \langle p \rangle p^{k-1} F^{-1}$. This is intrinsic to the étale sheaf $\mathrm{R}^1\widetilde{a}(\mathrm{Sym}^k f_{N*}\underline{\mathbf{Z}/l^n})$ on $\mathrm{Spec}\,\mathbf{F}_p$, as it involves only geometric maps defined directly over $\mathbf{F}_p$. There will be no appearance of $\overline{\mathbf{F}}_p$ in our analysis.

Consider the diagram over $\mathbf{F}_p$:

$$\begin{tikzcd} & Y_1(N)\coprod Y_1(N) \arrow[r, "\Phi_1\coprod\Phi_2"] \arrow[dl, "1\coprod F"'] \arrow[drr, "F\coprod I_p"' near end] & Y_1(N,p) \arrow[dll, "\pi_1" near end] \arrow[dr, "\pi_2"] & \\ Y_1(N) & & & Y_1(N) \end{tikzcd}$$

where $F = F_{Y_1(N)}$; see Corollary 4.2.8.3. Over this we have diagrams and cartesian squares of elliptic curves, as indicated in the two diagrams in the proof of Corollary 4.2.8.3, over $1 \coprod F$ and $F \coprod I_p$ (and over $\pi_1, \pi_2, \Phi_1, \Phi_2$ these diagrams are the ones from the *definitions* of these maps over $\mathbf{F}_p$).

Note that the universal p-isogeny $\varphi : E_1(N,p) \to E_1(N,p)/C$ over $Y_1(N,p)$ in the *definition* of T_p pulls back along $\Phi_1 \coprod \Phi_2$ to exactly the disjoint union of the p-isogenies $F_{E_1(N)/Y_1(N)}$ and its dual isogeny $V_{E_1(N)/Y_1(N)}$. It now follows from Deligne's rather general and formal compatibility lemma [**10**, Lemma 4.6] that the operator

$$T_p = \mathrm{tr}_{\pi_1} \circ \varphi^* \circ \pi_2^*$$

on $\mathrm{R}^1\widetilde{a}(\mathrm{Sym}^k \mathrm{R}^1 f_N * \underline{\mathbf{Z}/l^n})$ is *equal* to

$$\mathrm{tr}_{1\coprod F} \circ \left(F^*_{E_1(N)/Y_1(N)} \coprod V^*_{E_1(N)/Y_1(N)}\right) \circ \left(F \coprod I_p\right)^*.$$

This in turn is the sum of the pullback along the commutative diagram

$$\begin{tikzcd} E_1(N) \arrow[r, "F_{E_1(N)}"] \arrow[d, "f_N"'] & E_1(N) \arrow[d, "f_N"] \\ Y_1(N)_{/\mathbf{F}_p} \arrow[r, "F_{Y_1(N)}"'] & Y_1(N)_{/\mathbf{F}_p} \end{tikzcd}$$

(which is *exactly* the operator F in Lemma 5.3.2.2) and the composite of $\langle p\rangle$ followed by

$$\mathrm{R}^1\widetilde{a}\big(\mathrm{Sym}^k\,\mathrm{R}^1 f_{N*}\underline{\mathbf{Z}/l^n}\big) \overset{V^*_{E_1(N)/Y_1(N)}}{\longrightarrow} \mathrm{R}^1\widetilde{a}\big(\mathrm{Sym}^k\,\mathrm{R}^1 f^{(p)}_{N*}\underline{\mathbf{Z}/l^n}\big) \overset{\mathrm{tr}_{F_{Y_1(N)}}}{\longrightarrow} \mathrm{R}^1\widetilde{a}\big(\mathrm{Sym}^k\,\mathrm{R}^1 f_{N*}\underline{\mathbf{Z}/l^n}\big).$$

Here, $\mathrm{tr}_{F_{Y_1(N)}}$ is computed relative to the *cartesian* diagram

(5.3.3.3)
$$\begin{tikzcd} E_1(N)^{(p)} \arrow[r] \arrow[d, "f_N^{(p)}"'] & E_1(N) \arrow[d, "f_N"] \\ Y_1(N) \arrow[r, "F_{Y_1(N)}"'] & Y_1(N) \end{tikzcd}$$

over the *finite flat* map $F_{Y_1(N)}$, using the proper base change isomorphism

$$\mathrm{R}^1 f^{(p)}_{N*}\underline{\mathbf{Z}/l^n} \xrightarrow{\simeq} F^*_{Y_1(N)}\mathrm{R}^1 f_{N*}\underline{\mathbf{Z}/l^n}.$$

(Keep in mind that $\mathrm{R}^\bullet a_* \circ F_{Y_1(N)*} = \mathrm{R}^\bullet a_*$ and $\mathrm{R}^\bullet a_! \circ F_{Y_1(N)*} = \mathrm{R}^\bullet a_!$ since $aF_{Y_1(N)} = F_{\mathbf{F}_p}a$ and $F_{Y_1(N)}$ is finite.)

