Chapter XI

Arithmetic Part of Faltings's Proof

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1 Introduction

In this chapter we will follow §5 of [22] quite closely. We start in the following situation:

> k is a number field, A_k is an abelian variety over k, $X_k \subset A_k$ is a subvariety such that $X_{\bar{k}}$ does not contain any translate of a positive dimensional abelian subvariety of $A_{\bar{k}}$, m is a sufficiently large integer as in Chapter IX, Lemma 1

Let R be the ring of integers in k. Since we want to apply Faltings's version of Siegel's Lemma (see Ch. X, Lemma 4) we need lattices in things like $\Gamma(X_k^m, \text{line bundle})$. We obtain such lattices as:

$$\Gamma(\text{proper model of } X_k^m \text{ over } R, \text{extension of line bundle}).$$

2 Construction of Proper Models

First we extend A_k to a group scheme over $\mathrm{Spec}(R)$. Among all such extensions there is a canonical "best" one, the so-called Néron model $\mathcal{A} \to \mathrm{Spec}(R)$ of A_k. It is characterized by the property that $\mathcal{A} \to \mathrm{Spec}(R)$ is smooth and that for every smooth morphism of schemes $T \to \mathrm{Spec}(R)$ the induced map $\mathcal{A}(T) \to A_k(T_k)$ is bijective. For more information and a proof of existence we refer to [11]; there is an "explicit" construction of $\mathcal{A}/R$ when A_k is a product of elliptic curves. We will use the following properties of $\mathcal{A} \to \mathrm{Spec}(R)$:

(i) it is a group scheme,

(ii) every $x \in A_k(k)$ extends uniquely to an $x \in \mathcal{A}(\mathrm{Spec}(R))$,

(iii) $\mathcal{A} \to \mathrm{Spec}(R)$ is quasi-projective ([11] §6.4, Thm. 1) and smooth.

We choose an embedding $i\colon \mathcal{A} \hookrightarrow \mathbb{P}_R^M$. Let $\overline{\mathcal{A}}$ be the closure of $\mathcal{A} \hookrightarrow \mathbb{P}_R^M \times_R \mathbb{P}_R^M$ by $P \mapsto (iP, i(-P))$, and let $A \to \overline{\mathcal{A}}$ be its normalization. Since $\overline{\mathcal{A}} \to \mathrm{Spec}(R)$ is projective and $A \to \overline{\mathcal{A}}$ is finite (by Nagata's theorem [47], Thm. 31.H), $A \to \mathrm{Spec}(R)$ is projective. By construction A contains $\mathcal{A}$ as an open subset, $A \times_{\mathrm{Spec}(R)} \mathrm{Spec}(k) = A_k$, and the automorphism $[-1]$ of A_k extends to A. Let $\mathcal{L}_0$ be a very ample line bundle on A. Then $\mathcal{L}_1 := \mathcal{L}_0 \otimes [-1]^*\mathcal{L}_0$ is a very ample line bundle on A whose restriction to A_k is symmetric. Finally let $\mathcal{L} = \mathcal{L}_1 \otimes p^*0^*\mathcal{L}_1^{-1}$, where $p: A \to \mathrm{Spec}(R)$. Then $\mathcal{L}$ is very ample on A and we have given isomorphisms $\mathcal{L} \tilde{\to} [-1]^*\mathcal{L}$ and $\mathcal{O}_{\mathrm{Spec}(R)} \tilde{\to} 0^*\mathcal{L}$.

For $1 \leq i,j \leq m$ we have the line bundles $\mathcal{L}_i := \mathrm{pr}_i^*\mathcal{L}$ and $\mathcal{P}_{i,j} := (\mathrm{pr}_i + \mathrm{pr}_j)^*\mathcal{L} \otimes \mathrm{pr}_i^*\mathcal{L}^{-1} \otimes \mathrm{pr}_j^*\mathcal{L}^{-1}$ on A_k^m. We choose positive rational numbers $\varepsilon \leq 1$ and s as in Chapter IX, Thm. 4: if $s_1, \ldots, s_m$ are positive rational numbers with $s_1/s_2 \geq s, \ldots, s_{m-1}/s_m \geq s$ and $\varepsilon' \leq \varepsilon$ then the restriction to X_k^m of the $\mathbb{Q}$-line bundle

$$\mathcal{L}(-\varepsilon', s_1, \ldots, s_m) \ := \ \mathcal{L}_1^{(1-\varepsilon')s_1^2} \otimes \bigotimes_{i=2}^{m-1} \mathcal{L}_i^{(2-\varepsilon')s_i^2} \otimes \mathcal{L}_m^{(1-\varepsilon')s_m^2} \otimes \bigotimes_{i=1}^{m-1} \mathcal{P}_{i,i+1}^{-s_is_{i+1}}$$

on A_k^m is ample.

Let A^m denote the m-fold fibred product of $A \to \mathrm{Spec}(R)$ and let $\mathcal{L}_i := \mathrm{pr}_i^*\mathcal{L}$ on A^m. We extend the $\mathcal{P}_{i,j}$ to the open subset $\mathcal{A}^m$ of A^m by the same formula as above: $\mathcal{P}_{i,j} := (\mathrm{pr}_i + \mathrm{pr}_j)^*\mathcal{L} \otimes \mathrm{pr}_i^*\mathcal{L}^{-1} \otimes \mathrm{pr}_j^*\mathcal{L}^{-1}$. Using the construction described below it is easy to find a proper modification $B \to A^m$ such that 1) B is normal, 2) $B \to A^m$ is an isomorphism over $\mathcal{A}^m$ and 3) the $\mathcal{P}_{i,j}$ on $\mathcal{A}^m$ can be extended to line bundles on B.

