Appendix.

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1 Various descriptions of groups of connected components.

Let R be a complete discrete valuation ring, with fraction field K and algebraically closed residue field k. Let X_K be a smooth, proper, geometrically connected curve over K. Suppose that X is a nodal model of X_K over R, i.e., X is a proper flat R-scheme, its generic fibre is X_K and the only singularities of the special fibre X_k are ordinary double points. We will suppose, for simplicity, that the irreducible components of X_k are smooth. Let J_K be the jacobian of X_K, J the Néron model over R of J_K and $\Phi := J_k/J_k^o$ the group of connected components of J_k. Let $\widetilde{X} \to X$ be the minimal resolution of X. We associate a graph $\widetilde{G}$ to $\widetilde{X}$ as follows. The set of vertices is the set $\widetilde{\mathcal{C}}$ of irreducible components of $\widetilde{X}_k$, the set of edges is the set $\widetilde{\mathcal{S}}$ of singular points of $\widetilde{X}_k$. The vertices meeting an edge x are the two irreducible components containing x. The elements of $\widetilde{\mathcal{C}}$ are Cartier divisors on $\widetilde{X}$, hence for any two of them, say C and C', we have an intersection number $(C \cdot C')$. This intersection pairing defines a morphism of free $\mathbb{Z}$-modules:

$$\alpha: \mathbb{Z}^{\widetilde{\mathcal{C}}} \longrightarrow \mathbb{Z}^{\widetilde{\mathcal{C}}}, \qquad f \mapsto \left(C \mapsto \sum_{C'} (C \cdot C')\, f(C') \right). \tag{1.1}$$

A theorem of Raynaud (see [1, Thm. 9.6/1]) says that Φ is canonically isomorphic to the homology of the complex:

$$\mathbb{Z}^{\widetilde{\mathcal{C}}} \xrightarrow{\alpha} \mathbb{Z}^{\widetilde{\mathcal{C}}} \xrightarrow{+} \mathbb{Z}, \tag{1.2}$$

where the map "+" sends f to $\sum_C f(C)$. The isomorphism works as follows. Let P be in $J(K)$, let D be a divisor on $\widetilde{X}$ such that P is the class of the restriction of D to X_K. Then P specializes to the element of Φ given by $C \mapsto (C \cdot D)$.

Grothendieck gave another description of Φ in terms of the monodromy pairing in [6, Thm. 11.5]. Our aim is to compare these two descriptions.

We choose an orientation on the graph $\widetilde{G}$, i.e, we choose two maps s and t from $\widetilde{\mathcal{S}}$ to $\widetilde{\mathcal{C}}$ (s for source and t for target) such that for all edges x, $s(x)$ and $t(x)$ are the vertices meeting x. We get induced maps:

$$s_*, t_*: \mathbb{Z}^{\widetilde{\mathcal{S}}} \longrightarrow \mathbb{Z}^{\widetilde{\mathcal{C}}}, \qquad s^*, t^*: \mathbb{Z}^{\widetilde{\mathcal{C}}} \longrightarrow \mathbb{Z}^{\widetilde{\mathcal{S}}}, \tag{1.3}$$

where $(s_*f)C = \sum_{s(x)=C} f(x)$, $(s^*f)x = f(s(x))$, etc. Note that s^* is the adjoint, with respect to the standard inner products on $\mathbb{Z}^{\widetilde{\mathcal{S}}}$ and $\mathbb{Z}^{\widetilde{\mathcal{C}}}$, of s_*. With this notation, we can define the usual boundary and coboundary maps:

$$d_* := t_* - s_* \colon \mathbb{Z}^{\widetilde{\mathcal{S}}} \longrightarrow \mathbb{Z}^{\widetilde{\mathcal{C}}}, \qquad d^* := t^* - s^* \colon \mathbb{Z}^{\widetilde{\mathcal{C}}} \longrightarrow \mathbb{Z}^{\widetilde{\mathcal{S}}}. \tag{1.4}$$