In other words, we see that

$$T_p = F + \mathrm{tr}_{F_Y} \circ V^*_{E_1(N)/Y_1(N)}\,\langle p\rangle\,.$$

Since $\langle p\rangle$ and F commute (by the algebraic interpretation of F as geometric Frobenius and the fact that $\langle p\rangle$ is defined over $\mathbf{F}_p$) it remains to show that

$$\mathrm{tr}_{F_Y} \circ V^*_{E_1(N)/Y_1(N)} \overset{?}{=} p^{k+1}F^{-1}.$$

Put differently, we must show that

$$\mathrm{tr}_{F_Y} \circ V^*_{E_1(N)/Y_1(N)} \circ F^*_{E_1(N)/Y_1(N)} \circ F^*_Y \overset{?}{=} p^{k+1}$$

on $\mathrm{R}^1\widetilde{a}(\mathrm{Sym}^k\,\mathrm{R}^1 f_{N*}\underline{\mathbf{Z}/l^n})$. Here, the outer factors are the trace and pullback relative to the *cartesian* square (5.3.3.3). The composite isogeny $F_{E_1(N)/Y_1(N)} \circ V_{E_1(N)/Y_1(N)}$ is p on $E_1(N)$. By Lemma 5.3.3.2 below, $V^*_{E_1(N)/Y_1(N)} \circ F^*_{E_1(N)/Y_1(N)}$ therefore induces multiplication by p on $\mathrm{R}^1 f_{N*}\underline{\mathbf{Z}/l^n}$, hence p^k on $\mathrm{Sym}^k\,\mathrm{R}^1 f_{N*}\underline{\mathbf{Z}/l^n}$.

We are now left with checking that

$$\mathrm{tr}_{F_Y} \circ F^*_Y = p$$

on $\mathrm{R}^1\widetilde{a}(\mathrm{Sym}^k \mathrm{R}^1 f_{N*}\underline{\mathbf{Z}/l^n})$. Since $F_Y : Y_1(N) \to Y_1(N)$ is finite flat of degree p, we're reduced to checking that if

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\overline{a}"'] & & Y \arrow[dl, "a"] \\ & S & \end{tikzcd}$$

is a commutative diagram with a finite type separated, f finite flat of degree d, and if $\mathcal{F}$ is a torsion sheaf on Y, then the map

$$\mathrm{R}^i a_*\mathcal{F} \to \mathrm{R}^i(a \circ f)_* f^*\mathcal{F} \simeq \mathrm{R}^i a_*(f_* f^*\mathcal{F}) \xrightarrow{\mathrm{tr}_{\mathcal{F}}} \mathrm{R}^i a_*\mathcal{F}$$

is multiplication by d. This composite is $\mathrm{R}^i a_*$ applied to

$$\mathcal{F} \to f_* f^*\mathcal{F} \xrightarrow{\mathrm{tr}_{\mathcal{F}}} \mathcal{F}$$

which *is* multiplication by d via the stalk description (5.2.10.2) of the trace map. This completes the proof. □

It remains to prove the following lemma which we used above and which we will need later. We remind the reader that this completes the proof of Main Theorem 2.

LEMMA 5.3.3.2. *Let $f : A \to S$ be an abelian scheme and let n be any integer. Then for any integer d, the multiplication-by-d map $[d] : A \to A$ induces a pullback map*

$$\mathrm{R}^1 f_*\underline{\mathbf{Z}/n} \to \mathrm{R}^1 f_*\underline{\mathbf{Z}/n}$$

which is multiplication by d.

PROOF. By proper base change and Theorem 5.1.4.2(2), we may assume that $S = \mathrm{Spec}\, k$ for an algebraically closed field k. By Theorem 5.2.2.1 it suffices to show that the map $\pi_1(A, 0) \to \pi_1(A, 0)$ induced by $[d]$ is multiplication by d. Note that $\pi_1(A, 0)$ is canonically identified (as a topological group) with the (abelian) "$\widehat{\mathbf{Z}}$-Tate module" of A by [**45**, Section 18]. This makes the assertion clear. □

5.4. Application of the Weil conjectures

The goal of this section is to use the Riemann hypothesis part of the Weil conjectures to prove that the eigenvalues of F on ${}^k_N W_l$ (for $N \geq 5$) are algebraic integers with all complex absolute values equal to $p^{\frac{k+1}{2}}$. By Corollary 4.1.3.3, this yields the same results for $N \leq 4$ and thus completes the proof of Main Theorem 1(1).

5.4.1. The weight 2 case. We first dispose of the case $k = 0$, since later on we will want to work with the relative dimension k abelian scheme $E_1(N)^k \to Y_1(N)$. By Theorem 5.2.8.1,

$${}^0_N W_{\mathbf{Z}/l^n} = \mathrm{R}^1\widetilde{a}\big(\underline{\mathbf{Z}/l^n}\big)$$

has $\overline{\mathbf{F}}_p$-fiber equal to the image of

$$\mathrm{H}^1_{\mathrm{c,\acute{e}t}}\big(Y_1(N)_{/\overline{\mathbf{F}}_p}, \underline{\mathbf{Z}/l^n}\big) \to \mathrm{H}^1_{\mathrm{\acute{e}t}}\big(X_1(N)_{/\overline{\mathbf{F}}_p}, \underline{\mathbf{Z}/l^n}\big) \tag{5.4.1.1}$$