2.1 Construction. Let X be an integral scheme, of finite type over $\mathbb{Z}$, $U \subset X$ open and normal, $D \subset U$ a closed subscheme whose sheaf of ideals I_D is an invertible $\mathcal{O}_U$-module. Let $\bar{D}$ be the scheme theoretic closure of D in X, let $\pi: \tilde{X} \to X$ be the blow up in $I_{\bar{D}}$, let X' be the normalization of the closure of $\pi^{-1}U$ in $\tilde{X}$ and let $\bar{D}'$ be the closure of $D' := \pi^{-1}D$ in X'. Then X' is normal, $X' \to X$ is proper and an isomorphism over U, and $I_{\bar{D}'}$ is an invertible $\mathcal{O}_{X'}$-module. $\square$

We fix a choice of B and extensions of the $\mathcal{P}_{i,j}$ from $\mathcal{A}^m$ to B. The pullback of the $\mathcal{L}_i$ to B are still denoted $\mathcal{L}_i$. We let Y be the closure of X_k^m in B and we define $Y^o := Y \cap \mathcal{A}^m$. Note that every $x \in X_k^m(k)$ extends uniquely to an $x \in Y^o(R)$. We extend all line bundles $\mathcal{L}(-\varepsilon', s_1, \ldots, s_m)^d$ to B by their defining formula.

3 Applying Faltings's version of Siegel's Lemma

We fix norms on $\mathcal{L}$ at the infinite places. This gives norms on the $\mathcal{L}_i$, the $\mathcal{P}_{i,j}$ and the $\mathcal{L}(-\varepsilon', s_1, \ldots, s_m)^d$. In order to get some control over $\Gamma(Y, \mathcal{L}(-\varepsilon', s_1, \ldots, s_m)^d)$, using Faltings's version of Siegel's Lemma, we will embed $\mathcal{L}(-\varepsilon', s_1, \ldots, s_m)^d$ into $\oplus_1^a \otimes_{i=1}^m \mathcal{L}_i^{d_i}$ (a independent of the s_i, ε' and d) just as in the proof of Chapter IX, Thm. 4. However, this time we have to keep track of norms and denominators.

We fix a finite set of generators f_α, $\alpha \in I$, of the R-module $\Gamma(A, \mathcal{L})$. We choose an isomorphism ϕ on A_k^3:

$$(\mathrm{pr}_1 + \mathrm{pr}_2 + \mathrm{pr}_3)^*\mathcal{L} \xrightarrow{\sim} \bigotimes_{1 \leq i < j \leq 3} (\mathrm{pr}_i + \mathrm{pr}_j)^*\mathcal{L} \otimes \bigotimes_{1 \leq i \leq 3} \mathrm{pr}_i^*\mathcal{L}^{-1}$$

whose existence is guaranteed by the Theorem of the cube. Note that if ϕ' is another such isomorphism then $\phi' = u\phi$ for a unique $u \in k^*$. From ϕ we can construct isomorphisms $\phi_{a,b,i,j}$ on A_k^m for $a, b \in \mathbb{Z}$ and $1 \leq i,j \leq m$:

$$\phi_{a,b,i,j}: (a\mathrm{pr}_i + b\mathrm{pr}_j)^*\mathcal{L} \xrightarrow{\sim} \mathcal{L}_i^{a^2} \otimes \mathcal{L}_j^{b^2} \otimes \mathcal{P}_{i,j}^{ab}$$

as follows. Let $\mathcal{P} := (\mathrm{pr}_1 + \mathrm{pr}_2)^*\mathcal{L} \otimes \mathrm{pr}_1^*\mathcal{L}^{-1} \otimes \mathrm{pr}_2^*\mathcal{L}^{-1}$ on A_k^2. We have (on A_k^m and on A_k):

$$\phi\colon (\mathrm{pr}_i + \mathrm{pr}_j, \mathrm{pr}_l)^*\mathcal{P} \xrightarrow{\sim} (\mathrm{pr}_i, \mathrm{pr}_l)^*\mathcal{P} \otimes (\mathrm{pr}_j, \mathrm{pr}_l)^*\mathcal{P}$$

and

$$\phi^{-1}\colon [a+1]^*\mathcal{L} \xrightarrow{\sim} [a]^*\mathcal{L}^2 \otimes [a-1]^*\mathcal{L}^{-1} \otimes \mathcal{L}^2$$

And by definition: $(a\mathrm{pr}_i + b\mathrm{pr}_j)^*\mathcal{L} \tilde{\to} a\mathrm{pr}_i^*\mathcal{L} \otimes b\mathrm{pr}_j^*\mathcal{L} \otimes (a\mathrm{pr}_i, b\mathrm{pr}_j)^*\mathcal{P}$ on A_k^m.

3.1 Lemma. (Lemma 5.1 of [22]) *For any a, b, i, j let $\mathcal{F}_{a,b,i,j}$ be the subsheaf of $\mathcal{O}_B$-modules of the $\mathcal{O}_{A_k^m}$-module $\left(\mathcal{L}_i^{a^2} \otimes \mathcal{L}_j^{b^2} \otimes \mathcal{P}_{i,j}^{ab}\right) \otimes k$ generated by the $\phi_{a,b,i,j}(a\mathrm{pr}_i + b\mathrm{pr}_j)^* f_\alpha$, $\alpha \in I$. There exists $c_1 \in \mathbb{R}$ such that for all a, b, i, j there exists $r \in \mathbb{Z}$, $0 < r < \exp(c_1(1 + a^2 + b^2))$ with:*

1. *$r \cdot \mathcal{F}_{a,b,i,j} \subset \mathcal{L}_i^{a^2} \otimes \mathcal{L}_j^{b^2} \otimes \mathcal{P}_{i,j}^{ab}$ (on B)*
2. *on $\mathcal{A}^m$: $\mathcal{L}_i^{a^2} \otimes \mathcal{L}_j^{b^2} \otimes \mathcal{P}_{i,j}^{ab} \subset r^{-1} \cdot \mathcal{F}_{a,b,i,j}$*

Proof. Let us first check that the lemma holds for our choice of data if and only if it holds for any choice. If $\phi' = u^{-1}\phi$ with $u \in k^*$ then

$$\phi'_{a,b,i,j} = u^{a(a-1)/2 + b(b-1)/2 - a - b + 2}\phi_{a,b,i,j}$$

If $u\mathcal{L}_i \subset \mathcal{L}'_i \subset u^{-1}\mathcal{L}_i$ and $u\mathcal{P}_{i,j} \subset \mathcal{P}'_{i,j} \subset u^{-1}\mathcal{P}_{i,j}$ for all i, j then

$$u^{a^2+b^2+ab}\mathcal{L}_i^{a^2} \otimes \mathcal{L}_j^{b^2} \otimes \mathcal{P}_{i,j}^{ab} \subset \mathcal{L}_i'^{a^2} \otimes \mathcal{L}_j'^{b^2} \otimes \mathcal{P}_{i,j}'^{ab} \subset u^{-(a^2+b^2+ab)}\mathcal{L}_i^{a^2} \otimes \mathcal{L}_j^{b^2} \otimes \mathcal{P}_{i,j}^{ab}$$

If $\pi\colon B' \to B$ is another model satisfying the same conditions as B, then for any line bundle $\mathcal{M}$ on B we have: $\Gamma(B', \pi^*\mathcal{M}) = \Gamma(B, \pi_*\pi^*\mathcal{M}) = \Gamma(B, \mathcal{M})$; this shows that c_1 works for B' if and only if c_1 works for B (if we take $\mathcal{P}'_{i,j} = \pi^*\mathcal{P}_{i,j}$).