By definition, $\ker(d_*)$ is the homology group $\mathrm{H}_1(\widetilde{G}, \mathbb{Z})$ and $\mathrm{coker}(d^*) = \mathrm{H}^1(\widetilde{G}, \mathbb{Z})$ (the isomorphisms depend on the orientation chosen). Since $\widetilde{G}$ is connected, $\ker(d^*) = \mathrm{H}^0(\widetilde{G}, \mathbb{Z})$ is the diagonal in $\mathbb{Z}^{\widetilde{\mathcal{C}}}$, and $\mathrm{coker}(d_*) = \mathrm{H}_0(\widetilde{G}, \mathbb{Z})$ is $\mathbb{Z}$, via the map $+\colon \mathbb{Z}^{\widetilde{\mathcal{C}}} \to \mathbb{Z}$. A simple computation shows:

$$\alpha = t_*s^* + s_*t^* - s_*s^* - t_*t^* = -(t_* - s_*)(t^* - s^*) = -d_*d^*. \tag{1.5}$$

Let T be the maximal torus of J_k^o, and let $M := \mathrm{Hom}(T, \mathrm{G}_{\mathrm{m},k})$ be its character group. We denote the $\mathbb{Z}$-linear dual of M by $M^\vee$. It is well known, e.g. [6, 12.3.7] or [1, §9.2, Example 8], that M and $\mathrm{H}_1(\widetilde{G}, \mathbb{Z})$ are canonically isomorphic. Let us give an isomorphism. The torus T is the kernel of the pullback map $\mathrm{Pic}_{\widetilde{X}_k/k} \to \mathrm{Pic}_{\widetilde{X}_k^{\mathrm{nor}}/k}$, where $\mathrm{nor}\colon \widetilde{X}_k^{\mathrm{nor}} \to \widetilde{X}_k$ is the normalization map. For each x in $\widetilde{\mathcal{S}}$ we can distinguish the two elements of $\mathrm{nor}^{-1}\{x\}$ because one of them lies on $s(x)$ and one on $t(x)$; we denote these two points by s_x and t_x. Let f be a k^*-valued function on $\widetilde{\mathcal{S}}$. Then we can construct a line bundle on $\widetilde{X}_k$ whose class is in $T(k)$ as follows: take the trivial line bundle on $\widetilde{X}_k^{\mathrm{nor}}$, and, for each x in $\widetilde{\mathcal{S}}$, identify the fiber at s_x with the fibre at t_x via multiplication by $f(x)$. This construction induces an isomorphism between $M^\vee$ and $\mathrm{H}^1(\widetilde{G}, \mathbb{Z})$ which does not depend on the orientation chosen. Passing to duals gives the desired isomorphism. From (1.5) we get the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[dr] & & & 0 \arrow[d] & & \\
 & & \mathbb{Z} \arrow[rr, "0"] \arrow[dr, "\mathrm{diag}"] & & M \arrow[d] \arrow[dr, "i"] & & \\
0 \arrow[r] & \mathbb{Z} \arrow[ur, "\mathrm{id}"] \arrow[rr, "\mathrm{diag}"] & & \mathbb{Z}^{\widetilde{\mathcal{C}}} \arrow[r, "d^*"] \arrow[dr, "-\alpha"] & \mathbb{Z}^{\widetilde{\mathcal{S}}} \arrow[r] \arrow[d, "d_*"] & M^\vee \arrow[r] \arrow[d] & 0 \\
 & & & & \mathbb{Z}^{\widetilde{\mathcal{C}}}[+] \arrow[r] \arrow[d] & \Phi \arrow[dl] \arrow[dr] & \\
 & & & & 0 & & 0
\end{tikzcd} \tag{1.6}$$

As a consequence, we have the short exact sequence:

$$0 \longrightarrow M \xrightarrow{\ i\ } M^\vee \longrightarrow \Phi \longrightarrow 0. \tag{1.7}$$

A computation shows that for all m_1 and m_2 in M:

$$(i(m_1))\,(m_2) = \sum_{x\in\widetilde{\mathcal{S}}} m_1(x)\, m_2(x), \tag{1.8}$$

which is exactly the value at (m_1, m_2) of the monodromy pairing, see [6, Thm. 12.5]. Equations (1.7) and (1.8) together are Grothendieck's description of Φ. It follows that the diagram (1.6) gives a translation between Raynaud's and Grothendieck's descriptions of Φ.