where $X_1(N)_{/\overline{\mathbf{F}}_p}$ denotes the (geometrically connected) base change to $\overline{\mathbf{F}}_p$ of the unique proper smooth curve over $\mathbf{F}_p$ containing the (geometrically connected) smooth curve $Y_1(N)_{/\mathbf{F}_p}$ as a dense open. From the excision sequence for cohomology with proper supports and the fact that $(X_1(N) - Y_1(N))_{\overline{\mathbf{F}}_p}$ is finite and thus cohomologically trivial, (5.4.1.1) is an isomorphism. Passing to the limit, ${}^0_N W_l$ with its geometric Frobenius action is identified with

$$\mathbf{Q}_l \otimes_{\mathbf{Z}_l} \varprojlim_n \mathrm{H}^1_{\text{ét}}\big(X_1(N)_{/\overline{\mathbf{F}}_p}, \underline{\mathbf{Z}/l^n}\big)$$

with its canonical action of geometric Frobenius.

A special case of the general theory of constructible sheaves and l-adic sheaves [**17**, Ch I, §8.10 and Ch I, §12.15] is the following result (which includes Lemma 5.2.7.2).

THEOREM 5.4.1.1. *Let X be a separated scheme of finite type over a separably closed field k and let $\mathcal{F}$ be a locally constant constructible sheaf on X. Then $\mathrm{H}^i_{\mathrm{c},\text{ét}}(X, \mathcal{F})$ is a finite group. For any prime l,*

$$\varprojlim_n \mathrm{H}^i_{\mathrm{c},\text{ét}}(X, \underline{\mathbf{Z}/l^n})$$

is a finitely generated $\mathbf{Z}_l$-module.

REMARK 5.4.1.2. The finiteness rests on knowing that $\mathrm{H}^i_{\mathrm{c},\text{ét}}(X, \cdot)$ *vanishes* on *arbitrary* torsion sheaves for i larger than twice the dimension of X.

The Riemann hypothesis concerns the following situation. Let $X \to \operatorname{Spec}\kappa$ be a smooth projective scheme over a finite field κ. Choose a separable algebraic closure $\overline{\kappa}$ of κ and a prime l different from the characteristic of κ. Consider the finite $\mathbf{Z}_l$-module

$$\mathrm{H}^i_{\text{ét}}(X \times_{\operatorname{Spec}\kappa} \operatorname{Spec}\overline{\kappa}, \mathbf{Z}_l) \overset{\text{def}}{=} \varprojlim_n \mathrm{H}^i_{\text{ét}}\big(X \times_{\operatorname{Spec}\kappa} \operatorname{Spec}\overline{\kappa}, \underline{\mathbf{Z}/l^n}\big). \tag{5.4.1.2}$$

The profinite topology is the *unique* structure of compact topological $\mathbf{Z}_l$-module on (5.4.1.2), so it coincides with the natural l-adic topology. The geometric Frobenius $\phi_\kappa^{-1} \in \operatorname{Gal}(\overline{\kappa}/\kappa)$ acts continuously on (5.4.1.2), and the Riemann hypothesis is the following assertion which is the main result of [**11**] (and significantly generalized in [**12**], including the replacement of projectiveness by properness above):

THEOREM 5.4.1.3 (Deligne). *The characteristic polynomial of ϕ_κ^{-1} on*

$$\mathrm{H}^i_{\text{ét}}(X \times_{\operatorname{Spec}\kappa} \overline{\kappa}, \mathbf{Q}_l) \overset{\text{def}}{=} \mathbf{Q}_l \otimes_{\mathbf{Z}_l} \mathrm{H}^i_{\text{ét}}(X \times_{\operatorname{Spec}\kappa} \operatorname{Spec}\overline{\kappa}, \mathbf{Z}_l)$$

lies in $\mathbf{Z}[T]$, is independent of l and has roots which are algebraic integers with all complex absolute values $|\kappa|^{\frac{i}{2}}$.

Applying this to $X_1(N)_{/\mathbf{F}_p}$ (with $\kappa = \mathbf{F}_p$), the case $k = 0$ is settled.

5.4.2. Cohomology of relative curves. From now on we assume $k > 0$. For reasons of space, we have to bypass a general discussion of l-adic sheaves (though, as with constructible sheaves, they form an essential technical tool for an understanding of étale cohomology). We will therefore exploit some special facts about curves. This is admittedly ad hoc.

LEMMA 5.4.2.1. *Let $f : X \to S$ be proper smooth with geometrically connected fibers of dimension 1 and genus g. Let $f_k : X^k \to S$ be the k-fold fiber product of X over S. For any integer n invertible on S, there are canonical isomorphisms*

$$\underline{\mathbf{Z}/n} \simeq f_*\underline{\mathbf{Z}/n} \qquad \mathrm{R}^2 f_*\underline{\mathbf{Z}/n} \simeq \mu_n^{-1}$$

compatible with base change, and $\mathrm{R}^1 f_* \underline{\mathbf{Z}/n}$ *is locally free of rank* $2g$ *over* $\underline{\mathbf{Z}/n}$*. All of these are compatible with change of* n*.*