Let us now prove statements (1) and (2) on $\mathcal{A}^m$. By [50] Lemme II, 1.2.1, there exists $d_0 > 0$ such that $\mathcal{L}^{d_0}|_{A_k}$ can be extended to a $\mathcal{L}'$ on $\mathcal{A}$ for which some isomorphism ϕ' on A_k^3 extends to an isomorphism over $\mathcal{A}^3$. This means that the lemma is true for $\mathcal{L}'$ on $\mathcal{A}^m$, hence for $\mathcal{L}^{d_0}$ and hence for $\mathcal{L}$ itself (on $\mathcal{A}^m$).

To finish the proof we have to consider pole orders of the $\phi_{a,b,i,j}(a\mathrm{pr}_i + b\mathrm{pr}_j)^* f_\alpha$ along the (finitely many) divisors of B contained in $B - \mathcal{A}^m$; these divisors are irreducible components of closed fibres of $B \to \mathrm{Spec}(R)$. Let V be the local ring on B at the generic point of such a divisor. Then V is a discrete valuation ring and pr_i and pr_j define V-valued points of A. The lemma is then proved by base changing from R to V, replacing A_V by the Néron model of A over V, and applying the arguments above. Note that those arguments give upper bounds on pole and zero orders on the whole model; that upper bounds on the pole orders on the whole model give upper bounds on the pole orders on V-valued points but that the same is not true for the order of zeros since the divisors of zeros of the $\phi_{a,b,i,j}(a\mathrm{pr}_i + b\mathrm{pr}_j)^* f_\alpha$ are not necessarily supported on closed fibres. □

Let $s_1, \ldots, s_m$ be positive rational numbers, let $0 \le \varepsilon' \le \varepsilon$ and let $d \in \mathbb{Z}$, $d > 0$, be a sufficiently divisible square. Let $d_i := 4ds_i^2$. Let α_i $(1 \le i \le m-1)$ and β_i $(1 \le i \le m)$ be arbitrary elements of the set I of labels of the f_α. Multiplication by the product of the $r_i \cdot \phi_{s_i\sqrt{d}, s_{i+1}\sqrt{d}, i, i+1}(s_i\sqrt{d}\mathrm{pr}_i + s_{i+1}\sqrt{d}\mathrm{pr}_{i+1})^* f_{\alpha_i}$ (with the r_i as given by Lemma 3.1), the $\mathrm{pr}_i^* f_{\beta_i}^{\varepsilon' ds_i^2}$, $\mathrm{pr}_1^* f_{\beta_1}^{2ds_1^2}$ and $\mathrm{pr}_m^* f_{\beta_m}^{2ds_m^2}$ is a map from $\mathcal{L}(-\varepsilon', s_1, \ldots, s_m)^d$

to $\bigotimes_{i=1}^m \mathcal{L}_i^{d_i}$. Since the f_α ($\alpha \in I$) are generators of $\Gamma(A, \mathcal{L})$, all such maps together give an embedding on B:

$$\rho\colon \mathcal{L}(-\varepsilon', s_1, \dots, s_m)^d \hookrightarrow \bigoplus_{j=1}^{a} \bigotimes_{i=1}^{m} \mathcal{L}_i^{d_i}$$

where $a = \# I^{2m-1}$. By construction, this embedding remains an embedding after restriction to Y; however, only on $B_k = A_k^m$ we know that it is an embedding without zeros (that is, locally split).

3.2 Constructions. 1. Let A be a commutative ring, $a_1, \dots, a_n$, $r \in A$ such that $r \in \sum_i Aa_i$. Then the homology of the (start of the Koszul) complex:

$$0 \to A \to A^n \to A^{n^2}, \qquad x \mapsto (a_i x)_i, \qquad (x_i)_i \mapsto (a_j x_i - a_i x_j)_{i,j}$$

is annihilated by r. Proof: choose b_i such that $\sum a_i b_i = r$. Then use the "homotopy": $(x_i)_i \mapsto \sum_i b_i x_i$, $(x_{i,j})_{i,j} \mapsto (\sum_j b_j x_{i,j})_i$.

2. Let X be a scheme, $\mathcal{L}$ and $\mathcal{M}$ invertible $\mathcal{O}_X$-modules, $r \in \mathcal{O}_X(X)$, $\rho_j\colon \mathcal{L} \to \mathcal{M}$ ($1 \le j \le a$) a finite set of morphisms such that $r\mathcal{M} \subset \sum_j \rho_j(\mathcal{L})$. Twisting by $\mathcal{L}^{-1}$ gives $\rho_j\colon \mathcal{O}_X \to \mathcal{M} \otimes \mathcal{L}^{-1}$. As in (1) we get a complex $0 \to \mathcal{O}_X \to \oplus_{j=1}^a (\mathcal{M} \otimes \mathcal{L}^{-1}) \to \oplus_{j=1}^{a^2} (\mathcal{M} \otimes \mathcal{L}^{-1})^2$. Twisting by $\mathcal{L}$ gives:

$$0 \to \mathcal{L} \to \oplus_{j=1}^a \mathcal{M} \to \oplus_{j=1}^{a^2} \mathcal{M}^2 \otimes \mathcal{L}^{-1}$$