Our next objective is to give the relation between the descriptions of M in terms of $\widetilde{X}$ and X. Let G be the graph associated to X: the set of vertices is the set $\mathcal{C}$ of irreducible components of X_k, the set of edges is the set $\mathcal{S}$ of singular points of X_k, the vertices meeting an edge x are the two components containing x. Let $r\colon \widetilde{X} \to X$ denote the resolution morphism; it is the composition of blowing ups in the singular points. For x in $\mathcal{S}$ let $w(x)$ be the "width" of the singularity at x: if π is a uniformizer of R then, locally at x for the etale topology, X is given by the equation $uv = \pi^{w(x)}$. Let x be in $\mathcal{S}$. It is well known that $r^{-1}\{x\}$ is a chain of $w(x)-1$ projective lines. In terms of graphs, this means that $\widetilde{G}$ is obtained from G by replacing each edge x of G by a path of $w(x)$ edges:

(1.9)

It is clear that an orientation on G induces an orientation on $\widetilde{G}$. We will assume that the orientation that we already have on $\widetilde{G}$ comes from an orientation on G; the corresponding two maps $\mathcal{S} \to \mathcal{C}$ will also be denoted by s and t. It is also clear that $\mathrm{H}_1(G,\mathbb{Z})$ and $\mathrm{H}_1(\widetilde{G},\mathbb{Z})$ are canonically isomorphic, and that this isomorphism is compatible with the identifications of both of them with M (there is an isomorphism between M and $\mathrm{H}_1(G,\mathbb{Z})$ defined just as for $\widetilde{G}$).

The picture (1.9) suggests to consider the following maps:

$$h_{\mathcal{C}}\colon \mathbb{Z}^{\mathcal{C}} \longrightarrow \mathbb{Z}^{\widetilde{\mathcal{C}}},\quad C \mapsto \widetilde{C}, \qquad h_{\mathcal{S}}\colon \mathbb{Z}^{\mathcal{S}} \longrightarrow \mathbb{Z}^{\widetilde{\mathcal{S}}},\quad x \mapsto \sum_{y\in r^{-1}x} y, \tag{1.10}$$

with $\widetilde{C}$ the strict transform of C. By construction, we get a commutative diagram:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & M & \longrightarrow & \mathbb{Z}^{\mathcal{S}} & \xrightarrow{d_*} & \mathbb{Z}^{\mathcal{C}} & \xrightarrow{+} & \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle h_{\mathcal{S}}} & & \downarrow{\scriptstyle h_{\mathcal{C}}} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & M & \longrightarrow & \mathbb{Z}^{\widetilde{\mathcal{S}}} & \xrightarrow{d_*} & \mathbb{Z}^{\widetilde{\mathcal{C}}} & \xrightarrow{+} & \mathbb{Z} & \longrightarrow & 0
\end{array} \tag{1.11}$$

Dualizing gives another such diagram:

$$
(1.12)\qquad
\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\text{diag}} & \mathbb{Z}^{\widetilde{\mathcal{C}}} & \xrightarrow{d^*} & \mathbb{Z}^{\widetilde{\mathcal{S}}} & \longrightarrow & M^\vee & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle h_{\mathcal{C}}^\vee} & & \downarrow{\scriptstyle h_{\mathcal{S}}^\vee} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\text{diag}} & \mathbb{Z}^{\mathcal{C}} & \xrightarrow{d^*} & \mathbb{Z}^{\mathcal{S}} & \longrightarrow & M^\vee & \longrightarrow & 0
\end{array}
$$

In this diagram $h_{\mathcal{S}}^\vee$ and $h_{\mathcal{C}}^\vee$ are given by:

$$
(1.13)\qquad h_{\mathcal{S}}^\vee\colon \mathbb{Z}^{\widetilde{\mathcal{S}}} \longrightarrow \mathbb{Z}^{\mathcal{S}}, \quad y \mapsto r(y), \qquad h_{\mathcal{C}}^\vee\colon \mathbb{Z}^{\widetilde{\mathcal{C}}} \longrightarrow \mathbb{Z}^{\mathcal{C}}, \quad C \mapsto r_*(C),
$$

where $r_*(C)$ is zero if $r(C)$ is a point, and $r(C)$ otherwise.