The cup product pairing

$$\bigoplus_{i_1+\cdots+i_k=j} (\mathrm{R}^{i_1} f_* \underline{\mathbf{Z}/n} \otimes \cdots \otimes \mathrm{R}^{i_k} f_* \underline{\mathbf{Z}/n}) \to \mathrm{R}^j f_{k*} \underline{\mathbf{Z}/n},$$

which is compatible with base change and change in n*, is an isomorphism. In particular,* $\mathrm{R}^j f_{k*} \underline{\mathbf{Z}/n}$ *is locally free of finite rank over* $\underline{\mathbf{Z}/n}$*, compatibly with change in* n*.*

PROOF. By proper base change we may assume that $S = \operatorname{Spec} k$ for an algebraically closed field k, as Theorem 5.2.7.3 guarantees that the locally constant constructible condition and the isomorphism $\mathrm{R}^2 f_* \underline{\mathbf{Z}/n} \simeq \mu_n^{-1}$ follows from the general theory of the trace map, compatibly with change in n. Also, the isomorphism $\mathrm{H}^1_{\text{ét}}(X, \underline{\mathbf{Z}/n}) \simeq \mathrm{Pic}^0(X)[n]$ shows that $\mathrm{H}^1_{\text{ét}}(X, \underline{\mathbf{Z}/n})$ is indeed free of rank $2g$.

If k has positive characteristic p, then we can pick a complete discrete valuation ring W and by [**23**] we may find a proper smooth curve over $\mathrm{Spec}(W)$ whose closed fiber is X. By Theorem 5.2.7.3, it suffices to work on the geometric generic fiber. Thus, we may assume k has characteristic 0. By direct limit methods we may assume that $k = \mathbf{C}$, so by the comparison theorem we reduce to the well-known analogue for $\mathbf{Z}/n$-cohomology of compact Riemann surfaces. Alternatively, one can bypass lifting to characteristic 0 by using the general Künneth formula in étale cohomology (but beware that the extremely quick proof of this in [**17**, Ch I, 8.17] omits the rather long and difficult verification that the projection formula isomorphism there *is* the cup product map). ☐

From Lemma 5.4.2.1, we conclude that for any relative curve $f : X \to S$ over a $\mathbf{Z}[1/Nl]$-scheme S, the sheaf

$$\left(\mathrm{R}^1 f_* \underline{\mathbf{Z}/l^n}\right)^{\otimes k}$$

is a direct summand of $\mathrm{R}^k f_{k*} \underline{\mathbf{Z}/l^n}$, compatible with change in n. Here $f_k : X^k \to S$ is the k-fold fiber product of $X \to S$. Furthermore, the natural map of locally free $\mathbf{Z}/l^n$-modules of finite rank

$$\mathrm{R}^1 f_* (\underline{\mathbf{Z}/l^n})^{\otimes k} \to \mathrm{Sym}^k \mathrm{R}^1 f_* \underline{\mathbf{Z}/l^n}$$

is surjective, and its composite with the canonical "symmetrization" map

$$\mathrm{Sym}^k \mathrm{R}^1 f_* \underline{\mathbf{Z}/l^n} \to \mathrm{R}^1 f_* (\underline{\mathbf{Z}/l^n})^{\otimes k}$$

is just multiplication by $k!$.

We also have the following very special case of Grothendieck's Poincaré duality theorem [**59**, 4.8], whose proof ultimately reduces to the perfectness of (5.2.11.1). Recall also the finiteness result stated in Theorem 5.4.1.1.

THEOREM 5.4.2.2. *Let* X *be a smooth, connected, finite type, separated scheme over a separably closed field* k*. Let* $\mathcal{F}$ *be a locally constant constructible sheaf of* $\mathbf{Z}/n$*-modules on* X *which are locally free of finite rank over* $\underline{\mathbf{Z}/n}$*. Assume that* n *is not divisible by the characteristic of* k*. Then the cup product*

$$\mathrm{H}^i_{\text{ét}}\big(X, \mathcal{F}^\vee \otimes \mu_n^{\otimes d}\big) \otimes \mathrm{H}^{2d-i}_{\mathrm{c},\text{ét}}\big(X, \mathcal{F}\big) \to \mathrm{H}^{2d}_{\mathrm{c}}\big(X, \mu_n^{\otimes d}\big) \xrightarrow{\simeq} \mathbf{Z}/n$$

is a perfect pairing. In particular, $\mathrm{H}^i_{\text{ét}}(X, \mathcal{F}^\vee \otimes \mu_n^{\otimes d})$ *is finite.*