The homology of this complex is annihilated by r. □

We apply these constructions to the embeddings $\rho_j\colon \mathcal{L}(-\varepsilon', s_1, \dots, s_m)^d \to \otimes_{i=1}^m \mathcal{L}_i^{d_i}$. In this case we can even embed $\mathcal{L}(-\varepsilon', s_1, \dots, s_m)^{-d}$ into $\oplus_{j=1}^a \otimes_{i=1}^m \mathcal{L}_i^{d_i}$ by multiplication by the $r_i \cdot \phi_{s_i\sqrt{d}, -s_{i+1}\sqrt{d}, i, i+1}(s_i\sqrt{d}\mathrm{pr}_i - s_{i+1}\sqrt{d}\mathrm{pr}_{i+1})^* f_{\alpha_i}$ and by the $\mathrm{pr}_i^* f_{\beta_i}^{(4-\varepsilon')ds_i^2}$. All together we get a complex:

$$0 \to \mathcal{L}(-\varepsilon', s_1, \dots, s_m)^d \to \bigoplus_{j=1}^{a} \bigotimes_{i=1}^{m} \mathcal{L}_i^{d_i} \to \bigoplus_{j=1}^{a^3} \bigotimes_{i=1}^{m} \mathcal{L}_i^{3d_i}$$

of sheaves on Y whose homology on Y^o is annihilated by some $r \in \mathbb{Z}$ with $0 < r < \exp(c_1 \sum_{i=1}^m d_i)$.

3.3 Proposition. (Proposition 5.2 of [22]) *There exists $c_2 \in \mathbb{R}$ such that for all $s_1, \dots, s_m$, ε' positive rational numbers with $0 < \varepsilon' \le \varepsilon$, $d \in \mathbb{Z}$ square and sufficiently divisible there exists an exact sequence:*

$$0 \to \Gamma(X_k^m, \mathcal{L}(-\varepsilon', s_1, \dots, s_m)^d) \to \Gamma(X_k^m, \bigoplus_{j=1}^{a} \bigotimes_{i=1}^{m} \mathcal{L}_i^{d_i}) \to \Gamma(X_k^m, \bigoplus_{j=1}^{a^3} \bigotimes_{i=1}^{m} \mathcal{L}_i^{3d_i})$$

such that:

1. *the norms of the maps in the exact sequence and the difference between the norm on $\Gamma(X_k^m, \mathcal{L}(-\varepsilon', s_1, \dots, s_m)^d)$ and that induced on it from $\Gamma(X_k^m, \bigoplus_{j=1}^a \bigotimes_{i=1}^m \mathcal{L}_i^{d_i})$ are bounded by $\exp(c_2 \sum_i d_i)$,*
2. *if a section f of $\mathcal{L}(-\varepsilon', s_1, \dots, s_m)^d$ over X_k^m maps to $\Gamma(Y^o, \bigoplus_\alpha \bigotimes_{i=1}^m \mathcal{L}_i^{d_i})$ then there exists $r \in \mathbb{Z}$ with $0<r<\exp(c_2 \sum_i d_i)$ such that $rf \in \Gamma(Y^o, \mathcal{L}(-\varepsilon', s_1, \dots, s_m)^d)$.*

Proof. The exact sequence is obtained by applying $\Gamma(X_k^m, -)$ to the complex above. It remains to prove the statements concerning norms. To make sense of these statements we must say which norms we are talking about. The norms on the line bundles are obtained from those on $\mathcal{L}$. On k_v-vector spaces such as

$$\Gamma(X_k^m, \text{metrized vector bundle}) \otimes_k k_v$$

we take the supremum norm over the complex valued points. On finite direct sums of normed k_v-vector spaces we take the maximum norm. Finally, if V is a k-vector space with given norms on the $V \otimes_k k_v$ then we take the maximum norm on

$$V \otimes_{\mathbb{Q}} \mathbb{R} \;=\; V \otimes_k k \otimes_{\mathbb{Q}} \mathbb{R} \;=\; V \otimes_k (\oplus_v k_v) \;=\; \oplus_v (V \otimes_k k_v)$$

Note that the norm we have put on $(\oplus_j \Gamma(X_k^m, \mathcal{F}_j)) \otimes_{\mathbb{Q}} \mathbb{R}$ can be viewed as the supremum norm on $\Gamma(X_k^m \times_{\mathrm{Spec}(\mathbb{Q})} \mathrm{Spec}(\mathbb{R}), \oplus_j \mathcal{F}_j)$, the supremum being taken over the $P \in X_k^m(\mathbb{C})$. The proof concerning the statements about norms is analogous to the arguments to bound poles and zeros at the finite places. The analog of Lemma 3.1 is independent of the choice of metrics on $\mathcal{L}$, and for a particular choice (metrics with translation invariant curvature forms) ϕ is an isometry. □

We want to apply the following lemma (for a proof see Ch. X, Lemma 4).

3.4 Lemma. (Prop. 2.18 of [22]) *Let $\rho: V \to W$ be a linear map between normed finite dimensional $\mathbb{R}$-vector spaces with lattices M and N such that $\rho(M) \subset N$. Let $U := \ker(\rho)$ and $L := M \cap U$; then $L \subset U$ is a lattice. Assume that for some $C \geq 2$ we have: 1. ρ has norm $\leq C$, 2. M is generated by elements of norm $\leq C$, 3. any non-trivial element of M or of N has norm $\geq C^{-1}$. Then for any proper subspace $U_0 \subset U$ there exists an element $f \in L$, $f \notin U_0$ such that $\|f\| \leq (C^3 \dim V)^{\dim(V)/\mathrm{codim}(U_0)}$.* □