The map $h_{\mathcal{S}}$ from (1.10) is injective; the standard inner product on $\mathbb{Z}^{\widetilde{\mathcal{S}}}$ induces the following inner product on $\mathbb{Z}^{\mathcal{S}}$:

$$
(1.14)\qquad \langle\cdot,\cdot\rangle\colon \mathbb{Z}^{\mathcal{S}} \times \mathbb{Z}^{\mathcal{S}} \longrightarrow \mathbb{Z}, \quad (f,g) \mapsto \sum_{x\in\mathcal{S}} w(x)\, f(x)\, g(x).
$$

It follows that, when we view M as a submodule of $\mathbb{Z}^{\mathcal{S}}$ and $M^\vee$ as a quotient of it, the map (1.7) $i\colon M \to M^\vee$ is given by:

$$
(1.15)\qquad (i(m_1))(m_2) = \sum_{x\in\mathcal{S}} w(x)\, m_1(x)\, m_2(x).
$$

The discussion up to here allows us to translate between Raynaud's description of Φ in terms of $\widetilde{X}$ and Grothendieck's description in terms of the Picard–Lefschetz formula for X.

2 Specialization of divisors of degree zero.

We keep the notation of the previous section. Since X and $\widetilde{X}$ are proper over R, we can identify $X(K)$, $X(R)$ and $\widetilde{X}(R)$. The closure in $\widetilde{X}$ of an effective divisor on X_K is a Cartier divisor. Extending this by linearity we associate to each divisor D on X_K a Cartier divisor $\widetilde{D}$ on $\widetilde{X}$.

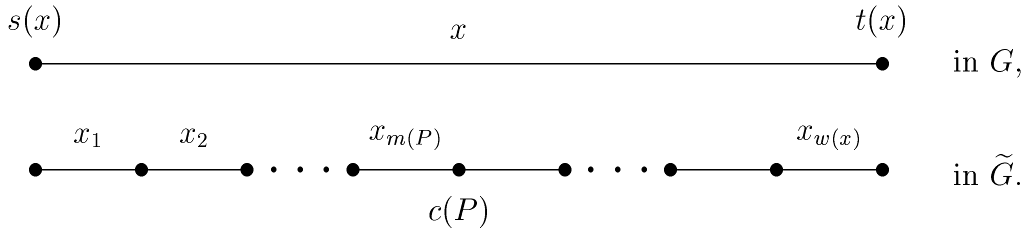
Suppose now that D is a divisor of degree zero on X_K. Then the class of D under linear equivalence corresponds to an element, say $[D]$, of $J(K) = J(R)$. Let $\phi(D)$ denote the image of $[D]$ in Φ. By Raynaud's description of Φ, see (1.2), $\phi(D)$ is given by the element $(C \mapsto (C\cdot\widetilde{D}))$ of $\mathbb{Z}^{\widetilde{\mathcal{C}}}$. Our aim is now to get an expression for $\phi(D)$ in terms of $\mathbb{Z}^{\mathcal{C}}$ and $\mathbb{Z}^{\mathcal{S}}$. For simplicity, we suppose that D has support in the set of K-rational points of X (note that since X is semi-stable, one can always reduce to this case by extending K). Let us write:

$$
(2.1)\qquad D = \sum_P n_P\, P,
$$

where the sum ranges through the set $X(K)$ (of course, almost all n_P are zero, and their sum is zero). Each P in $X(K)$ specializes to a unique element $c(P)$ of $\widetilde{\mathcal{C}}$, since $\widetilde{X}$ is regular. With this notation, $\phi(D)$ is given by the element $\sum_P n_P\, c(P)$ of $\mathbb{Z}^{\widetilde{\mathcal{C}}}$.