5.4.3. Conclusion of the proof. Returning to the previous situation involving a relative curve $f : X \to S$, assume that S is smooth, separated, and of finite type over a separably closed field. We conclude from the Mittag-Leffler lemma (the exactness of formation of inverse limits for projective systems of finite abelian groups) that

$$\varprojlim_n \widetilde{\mathrm{H}}^1_{\text{ét}}\big(S, \operatorname{Sym}^k \mathrm{R}^1 f_* \underline{\mathbf{Z}/l^n}\big)$$

is the image of the map

$$\varprojlim_n \mathrm{H}^1_{\mathrm{c},\text{ét}}\big(S, \operatorname{Sym}^k \mathrm{R}^1 f_* \underline{\mathbf{Z}/l^n}\big) \longrightarrow \varprojlim_n \mathrm{H}^1_{\text{ét}}\big(S, \operatorname{Sym}^k \mathrm{R}^1 f_* \underline{\mathbf{Z}/l^n}\big)$$

and likewise for $\mathrm{R}^1 f_*(\underline{\mathbf{Z}/l^n})^{\otimes k}$ in place of $\operatorname{Sym}^k(\mathrm{R}^1 f_*(\underline{\mathbf{Z}/l^n}))$. Thus,

$$\mathbf{Q}_l \otimes_{\mathbf{Z}_l} \varprojlim \ \widetilde{\mathrm{H}}^1_{\text{ét}}\big(S, \operatorname{Sym}^k \mathrm{R}^1 f_* \underline{\mathbf{Z}/l^n}\big)$$

is naturally a direct summand of

$$\mathbf{Q}_l \otimes_{\mathbf{Z}_l} \varprojlim \ \widetilde{\mathrm{H}}^1_{\text{ét}}\big(S, \mathrm{R}^1 f_* (\underline{\mathbf{Z}/l^n})^{\otimes k}\big).$$

(This uses the fact that $k!$ lies in $\mathbf{Q}_l^\times$.) This in turn is a direct summand of

$$\mathbf{Q}_l \otimes_{\mathbf{Z}_l} \varprojlim \ \widetilde{\mathrm{H}}^1_{\text{ét}}\big(S, \mathrm{R}^k f_{k*} \underline{\mathbf{Z}/l^n}\big).$$

This is all functorial in $f : X \to S$, and in particular if S is an $\mathbf{F}_p$-scheme then the equality $f \circ F_X = F_S \circ f$ implies that the natural Frobenius actions (induced by F_S, F_X) respect all of the maps above.

Specializing to the case of

$$f_N : E_1(N)_{/\overline{\mathbf{F}}_p} \to Y_1(N)_{\overline{\mathbf{F}}_p},$$

we conclude that

$${}^k_N W_l = \mathbf{Q}_l \otimes_{\mathbf{Z}_l} \varprojlim_n \widetilde{\mathrm{H}}^1_{\text{ét}}\big(Y_1(N)_{/\overline{\mathbf{F}}_p}, \operatorname{Sym}^k \mathrm{R}^1 f_{N*} \underline{\mathbf{Z}/l^n}\big)$$

with its action of geometric Frobenius is a direct summand of

$$\mathbf{Q}_l \otimes_{\mathbf{Z}_l} \varprojlim_n \widetilde{\mathrm{H}}^1_{\text{ét}}\big(Y_1(N)_{/\overline{\mathbf{F}}_p}, \mathrm{R}^k f^k_{N*} \underline{\mathbf{Z}/l^n}\big).$$

Here $f^k_N : E_1(N)^k \to Y_1(N)$ is the k-fold fiber product abelian scheme over $\mathbf{Z}[1/Nl]$ (or over $\mathbf{F}_p$ or $\overline{\mathbf{F}}_p$).

LEMMA 5.4.3.1. *The $\mathbf{Q}_l$-vector space*

$$\mathbf{Q}_l \otimes_{\mathbf{Z}_l} \varprojlim \ \widetilde{\mathrm{H}}^{i+j}_{\text{ét}}\big(E_1(N)^k_{/\overline{\mathbf{F}}_p}, \underline{\mathbf{Z}/l^n}\big)$$

contains

$$\mathbf{Q}_l \otimes_{\mathbf{Z}_l} \varprojlim \ \widetilde{\mathrm{H}}^i_{\text{ét}}\big(Y_1(N)_{/\overline{\mathbf{F}}_p}, \mathrm{R}^j f^k_{N*} \underline{\mathbf{Z}/l^n}\big)$$

for all i, j, compatibly with the action of geometric Frobenius. Here $\widetilde{\mathrm{H}}^i_{\text{ét}}$ denotes the image of $\mathrm{H}^i_{\mathrm{c},\text{ét}}$ in $\mathrm{H}^i_{\text{ét}}$.

PROOF. For fixed n, consider the Leray spectral sequences

$$E_n : E^{i,j}_{2,n} = \mathrm{H}^i_{\text{ét}}\big(Y_1(N)_{/\overline{\mathbf{F}}_p}, \mathrm{R}^j f^k_{N*} \underline{\mathbf{Z}/l^n}\big) \Rightarrow \mathrm{H}^{i+j}_{\text{ét}}\big(E_1(N)^k_{\overline{\mathbf{F}}_p}, \underline{\mathbf{Z}/l^n}\big)$$

and

$${}_cE_n : {}_cE^{i,j}_{2,n} = \mathrm{H}^i_{\mathrm{c},\text{ét}}\big(Y_1(N)_{/\overline{\mathbf{F}}_p}, \mathrm{R}^j f^k_{N*} \underline{\mathbf{Z}/l^n}\big) \Rightarrow \mathrm{H}^{i+j}_{\mathrm{c},\text{ét}}\big(E_1(N)^k_{\overline{\mathbf{F}}_p}, \underline{\mathbf{Z}/l^n}\big).$$