Now let $U := \Gamma(X_k^m, \mathcal{L}(-\varepsilon', s_1, \ldots, s_m)^d) \otimes_{\mathbb{Q}} \mathbb{R}$, $V := \Gamma(X_k^m, \bigoplus_{j=1}^a \bigotimes_{i=1}^m \mathcal{L}_i^{d_i}) \otimes_{\mathbb{Q}} \mathbb{R}$ and $W := \Gamma(X_k^m, \bigoplus_{j=1}^{a^3} \bigotimes_{i=1}^m \mathcal{L}_i^{3d_i}) \otimes_{\mathbb{Q}} \mathbb{R}$. The lattices M and N are $\Gamma(Y, \bigoplus_{j=1}^a \bigotimes_{i=1}^m \mathcal{L}_i^{d_i})$ and $\Gamma(Y, \bigoplus_{j=1}^{a^3} \bigotimes_{i=1}^m \mathcal{L}_i^{3d_i})$. Condition 1 is satisfied with $C = \exp(c_2 \sum_i d_i)$. Condition 2 is satisfied with C of the form $\exp(c_3 \sum_i d_i)$ (with c_3 independent of the s_i) because $S := \oplus_{d_1,\ldots,d_m \geq 0} \Gamma(Y, \bigotimes_{i=1}^m \mathcal{L}_i^{d_i})$ is a finitely generated R-algebra by the following argument. The morphism $f: Y \to X^m$ is proper and each $\mathcal{L}_i$ is of the form $f^*\mathcal{L}'_i$, hence $S = \oplus_{d_1,\ldots,d_m \geq 0} \Gamma(X^m, (f_*\mathcal{O}_Y) \otimes \bigotimes_{i=1}^m \mathcal{L}'^{d_i}_i)$ by the projection formula ([27], Ch. 2, Exc. 5.1). Let $\iota: X^m \hookrightarrow P := (\mathbb{P}_R^M)^m$ denote a closed immersion induced by $\mathcal{L}$ on each factor. Then $S = \oplus_{d_1,\ldots,d_m \geq 0} \Gamma(P, \mathcal{F}(d_1, \ldots, d_m))$, with $\mathcal{F}$ the coherent $\mathcal{O}_P$-module $\iota_* f_* \mathcal{O}_Y$. One then shows that S is a finitely generated module over the multi-homogeneous coordinate ring of P.

The proof of condition 3 that we give is copied from [81], Lemma 13.9.

3.5 Lemma. *There exists $c_4 \in \mathbb{R}$ such that for all $d_i \geq 0$ and for all non-zero $f \in \Gamma(Y, \bigotimes_{i=1}^m \mathcal{L}_i^{d_i})$ we have $\|f\| \geq \exp(-c_4 \sum_i d_i)$.*

Proof. Let $[k' : k] < \infty$, let R' be the ring of integers in k', $P = (P_1, \ldots, P_m) \in X_k^m(k') = Y(R')$ such that $P^*(f) \neq 0$. Then:

$$h_{\otimes \mathcal{L}_i^{d_i}}(P) = [k' : \mathbb{Q}]^{-1} \deg_{R'}(P^* \otimes_i \mathcal{L}_i^{d_i}) = \sum_{i=1}^m d_i h_{\mathcal{L}}(P_i)$$

On the other hand:

$$h_{\otimes \mathcal{L}_i^{d_i}}(P) = [k' : \mathbb{Q}]^{-1} \left(\log \# \left(\left(P^* \otimes_i \mathcal{L}_i^{d_i} \right) / R' \cdot P^*(f) \right) - \sum_{v'|\infty} \epsilon(v') \log \|f(P)\|_{v'} \right)$$

where $\epsilon(v') = 1$ if v' is real and $\epsilon(v') = 2$ if v' is complex. Hence:

$$[k' : \mathbb{Q}]^{-1} \sum_{v'|\infty} \epsilon(v') \log \|f(P)\|_{v'} \geq - \sum_{i=1}^{m} d_i h_{\mathcal{L}}(P_i)$$

Let $\pi_i : X_k \to \mathbb{P}^{\dim X_k}$ be associated to $\mathcal{L}$: $\pi_i^* \mathcal{O}(1) \cong \mathcal{L}$, and let h be the naive height function on $\mathbb{P}^{\dim X_k}(\bar{k})$. Then there exists c_4 such that $|h_{\mathcal{L}}(P_i) - h(\pi_i(P_i))| \leq c_4$ for all k' and all P_i. Now we take the P_i such that all coordinates of the $\pi_i(P_i)$ are roots of unity; then $h(\pi_i P_i) = 0$. Note that these points are Zariski dense. So we have:

$$[k' : \mathbb{Q}]^{-1} \sum_{v'|\infty} \epsilon(v') \log \|f(P)\|_{v'} \geq -c_4 \sum_{i=1}^{m} d_i$$

It follows that there exists an infinite place v of k with $\|f\|_{v,\sup} \geq \exp(-c_4 \sum_i d_i)$. □

To get a suitable subspace U_0 of U we put $\varepsilon' = \varepsilon - \sigma$ with $0 < \sigma < \varepsilon$, let $x = (x_1, \ldots, x_m) \in X_k^m(k)$ be a smooth point and let

$$U_0 := \{ f \in \Gamma(X_k^m, \mathcal{L}(\sigma - \varepsilon, s_1, \ldots, s_m)^d) \mid \operatorname{index}(x, f) \geq \sigma \}$$

For this to be useful we need a lower bound for $\operatorname{codim}(U_0)$. Note that:

$$\mathcal{L}(\sigma - \varepsilon, s_1, \ldots, s_m)^d \; = \; \mathcal{L}(-\varepsilon, s_1, \ldots, s_m)^d \otimes \bigotimes_{i=1}^{m} \mathcal{L}_i^{d\sigma s_i^2}$$

Because $\mathcal{L}(-\varepsilon, s_1, \ldots, s_m)$ is ample on X_k^m, $\mathcal{L}(-\varepsilon, s_1, \ldots, s_m)^d$ has a global section that does not vanish at x. Because $\mathcal{L}$ is very ample on X_k there are global sections of $\mathcal{L}^{d\sigma s_i^2}$ on X_k with prescribed Taylor expansion at x_i up to order $d\sigma s_i^2 = d_i \sigma / 4$. Namely: "$\mathcal{L}$ very ample on X_k" is equivalent to "$\Gamma(X_k, \mathcal{L})$ separates points and tangent vectors" which implies "$\Gamma(X_k, \mathcal{L}) \to \mathcal{O}_{X_k, x_i} / m_{x_i}^2$ is surjective for all i" which implies "$\Gamma(X_k, \mathcal{L}^{\sigma d_i / 4}) \to \mathcal{O}_{X_k, x_i} / m_{x_i}^{1 + \sigma d_i / 4}$ is surjective". It follows that $\operatorname{codim} U_0 \geq c_5 (\sigma^m \prod_i d_i)^{\dim X_k}$, with $c_5 > 0$ depending only on m and $\dim X_k$. In Chapter VII it was proved that $\dim V \leq c_6 (\prod_i d_i)^{\dim X_k}$ with c_6 independent of the d_i (and of σ of course). It follows that $(\dim V)/(\operatorname{codim} U_0) \leq c_7 \sigma^{-m \dim X_k}$ with c_7 independent of the d_i and of σ. Prop. 3.3 and Lemma 3.4 give us $f \neq 0$ in $\Gamma(Y, \mathcal{L}(\sigma - \varepsilon, s_1, \ldots, s_m)^d)$ with index $< \sigma$ at x and $\log \|f\| \leq c_8 \sigma^{-m \dim X_k} \sum_{i=1}^m d_i$ (where c_8 is independent of the d_i and of σ). We have proved:

