Complete intersections in projective spaces

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The aim of this appendix is to provide a proof of the following theorem. The proof is very standard, and the theorem is probably known (at least implicitly) by those who compute the cohomology of twisted ideal sheaves of complete intersections in projective spaces.

Theorem 1 *Let S be a complex analytic variety, with a given point s. Let $n \geq 1$ be an integer, and $X \to S$ be a smooth analytic family of projective complex analytic varieties of dimension $d > 0$, embedded in $\mathbb{P}^n_S := \mathbb{P}^n(\mathbb{C}) \times S$. Suppose that the fibre X_s at s is a complete intersection. Then there is an open neighborhood of s in S over which X is a complete intersection.*

Proof. Let I be the ideal sheaf of X in the structure sheaf $\mathcal{O}$ of $\mathbb{P}^n_S$. Let $f\colon \mathbb{P}^n_S \to S$ denote the projection, let $c := n - d$, and $r_1, \ldots, r_c$ the degrees of a set of hypersurfaces H_i in $\mathbb{P}^n_s$ of which X_s is the intersection. Each H_i is the zero locus of a section of $\mathcal{O}_{\mathbb{P}^n_s}(r_i)$. All we have to show is that the sections defining the H_i can be lifted to sections of $f_*I(r_i)$ over some neighborhood of s, since then X is contained in the complete intersection defined by these lifts, hence equal to it. We will show in fact that $f_*I(r)$ is free as $\mathcal{O}_S$-module on a neighborhood of s for each integer r.

In order to show that $f_*I(r)$ is locally free at s, it is enough, by the standard results on base change and coherent cohomology (see [2, III, Thm. 12.11] for the case of schemes), to show that $\mathrm{H}^1(\mathbb{P}^n_s, I_s(r))$ is zero, with $I_s = I/m_sI$ and m_s the ideal of s in $\mathcal{O}_S$. Since the holomorphic tangent map of $X \to S$ is surjective, $\mathcal{O}_X$ is flat over $\mathcal{O}_S$, hence tensoring the short exact sequence:

$$0 \longrightarrow I(r) \longrightarrow \mathcal{O}(r) \longrightarrow \mathcal{O}_X(r) \longrightarrow 0$$

over $\mathcal{O}_S$ with $\mathbb{C} = \mathcal{O}_S/m_s$ gives a short exact sequence:

$$0 \longrightarrow I_s(r) \longrightarrow \mathcal{O}_s(r) \longrightarrow \mathcal{O}_{X_s}(r) \longrightarrow 0.$$

This shows that $I_s(r)$ is the rth twist of the ideal sheaf of X_s in $\mathbb{P}^n_s$. The hypothesis that X_s is a complete intersection, with degrees $r_1, \ldots, r_c$, gives a surjection of $\mathcal{O}_s$-modules:

$$\bigoplus_{i=1}^{c} \mathcal{O}_s(r - r_i) \longrightarrow I_s(r).$$

This surjection can then be extended to a Koszul type resolution (see for example [1, 5.3]):

$$K \longrightarrow I_s(r) \longrightarrow 0,$$

with K a complex (placed in degrees ≤ 0) such that, for $p \leq 0$:

$$K^p = \left(\bigwedge^{1-p} \left(\bigoplus_{i=1}^{c} \mathcal{O}_s(-r_i) \right) \right) (r).$$

In particular, $K^p = 0$ for $p > 0$ and for $p \leq -c$. We view this resolution as a quasi-isomorphism from the complex K to the complex consisting of just $I_s(r)$, placed in degree 0. Then it follows that the cohomology of $I_s(r)$ is the hypercohomology of K, and that there is a spectral sequence E with $E_1^{p,q} = \mathrm{H}^q(\mathbb{P}^n_s, K^p)$, converging to the cohomology of $I_s(r)$ (see for example [1, 3.5]). Since each K^p is a direct sum of invertible $\mathcal{O}_s$-modules, we have $E_1^{p,q} = 0$ for q not equal to 0 or n (see for example [2, III, Thm. 5.1]; this is an interesting property of projective spaces). Hence the E_1-term has the following form:

$$\begin{array}{ccccc}
\mathrm{H}^n(K^{1-c}) & \to & \cdots & \to & \mathrm{H}^n(K^0) \\
0 & \to & \cdots & \to & 0 \\
\vdots & & & & \vdots \\
0 & \to & \cdots & \to & 0 \\
\mathrm{H}^0(K^{1-c}) & \to & \cdots & \to & \mathrm{H}^0(K^0)
\end{array}$$

It follows that $E_1^{p,q} = 0$ for all (p,q) with $p + q = 1$, hence that $E_r^{p,q} = 0$ for all $r \geq 1$ and such (p,q). (Note that we have used here that $d > 0$, which is easily seen to be a necessary condition.) To finish the proof, note that $\mathrm{H}^1(\mathbb{P}^n_s, I_s(r)$ is the direct sum of the $E_\infty^{p,q}$ with $p + q = 1$, hence is zero. $\square$

Remark 2 As the proof shows, the hypothesis that $X \to S$ be smooth can be replaced by the one that $X \to S$ be flat. The proof works also in the contexts of analytic spaces (not necessarily reduced) and schemes. The spectral sequence above degerates at E_2, and can be used to compute the cohomology groups of I_s. This last fact makes it very plausible that this computation is not new.

References

[1] P. Griffiths and J. Harris. *Principles of algebraic geometry.* Wiley Interscience, 1978.

[2] R. Hartshorne. *Algebraic geometry.* Graduate Texts in Mathematics 52. Springer-Verlag, 1977.