These are both compatible with change in n and are *functorial* with respect to the map $f_N^k : E_1(N)^k_{/\overline{\mathbf{F}}_p} \to Y_1(N)_{/\overline{\mathbf{F}}_p}$. In particular, they are compatible with geometric Frobenius actions. These spectral sequences are also compatible with the natural maps $\mathrm{H}^\bullet_{\mathrm{c,\acute{e}t}} \to \mathrm{H}^\bullet_{\mathrm{\acute{e}t}}$, so we have a map of *spectral sequences*

$$ {}_cE_n \to E_n. $$

For $r \geq 2$, $\mathrm{d}_{r,n}^{p,q}$ (resp. ${}_c\mathrm{d}_{r,n}^{p,q}$) maps $E^{i,j}$ to $E^{i+r,j-r+1}$ (resp. ${}_cE^{i,j}$ to ${}_cE^{i+r,j-r+1}$) with $j \leq j-r+1$. However, for any integer m, the multiplication-by-m map $[m] : E_1(N)^k \to E_1(N)^k$ over $Y_1(N)$ induces a commutative square of spectral sequences

$$\begin{tikzcd} {}_cE_n \arrow[r] \arrow[d, "{[m]}"'] & E_n \arrow[d, "{[m]}"] \\ {}_cE_n \arrow[r] & E_n \end{tikzcd}$$

by *functoriality*. By Lemma 5.3.3.2, the maps on $E_{2,n}^{i,j}$ and ${}_cE_{2,n}^{i,j}$ induced by $[m]$ are just multiplication by m^j, yet the d_r-maps and ${}_c\mathrm{d}_r$-maps respect this!

Now fix $m > 1$. Passing to the inverse limit over n (which respects the formation of spectral sequences, by Mittag-Leffler) and tensoring with $\mathbf{Q}_l$ over $\mathbf{Z}_l$, the compatibility of $\varprojlim\ \mathrm{d}_{r,n}^{i,j}$, $\varprojlim\ {}_c\mathrm{d}_{r,n}^{i,j}$ with m^j and m^{j-r+1} forces these inverse limit maps to be *zero*. Thus $\mathbf{Q}_l \otimes_{\mathbf{Z}_l} \varprojlim\ E_{2,n}^{i,j}$ is canonically identified with the m^j-eigenspace of the $[m]$-action on

$$ \mathbf{Q}_l \otimes_{\mathbf{Z}_l} \varprojlim\ \mathrm{H}^{i+j}_{\mathrm{\acute{e}t}}\big(E_1(N)^k_{/\overline{\mathbf{F}}_p}, \underline{\mathbf{Z}/l^n}\big) $$

and the same conclusions hold for the ${}_c\widetilde{\mathrm{H}}^\bullet_{\mathrm{\acute{e}t}}$'s and hence the $\widetilde{\mathrm{H}}^\bullet_{\mathrm{\acute{e}t}}$. This is the statement of the lemma. This respects the action of geometric Frobenius since $[m]$ is defined on $E_1(N)^k_{/\mathbf{F}_p}$ over $Y_1(N)_{/\mathbf{F}_p}$. $\square$

We conclude that ${}^k_NW_l$ with its action of geometric Frobenius is naturally a subspace of

$$ \mathbf{Q}_l \otimes_{\mathbf{Z}_l} \varprojlim\ \widetilde{\mathrm{H}}^{k+1}_{\mathrm{\acute{e}t}}\big(E_1(N)^k_{/\overline{\mathbf{F}}_p}, \underline{\mathbf{Z}/l^n}\big). $$

But by Theorem 4.3.1.1, there is an open immersion

$$ E_1(N)^k \hookrightarrow X $$

with X projective smooth over $\mathbf{F}_p$. By Theorem 5.4.2.2 and Mittag-Leffler, we conclude that

$$ \mathbf{Q}_l \otimes_{\mathbf{Z}_l} \varprojlim\ \widetilde{\mathrm{H}}^{k+1}_{\mathrm{\acute{e}t}}\big(E_1(N)^k_{/\overline{\mathbf{F}}_p}, \underline{\mathbf{Z}/l^n}\big) $$

is the image of

$$ \mathbf{Q}_l \otimes_{\mathbf{Z}_l} \varprojlim\ \mathrm{H}^{k+1}_{\mathrm{c,\acute{e}t}}\big(E_1(N)^k_{/\overline{\mathbf{F}}_p}, \underline{\mathbf{Z}/l^n}\big) \longrightarrow \mathbf{Q}_l \otimes_{\mathbf{Z}_l} \varprojlim\ \mathrm{H}^{k+1}_{\mathrm{\acute{e}t}}\big(E_1(N)^k_{/\overline{\mathbf{F}}_p}, \underline{\mathbf{Z}/l^n}\big). $$