3.6 Theorem. (Theorem 5.3 of [22]) *Let σ be a rational number with $0 < \sigma < \varepsilon$. There exists a real number $c(\sigma)$, depending only on σ (recall that ε is fixed), with the following property. For any smooth k-rational point x in X_k^m, any sequence $s_1, \ldots, s_m$ of positive rational numbers with $s_1/s_2 \geq s, \ldots, s_{m-1}/s_m \geq s$, and any square integer d which is big enough and sufficiently divisible there exists $f \in \Gamma\left(Y^o, \mathcal{L}(\sigma - \varepsilon, s_1, \ldots, s_m)^d\right)$ with index less than σ at x and norm at the infinite places bounded by $\exp(c(\sigma) \sum_{i=1}^m d_i)$, where $d_i = 4 d s_i^2$.* □

4 Leading Terms and Differential Operators

This section more or less follows part of a manuscript [33] on [22] written by L. Lafforgue in 1991 (not published), see also [34] and [35]. We will need some properties of leading terms in Taylor expansions in the following situation. Let X be an integral separated R-scheme and $x\colon \mathrm{Spec}(R) \to X$ an R-valued point of X. Let $\partial_1, \ldots, \partial_m$ be R-derivations of $\mathcal{O}_X$, let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module and f a section of $\mathcal{L}$ vanishing up to order e along x, that is, $f \in \Gamma(X, I^e\mathcal{L})$, where $I \subset \mathcal{O}_X$ is the ideal sheaf of x. Then for integers $e_1, \ldots, e_m \geq 0$ with $\sum_i e_i = e$ we can define $(\partial_1^{e_1} \cdots \partial_m^{e_m} f)(x)$ in $\Gamma(\mathrm{Spec}(R), x^*\mathcal{L})$ as follows: locally on X write $f = gf_1$ with f_1 a generator of $\mathcal{L}$ and put:

$$(\partial_1^{e_1} \cdots \partial_m^{e_m} f)(x) := (\partial_1^{e_1} \cdots \partial_m^{e_m} g)(x) f_1(x)$$

(one easily checks that the right hand side does not depend on the choice of f_1, so the local construction "glues"). Also, it is important to note that $(\partial_1^{e_1} \cdots \partial_m^{e_m} f)(x)$ does not depend on the order in which the derivations are applied, so that we can write $(\prod_i \partial_i^{e_i} {\cdot} f)(x)$ for it, and that for derivations $\partial'_1, \ldots, \partial'_m$ locally of the form $\partial'_i = \partial_i +$ terms like $g{\cdot}\partial$ with $g \in \Gamma(X, I)$ and $\partial\colon \mathcal{O}_X \to \mathcal{O}_X$ an R-derivation, one has $(\prod_i \partial_i'^{e_i} {\cdot} f)(x) = (\prod_i \partial_i^{e_i} {\cdot} f)(x)$.

4.1 Lemma. *In this situation,* $(\prod_i (\partial_i^{e_i}/e_i!){\cdot} f)(x)$*, which is a priori only a section of* $x^*\mathcal{L}$ *over* $\mathrm{Spec}(k)$*, is actually a section of* $x^*\mathcal{L}$ *over* $\mathrm{Spec}(R)$*.*

Proof. It suffices to verify this locally, so we may assume that X is affine, that $\mathcal{L}$ is trivialised by f_1 and that $f = gf_1$. Since f vanishes up to order e at x, g can be written as a sum of products $g_1 \cdots g_e$ with the g_i in $\Gamma(X, I)$. It suffices to verify the claim for each term, so we may assume that g is one such term, say $g = g_1 \cdots g_e$. Now consider $((\prod_{i=1}^m \partial_i^{e_i}) g_1 \cdots g_e)(x)$. This expression can be expanded by applying the product rule $(\partial(fg) = f\partial(g) + g\partial(f))$ as many times as possible. Since we evaluate at x, and the $g_i(x)$ are zero, only terms in which all g_i have been derived can give a non-zero contribution. As $\sum_i e_i = e$, we see that $((\prod_{i=1}^m \partial_i^{e_i}) g_1 \cdots g_e)(x)$ equals the sum over all partitions of $\{1, 2, \ldots, e\}$ into sets $S_1, \ldots, S_m$ of cardinalities $e_1, \ldots, e_m$, of the expresion

$$\prod_{i=1}^m \left(\partial_i^{e_i} (\prod_{j \in S_i} g_j) \right)(x) = \prod_{i=1}^m \left(e_i! \prod_{j \in S_i} (\partial_i g_j)(x) \right)$$

so the claim that $(\prod_i (\partial_i^{e_i}/e_i!){\cdot} f)(x)$ is integral has now been proved. □

However, in §5 we cannot apply this result directly, because we will only know that f vanishes at x up to order e on the generic fibre X_k of X. The problem is then that in general the scheme theoretic closure in X of the closed subscheme of X_k defined by $(I{\cdot}\mathcal{O}_{X_k})^e$ is strictly contained in the closed subscheme of X defined by I^e (in other words: $I^e \neq \mathcal{O}_X \cap (I{\cdot}\mathcal{O}_{X_k})^e$). Yet another complication will be that we will work in a product situation with weighted degrees for differential operators. The result we need is the following.