Suppose now that P is such that $c(P)$ is not in $\mathcal{C}$. Let x in $\mathcal{S}$ be the image of $c(P)$ under the morphism $r\colon \widetilde{X} \to X$. Then we have the following situation:

(2.2)



Note that the integer $m(P)$ is defined by the condition that $c(P)$ is the $m(P)$th projective line in the chain of projective lines from $s(x)$ to $t(x)$ in $\widetilde{X}$.

Using (1.6) it follows that $\phi(D)$ is represented by the sum of the following two contributions:

$$\sum_{c(P)\in\mathcal{C}} n_P\, c(P) + \sum_{c(P)\notin\mathcal{C}} n_P\, s(r(c(P))) \quad \text{and} \quad \sum_{c(P)\notin\mathcal{C}} n_P\, m(P)\, r(c(P)), \tag{2.3}$$

where the first contribution comes from the subgroup $\mathbb{Z}^{\mathcal{C}}[+]$ of $\mathbb{Z}^{\widetilde{\mathcal{C}}}[+]$ and the second from $\mathbb{Z}^{\mathcal{S}}$.

The last thing we want to do in this section is to give a useful interpretation of the integers $m(P)$. Suppose that P in $X(K)$ is such that $c(P)$ is not in $\mathcal{C}$. Let x denote the element $r(c(P))$ of $\mathcal{S}$. Let π be a uniformizer of R. Let u and v be elements of the complete local ring $\widehat{\mathcal{O}}_{X,x}$ such that $\widehat{\mathcal{O}}_{X,x} = R[[u,v]]/(uv - \pi^{w(x)})$. By interchanging u and v, if necessary, we may and do suppose that (u,π) is the ideal of the branch $s(x)$. Let $c_0, \ldots, c_{w(x)}$ be the irreducible components of $r^{-1}x$, numbered such that $s(x_i) = c_{i-1}$ and $t(x_i) = c_i$. For the divisor $\mathrm{div}(v)$ of v on the completion of $\widetilde{X}$ along $r^{-1}x$ one has:

$$\mathrm{div}(v) = \sum_{i=0}^{w(x)} i\, c_i. \tag{2.4}$$

To see this, note that blowing up $R[u,v]/(uv - \pi^n)$ in the ideal (u,v,π) gives the singularity $R[\pi^{-1}u, \pi^{-1}v]/(\pi^{-1}u\pi^{-1}v - \pi^{n-2})$. Equation (2.4) gives the following interpretation of $m(P)$:

$$m(P) = P\cdot(m(P)c(P)) = P\cdot\sum_{i=0}^{w(x)} i\, c_i = P\cdot \mathrm{div}(v) = \mathrm{val}_R(P^*(v)), \tag{2.5}$$

where val_R denotes the valuation on R such that $\mathrm{val}_R(\pi) = 1$, and where P^* is the morphism of rings $\widehat{\mathcal{O}}_{X,x} \to R$ induced by the morphism of schemes $P\colon \mathrm{Spec}(R) \to X$.

3 Modular curves.

The aim of this section is to give a moduli theoretic interpretation of the integer $m(P)$ in (2.5), in the case where X_K is a certain modular curve.

From now on, let p be a prime number. We suppose that K has characteristic zero and that k has characteristic p. Let X be the compactified coarse moduli scheme for the algebraic stack with objects $((\alpha: E \to E')/S/R, \beta)$, with S an R-scheme, E and E' elliptic curves over S, α an isogeny of degree p, and β some prime-to-p level structure (of a fixed type) on E. By [5, V.1] or [8, 13.4], the R-scheme X is proper and flat, X_K is smooth and the only singularities of X_k are ordinary double points. We suppose that the level structure β is such that X_K is geometrically irreducible. Then X satisfies the hypotheses of the beginning of Section 1. The special fibre X_k has two irreducible components; we call them C_∞ and C_0, with C_∞ containing the cusp ∞ and C_0 the cusp 0. Over C_∞, the isogeny α is isomorphic to the relative Frobenius $F: E \to E^{(p)}$. Over C_0, α is isomorphic to the Verschiebung $V: (E')^{(p)} \to E'$. We choose the following orientation on the graph G of X: for every double point x of X_k, we have $s(x) = C_0$ and $t(x) = C_\infty$.