However, this map *factors through*

$$ \mathbf{Q}_l \otimes_{\mathbf{Z}_l} \varprojlim\ \mathrm{H}^{k+1}_{\mathrm{c,\acute{e}t}}\big(X_{/\overline{\mathbf{F}}_p}, \underline{\mathbf{Z}/l^n}\big) = \mathbf{Q}_l \otimes_{\mathbf{Z}_l} \varprojlim\ \mathrm{H}^{k+1}_{\mathrm{\acute{e}t}}\big(X_{/\overline{\mathbf{F}}_p}, \underline{\mathbf{Z}/l^n}\big) $$

since

$$\begin{tikzcd}
\mathrm{H}^{\bullet}_{\mathrm{c},\text{ét}}(E_1(N)^k_{/\overline{\mathbf{F}}_p}, \cdot) \arrow[r] \arrow[d] & \mathrm{H}^{\bullet}_{\text{ét}}(E_1(N)^k_{/\overline{\mathbf{F}}_p}, \cdot) \\
\mathrm{H}^{\bullet}_{\mathrm{c},\text{ét}}(X_{/\overline{\mathbf{F}}_p}, \cdot) \arrow[r, equal] & \mathrm{H}^{\bullet}_{\text{ét}}(X_{/\overline{\mathbf{F}}_p}, \cdot) \arrow[u]
\end{tikzcd}$$

commutes. We conclude that ${}^k_N W_l$ with its action of geometric Frobenius is a subquotient of

$$\mathbf{Q}_l \otimes_{\mathbf{Z}_l} \varprojlim \mathrm{H}^{k+1}_{\text{ét}}\big(X_{/\overline{\mathbf{F}}_p}, \underline{\mathbf{Z}/l^n}\big)$$

with its action of geometric Frobenius. By the Riemann hypothesis (Theorem 5.4.1.3), this completes the proof of Main Theorem 1!

Bibliography

[1] Vladimir Berkovich. Étale cohomology for non-archimedean analytic spaces. *Institut des Hautes Etudes Scientifiques, Publications Mathematiques*, 78:5–161, 1993.

[2] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud. *Néron models*. Springer-Verlag, Berlin, 1990.

[3] Nicholas Bourbaki. *Éléments de mathématqiue. Topologie générale. Chapitres 1 à 4*. Hermann, Paris, 1971.

[4] Ching-Li Chai. Siegel moduli schemes and their compactifications over **C**. In Cornell and Silverman [**8**], pages 231–251.

[5] Ted Chinburg. Minimal models for curves over Dedekind rings. In Cornell and Silverman [**8**], pages 309–326.

[6] Brian Conrad. Modular curves and rigid analytic spaces. (in preparation).

[7] Brian Conrad. *Grothendieck duality and base change*. Springer-Verlag, Berlin-New York, 2000.

[8] Gary Cornell and Joseph Silverman, editors. *Arithmetic geometry* (*Storrs, Connecticut,* 1984). Springer-Verlag, 1986.

[9] A.J. de Jong. Crystalline Dieudonné module theory via formal and rigid geometry. *Institut des Hautes Etudes Scientifiques, Publications Mathematiques*, 82:5–96, 1995.

[10] Pierre Deligne. Formes modulaires et représentations l-adiques. *Séminaire Bourbaki*, 355:139–172, 1971.

[11] Pierre Deligne. La conjecture de Weil I. *Institut des Hautes Etudes Scientifiques, Publications Mathematiques*, 43:273–307, 1974.

[12] Pierre Deligne. La conjecture de Weil II. *Institut des Hautes Etudes Scientifiques, Publications Mathematiques*, 52:137–252, 1980.

[13] Pierre Deligne and Willem Kuyk, editors. *Modular functions of one variable II (Antwerp 1972)*, number 349 in Lecture Notes in Mathematics. Springer-Verlag, 1973.

[14] Pierre Deligne and Michael Rapoport. Schémas de modules de courbes elliptiques. In Deligne and Kuyk [**13**], pages 143–316.

[15] Pierre Deligne and Jean-Pierre Serre. Formes modulaires de poids 1. *Annales scientifiques de l'École Normale Superieure (4)*, 7:507–530, 1974.

[16] Martin Eichler. Quaternäre quadratische Formen und die Riemannsche Vermutung für die Kongruenzzetafunktion. *Arch. Math.*, 5:355–366, 1954.

[17] Eberhard Freitag and Reinhardt Kiehl. *Etale cohomology and the Weil conjecture*. Springer-Verlag, Berlin, 1988.

[18] Roger Godement. *Topologie algébrique et théorie des faisceaux*. Hermann, Paris, 1973.

[19] Hans Grauert and Reinhold Remmert. *Theory of Stein spaces*. Springer-Verlag, Berlin-New York, 1979.

[20] Hans Grauert and Reinhold Remmert. *Coherent analytic sheaves*. Springer-Verlag, Berlin-New York, 1984.

[21] Alexander Grothendieck. *Techniques de construction en géométrie analytique IX: quelques problémes de modules*. Exposé 16, Séminaire Cartan. 1960/61.

[22] Alexander Grothendieck. *Familles d'espaces complexes et fondements de la géométrie analytique*. Exposés au Séminaire Cartan 1960/61. W.A. Benjamin, Inc., New York, 1967.

[23] Alexander Grothendieck. *Revêtements étales et groupe fondamental (1960-1961)*. Number 224 in Lecture Notes in Mathematics. Springer-Verlag, 1971.

[24] Alexander Grothendieck, Michael Artin, and Jean-Louis Verdier. *Théorie des topos et cohomologie étale des schémas (1963-1964)*. Number 269, 270, 305 in Lecture Notes in Mathematics. Springer-Verlag, 1972-73.