Let $m \geq 1$. For $1 \leq i \leq m$ let $p_i\colon X_i \to P_i$ be a morphism of integral separated R-schemes of finite type, $\mathcal{M}_i$ an invertible $\mathcal{O}_{X_i}$-module, $G_i \in \Gamma(X_i, \mathcal{M}_i)$ a global section annihilating $\Omega^1_{X_i/P_i}$ and $\partial_{i,j}$, $1 \leq j \leq n_i$, some set of R-derivations on P_i. Let

$X := X_1 \times \cdots \times X_m$, $P := P_1 \times \cdots \times P_m$ (fibred products over $\mathrm{Spec}(R)$), and $p\colon X \to P$ the product of the p_i. Let Y be a separated R-scheme with a R-morphism $q\colon Y \to X$ that is an isomorphism on the generic fibres: $q_k\colon Y_k \tilde{\to} X_k$. Let $\mathcal{N}$ be an invertible $\mathcal{O}_Y$-module and $N \in \Gamma(Y, \mathcal{N})$ a global section annihilating $\Omega^1_{Y/X}$. Let $x \in Y(R)$ and write $q(x) = (x_1, \ldots, x_m)$ with $x_i \in X_i(R)$. Suppose that for all i the morphism of k-schemes $p_k\colon X_{i,k} \to P_{i,k}$ is etale at $x_{i,k}$, and that $P_i \to \mathrm{Spec}(R)$ is smooth along $p_i(x_i)$. Let $\mathcal{L}$ be an invertible $\mathcal{O}_Y$-module, $f \in \Gamma(Y, \mathcal{L})$ and let σ be the index of $f|_{Y_k}$ at x_k with respect to some weights $d_1, \ldots, d_m > 0$. Finally let $e_{i,j} \geq 0$ be given with $\sum_{i,j} e_{i,j}/d_i = \sigma$. Since the $p_{i,k}\colon X_{i,k} \to P_{i,k}$ are etale at $x_{i,k}$, the $\partial_{i,j}$ can be uniquely lifted to derivations $\partial_{i,j}$ on a neighborhood of $x_{i,k}$ in $X_{i,k}$. Since $Y_k = X_k = \prod_i X_{i,k}$ we can view the $\partial_{i,j}$ as derivations in the ith direction on a neighborhood of x_k in Y_k. As usual, one can define $(\prod_{i,j} \partial_{i,j}^{e_{i,j}} f)(x_k)$ in $x_k^*\mathcal{L}$ to be $(\prod_{i,j} \partial_{i,j}^{e_{i,j}} g)(x_k) \cdot f_1(x_k)$, where f_1 is a generator of $\mathcal{L}_{x_k}$ and $f = g f_1$.

4.2 Lemma. *In this situation,*

$$\prod_{i,j} ((NG_i)^{e_{i,j}}/e_{i,j}!) \cdot \left(\left(\prod_{i,j} \partial_{i,j}^{e_{i,j}} \right) f \right) (x_k)$$

extends to a global section over $\mathrm{Spec}(R)$ *of* $x^*\mathcal{L} \otimes x^*\mathcal{N}^e \otimes \bigotimes_{i=1}^m x_i^*\mathcal{M}_i^{e_i}$, *where* $e_i = \sum_j e_{i,j}$ *and* $e = \sum_i e_i$.

Proof. It suffices to check the statement locally, so after localizing R we may assume that $\mathcal{L}$ has generator f_1, that $f = g f_1$ and that the $\mathcal{M}_i$ and $\mathcal{N}$ are trivial. Let $J_i \subset \mathcal{O}_{P_i}$ be the ideal sheaf of $p_i(x_i)$ and let $J \subset \mathcal{O}_P$ be the ideal sheaf of $p(q(x))$. Let $M_i \subset \mathcal{O}_{X_i}$ be the ideal of x_i and $M \subset \mathcal{O}_X$ the ideal of $q(x)$. Let $I \subset \mathcal{O}_Y$ be the ideal of x and $I_i := \mathcal{O}_Y \cdot q^\# M_i$. The closed subscheme of Y defined by the ideal $I^{e+1} + \sum_{i=1}^m I_i^{e_i+1}$ is finite over $\mathrm{Spec}(R)$, hence affine, so isomorphic to $\mathrm{Spec}(A)$ for some R-algebra A. The ideals of A induced by I and the I_i will be denoted by the same symbols. For each i, let $\mathrm{Spec}(B_i)$ denote the closed subscheme of X_i defined by $M_i^{e_i+1}$. Let $B := \bigotimes_{i=1}^m B_i$ (tensor product over R); then $\mathrm{Spec}(B)$ is the closed subscheme of X defined by $\sum_i M_i^{e_i+1}$. Finally, let $\mathrm{Spec}(C_i)$ be the closed subscheme of P_i defined by $J_i^{e_i+1}$ and let $C := \bigotimes_{i=1}^m C_i$ (tensor product over R). Then we have a commutative diagram

$$\begin{array}{ccccc} Y & \to & X & \to & P \\ \uparrow & & \uparrow & & \uparrow \\ \mathrm{Spec}(A) & \to & \mathrm{Spec}(B) & \to & \mathrm{Spec}(C) \end{array}$$

The rings A, B and C are R-modules of finite type, the morphisms $C \to B \to A$ induce isomorphisms $C \otimes_R k \tilde{\to} B \otimes_R k \tilde{\to} A \otimes_R k$. Because of the smoothness of the $P_i \to \mathrm{Spec}(R)$ at the $p(x_i)$, the rings C_i and C are torsion free R-modules.