Let $((\alpha: E \to E')/R, \beta)$ be an R-valued point of the stack mentioned above, and let P in $X(R)$ denote the induced R-valued point of X. We suppose that P specializes to a singular point x of X_k; this just means that E_k is supersingular. Let A be the deformation ring of the object corresponding to x (we consider deformations over complete local R-algebras with residue field k). By [5, V, 1.19] or [8, Thms. 5.1.1 and 13.4.7], we can take y and z in A such that $A = R[[y,z]]/(yz-p)$. Let G be the group of k-automorphisms of the object corresponding to x. Then G acts on A, and by [5, I, 8.2.1] we have $\widehat{\mathcal{O}}_{X,x} = A^G$. The two branches at x of X_k are fixed by G, hence for all g in G, $g(y)/y$ and $g(x)/x$ are units in A. Let $\overline{G}$ be the image of G in $\mathrm{Aut}(A)$, and $|\overline{G}|$ its order. We get a subring $R[[u,v]]/(uv - p^{|\overline{G}|})$ of A^G by sending u to $\prod_{g \in \overline{G}} g(y)$ and v to $\prod_{g \in \overline{G}} g(z)$. This subring is normal, and the field extension given by the fraction fields of this subring and of A has degree $|\overline{G}|$. It follows that:

$$\widehat{\mathcal{O}}_{X,x} = R[[u,v]]/(uv - p^{|\overline{G}|}). \tag{3.1}$$

As in Section 2, we assume that u vanishes on the branch C_0, and v on C_∞. Let $((\alpha_{\mathrm{univ}}: E_{\mathrm{univ}} \to E'_{\mathrm{univ}})/A, \beta_{\mathrm{univ}})$ denote the universal deformation. Then z is an equation for the maximal closed subscheme of $\mathrm{Spec}(A/pA)$ over which E'_{univ} is isomorphic to $E^{(p)}_{\mathrm{univ}}$ (use that the analogous statement is true for the deformation ring over the Witt vectors of k; see [8, Thm. 13.4.7]). Since P in $X(R)$ arises from the object $((\alpha: E \to E')/R, \beta)$, the morphism $P^*: \widehat{\mathcal{O}}_{X,x} \to R$ is induced from a morphism $\widetilde{P}^*: A \to R$. The element $\widetilde{P}^*(z)$ is an equation for the maximal closed subscheme of $\mathrm{Spec}(R/pR)$ over which E' and $E^{(p)}$ are isomorphic. In other words, $\mathrm{val}_R(\widetilde{P}^*(z))$ is the maximal integer n such that E' and $E^{(p)}$ are isomorphic over $R/\pi^n R$ (as before, π is a uniformizer of R). We have:

$$\mathrm{val}_R(P^*(v)) = \mathrm{val}_R(\widetilde{P}^*(v)) = \sum_{g \in \overline{G}} \mathrm{val}_R(\widetilde{P}^*(g(z))) = |\overline{G}| \cdot \mathrm{val}_R(\widetilde{P}^*(z)). \tag{3.2}$$

Equation 3.1 implies that $w(x) = |\overline{G}| \cdot \mathrm{val}_R(p)$. Putting everything together gives:

$$m(P) = \frac{w(x)}{\mathrm{val}_R(p)} \cdot \max\{n \le \mathrm{val}_R(p) \mid E' \cong E^{(p)} \text{ over } R/\pi^n R\}. \tag{3.3}$$

4 Shimura curves.

We will now adapt the arguments of Section 3 to the case of certain Shimura curves.