[25] Alexander Grothendieck and M. Demazure. *Schémas en groupes (1962-1964)*. Number 151–153 in Lecture Notes in Mathematics. Springer-Verlag, 1964.

[26] Alexander Grothendieck and Jean Dieudonné. Etude cohomologique des faisceaux cohérents. Institut des Hautes Etudes Scientifiques, Publications Mathematiques. vol. 11 (1961), vol. 17 (1963).

[27] Alexander Grothendieck and Jean Dieudonné. Etude globale élémentaire de quelques classes de morphismes. Institut des Hautes Etudes Scientifiques, Publications Mathematiques. vol. 8 (1961).

[28] Alexander Grothendieck and Jean Dieudonné. Etude locale des schémas et des morphismes de schémas. Institut des Hautes Etudes Scientifiques, Publications Mathematiques. vol. 20 (1964), vol. 24 (1965), vol. 28 (1966), vol. 32 (1967).

[29] Alexander Grothendieck and Jean Dieudonné. Le langage des schémas. Institut des Hautes Etudes Scientifiques, Publications Mathematiques. vol. 4 (1960).

[30] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, Berlin-New York, 1977.

[31] Samuel Eilenberg Henri Cartan. *Homological algebra*. Princeton University Press, Princeton, 1956.

[32] Jun-Ichi Igusa. Kroneckerian models of field of elliptic modular functions. *Amer. J. Math.*, 81:561–577, 1959.

[33] Birger Iversen. *Cohomology of sheaves*. Springer-Verlag, Berlin-New York, 1986.

[34] Jean-Pierre Jouanolou. *Théorèmes de Bertini et applications*. Birkhauser, Boston, 1983.

[35] Nicholas Katz and Barry Mazur. *Arithmetic moduli of elliptic curves*. Princeton University Press, Princeton, New Jersey, 1985.

[36] Nicholas M. Katz. Algebraic solutions of differential equations (p-curvature and the Hodge filtration). *Inventiones Mathematicae*, 18:1–118, 1972.

[37] Reinhardt Kiehl. Theorem A und Theorem B in der nichtarchimedischen Funktionentheorie. *Inventiones Mathematicae*, 2:191–214, 1967.

[38] Serge Lang. *Real analysis*. Addison-Wesley, Reading, 1985.

[39] Serge Lang. *Elliptic functions*. Springer-Verlag, Berlin-New York, 1987.

[40] Stephen Lichtenbaum. Curves over discrete valuation rings. *American Journal of Mathematics*, 90:380–405, 1968.

[41] Werner Lütkebohmert. On compactification of schemes. *Manuscripta Mathematica*, 80:95–111, 1993.

[42] Hideyuki Matsumura. *Commutative ring theory*. Cambridge University Press, Cambridge, 1989.

[43] James Milne. Jacobian varieties. In Cornell and Silverman [**8**], pages 167–212.

[44] Toshitsune Miyake. *Modular forms*. Springer-Verlag, Berlin-New York, 1989.

[45] David Mumford. *Abelian varieties*. Oxford University Press, Bombay, 1970.

[46] David Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*. Springer-Verlag, Berlin, 1994.

[47] James Munkres. *Topology*. Prenctice-Hall, Englewood Cliffs, 1975.

[48] Tadao Oda. *The first de Rham cohomology group and Dieudonné modules*. *Ann. Sci. Ec. Norm. Sup.*, 5e série **2**:63–135, 1969.

[49] Ken Ribet. "Galois representations attached to eigenforms with nebentypus", in *Modular functions of one variable V* (Bonn, 1976), Lecture Notes in Mathematics, no. 601. Springer-Verlag, Berlin, 1977.

[50] Michael Rosen. Abelian varieties over $\mathbf{C}$. In Cornell and Silverman [**8**], pages 79–101.

[51] Michael Schlessinger. Functors on artin rings. *Transaction of the American Mathematical Society*, 130:208–222, 1968.

[52] Anthony J. Scholl. Motives for modular forms. *Inventiones mathematicae*, 100:419–430, 1990.

[53] Jean-Pierre Serre. *A course in arithmetic*. Springer-Verlag, Berlin-New York, 1973.

[54] Jean-Pierre Serre. *Galois cohomology*. Springer-Verlag, Berlin-New York, 1997.

[55] Goro Shimura. Correspondances modulaire et les fonctions ζ de courbes algébriques. *J. Math. Soc. Japan*, 10:1–28, 1958.

[56] Goro Shimura. *Introduction to the arithmetic theory of automorphic functions*. Princeton University Press, Princeton, 1971.

[57] Joseph Silverman. *The arithmetic of elliptic curves*. Springer Verlag, Berlin-New York, 1986.

[58] Joseph Silverman. *Advanced topics in the arithmetic of elliptic curves*. Springer Verlag, Berlin-New York, 1992.

[59] Jean-Louis Verdier. A duality theorem in the étale cohomology of schemes, in *Proc. Conf. Local Fields* (Dreibergen, 1966). Springer-Verlag, New York, 1967.

[60] Frank Warner. *Foundations of differentiable manifolds and Lie groups.* Springer-Verlag, Berlin-New York, 1983.