Let p and K be as in Section 3. Let $B_{\mathbb{Q}}$ be an indefinite quaternion algebra over $\mathbb{Q}$, of discriminant prime to p. Let B be a maximal order in $B_{\mathbb{Q}}$. First we consider the algebraic stack $\mathcal{M}$ with objects $(E/S/R, i\colon B \to \mathrm{End}_S(E), \beta)$, where S is an R-scheme, E/S an abelian scheme of relative dimension two, i a morphism of rings, and β some prime-to-p level structure. See [2] for the proof that this category is indeed an algebraic stack, and that it is proper and smooth of relative dimension one over R. We suppose that the level structure β is such that the fibres of $\mathcal{M}/R$ are geometrically irreducible. Combining [2, p. 54] with [3, III, 1.5] gives that each such E/S with B-action has a canonical principal polarization (the unique principal $*$-polarization).

In [5, Introduction, §7] it is explained how the results of [5] and [8] for modular curves imply results for $\mathcal{M}$. Let e be a non-trivial idempotent in $\mathbb{Z}_p \otimes B$ (recall that B is split at p). Let $(E/S, i, \beta)$ be an object of $\mathcal{M}$. Since $\mathbb{Z}_p \otimes B$ acts on the p-divisible group $E[p^\infty]$ of E, we get a direct sum decomposition

$$E[p^\infty] = eE[p^\infty] \oplus (1-e)E[p^\infty]. \tag{4.1}$$

The two terms in this decomposition are isomorphic, since e and $1-e$ are conjugates. The construction $G \mapsto eG$ defines an equivalence between the category of p-divisible S-groups with an action by B and the category of p-divisible S-groups. Via this equivalence, we can apply the Drinfeldian level structures of [8, §1.9] to $\mathcal{M}$. We will only consider the level structure corresponding to the $\Gamma_0(p)$ level structure in the modular curve case: it associates to $(E/S/R, i)$ the set of finite locally free subgroup schemes of rank p of $eE[p^\infty]$, or, equivalently, the set of isogenies $\alpha\colon (E, i) \to (E', i')$ (up to isomorphism) of degree p^2, with $i'\colon B \to \mathrm{End}(E')$ and α compatible with i and i'.

Let X be the coarse moduli space for the algebraic stack with objects $((\alpha\colon E \to E')/S/R, \beta)$, with S an R-scheme, E and E' abelian S-schemes of relative dimension two, equipped with an action by B, α as isogeny of degree p^2 and β a prime-to-p level structure as above. Let x be a closed point of X_k. Let $\widetilde{x} := ((\alpha\colon E \to E')/k, \beta)$ be the object corresponding to x. Then $eE[p^\infty]$ is a p-divisible group over k, of height two and with associated formal group of dimension one. The classification of p-divisible groups over algebraically closed fields implies that $eE[p^\infty]$ is isomorphic to the p-divisible group of some elliptic curve F over k (in fact, if $eE[p^\infty]$ is purely local, then any supersingular elliptic curve will do, otherwise any ordinary elliptic curve does the job). By the theorem of Serre and Tate [7] and the equivalence above, the deformation theory of $\widetilde{x}$ is equivalent to that of $(F/k, e\ker(\alpha))$ (again, we consider deformations to complete local R-algebras). It follows that the deformation ring of $\widetilde{x}$ is isomorphic to that of $(F/k, e\ker(\alpha))$. It follows that the only singularities of X_k are ordinary double points. One now easily shows, by constructing an ample divisor, that the algebraic space X is in fact a projective curve over R, satisfying the hypotheses of the beginning of Section 1. (An other way to see that X is a quasi-projective scheme is to prove that the morphism from $\mathcal{M}$ to the stack of two-dimensional principally polarized abelian varieties is representable and projective (use Hilbert schemes); see [4, §5.3].) We orient the graph G of X by demanding that for each singular point x of X_k, $s(x)$ is the branch on which α is generically etale. The following proposition is now a direct consequence of the results of the preceding sections.

4.2 Proposition. *Let P in $X(R)$ be induced by an object $((\alpha\colon E \to E')/R, \beta)$. Suppose that P specializes to a singular point x of X_k. Then the integer $m(P)$ of (2.5) is given by:*

$$m(P) = \frac{w(x)}{\mathrm{val}_R(p)} \cdot \max\{n \leq \mathrm{val}_R(p) \mid E' \cong E^{(p)} \text{ over } R/\pi^n R\}.$$

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