4.3 Lemma. *Let $\overline{B}$ be the image of B in A, then $A/\overline{B}$ is annihilated by N^e.*

Proof. We know that N annihilates $\Omega^1_{Y/X}$, hence also $x^*\Omega^1_{Y/X} = I/(I^2 + A\overline{M})$, where $\overline{M}$ is the image of M in A; note that $\overline{M} \subset I$. Since $A = R + I$, we have $I^2 + A \cdot \overline{M} = I^2 + \overline{M} + I\overline{M} = I^2 + \overline{M}$. It follows that $NI \subset I^2 + \overline{M}$. Induction shows $N^e I \subset I^{e+1} + \overline{M} = \overline{M}$, hence $N^e A = N^e(R + I) \subset R + \overline{M} = \overline{B}$. $\square$

4.4 Lemma. *For all i, the morphism $C_i \to B_i$ is injective, and B_i/C_i is annihilated by $G_i^{e_i}$.*

Proof. The morphism $C_i \to B_i$ is injective since C_i is torsion free and $C_{i,k} \tilde{\to} B_{i,k}$. The proof that $G_i^{e_i}$ annihilates B_i/C_i is the same as the proof of the previous lemma. $\square$

4.5 Lemma. *The morphism $C \to B$ is injective, and B/C is annihilated by $\prod_{i=1}^m G_i^{e_i}$.*

Proof. The morphism $C \to B$ is injective since C is torsion free and $C_k \tilde{\to} B_k$. Let us consider the filtration

$$B = B^1 \supset B^2 \supset \cdots \supset B^m = C$$

where B^j is the image in $B = B_1 \otimes \cdots \otimes B_m$ of $\bigotimes_{i<j} C_i \otimes \bigotimes_{i \geq j} B_i$. The successive quotient B^j/B^{j+1} of this filtration is a quotient of $\bigotimes_{i<j} C_i \otimes (B_j/C_j) \otimes \bigotimes_{i>j} B_j$, hence by the previous lemma it is annihilated by $G_j^{e_j}$. The lemma follows. $\square$

We have now proved that $h := (N^e \prod_{i=1}^m G_i^{e_i})g$, modulo $I^{e+1} + \sum_{i=1}^m I_i^{e_i+1}$, is in C. Note that $(\prod_{i,j} \partial_{i,j}^{e_{i,j}} g')(x)$ is zero for all $g' \in I^{e+1} + \sum_{i=1}^m I_i^{e_i+1}$. Let us consider h as an element of C. Since the index at x_k of $f|_{Y_k}$ is σ, it follows that the image of h in C_k is in the ideal $C_k \cdot J_1^{e_1} \cdot \cdots \cdot J_m^{e_m}$. Since the ring $C/(C \cdot J_1^{e_1} \cdot \cdots \cdot J_m^{e_m})$ is torsion free (this follows again from the smoothness of $P \to \mathrm{Spec}(R)$ at $p(q(x))$), h is actually an element of $C \cdot J_1^{e_1} \cdot \cdots \cdot J_m^{e_m}$. The method of the proof of Lemma 4.1, plus the fact that the $\partial_{i,j}$ are derivations on P in the direction of P_i, show that $((\prod_{i,j} \partial_{i,j}^{e_{i,j}}/e_{i,j}!)h)(p(q(x)))$ is integral. $\square$

5 Proof of the Main Theorem

5.1 Theorem. (Theorem I of [22]) *Let k be a number field, A_k an abelian variety over k and $X_k \subset A_k$ a closed subvariety such that $X_{\bar{k}}$ does not contain any translate of a positive dimensional abelian subvariety of $A_{\bar{k}}$. Then $X_k(k)$ is finite.*

Proof. The proof is by contradiction: we suppose that $X_k(k)$ is *not* finite. We may also suppose that $X_k(k)$ is Zariski-dense in X_k, because we can replace X_k by a positive dimensional irreducible component of the Zariski-closure of $X_k(k)$.

The situation is now as in §§1–3: A, $\mathcal{L}$, m, B, Y, the $\mathcal{P}_{i,j}$, ε and s are fixed. We embed X_k into some projective space over k by choosing a basis of $\Gamma(X_k, \mathcal{L}|_{X_k})$. Composing this embedding with a suitable projection as in Chapter IX, Lemma 5 we get a finite morphism $\mathrm{pr}\colon X_k \to \mathbb{P}_k^n$, where $n = \dim X_k$. Note that $\mathcal{L} = \mathrm{pr}^*\mathcal{O}(1)$. Let X denote the closure of X_k in A and let $\widetilde{X} \to X$ be a proper modification such that $\widetilde{X}_k = X_k$ and that $\mathrm{pr}\colon X_k \to \mathbb{P}_k^n$ extends to $\mathrm{pr}\colon \widetilde{X} \to \mathbb{P}_R^n$ (e.g., one can take $\widetilde{X}$ to be the closure in $X \times_{\mathrm{Spec}(R)} \mathbb{P}_R^n$ of the graph of $\mathrm{pr}\colon X_k \to \mathbb{P}_k^n$). We choose an integer g and a non-zero G in $\Gamma(\mathbb{P}_R^n, \mathcal{O}(g))$ which annihilates $\Omega^1_{\widetilde{X}/\mathbb{P}_R^n}$ (Lemma 5 of Chapter IX shows the existence of such a G annihilating $\Omega^1_{X_k/\mathbb{P}_k^n}$; multiplying such a G by a suitable integer gives the G we want). Let $\widetilde{Y} \to Y$ be a proper modification such that $\widetilde{Y}_k = Y_k$ and that the projections $\mathrm{pr}_i\colon Y_k = X_k^m \to X_k$ on the ith factor extend to $\mathrm{pr}_i\colon \widetilde{Y} \to \widetilde{X}$.

In particular, $\widetilde{Y}$ is a model of X_k^m dominating both Y and $\widetilde{X}^m$ ($m$fold fibred product over $\mathrm{Spec}(R)$). Since $\widetilde{Y} \to \widetilde{X}^m$ is an isomorphism on the generic fibres, we can choose an integer $N > 0$ such that N annihilates $\Omega^1_{\widetilde{Y}/\widetilde{X}^m}$. For $1 \leq i \leq m$ let $P_i := \mathbb{P}^n_k$ and $P_{i,R} := \mathbb{P}^n_R$. Let $p_i : \widetilde{Y} \to P_{i,R}$ be the composition of $\mathrm{pr}_i : \widetilde{Y} \to \widetilde{X}$ and $\mathrm{pr} : \widetilde{X} \to \mathbb{P}^n_R$. Let $P := \prod_{i=1}^m P_i$ and $P_R := \prod_{i=1}^m P_{i,R}$ (fibred products over k and R) and let $p : \widetilde{Y} \to P_R$ be $(p_1, \ldots, p_m)$. For $1 \leq i \leq m$ and $1 \leq j \leq n$ let $\partial_{i,j}$ be the derivation $x_{i,j}\partial/\partial x_{i,j}$ on P_R, where the $x_{i,j}$, $1 \leq j \leq n$, are the coordinates of the standard affine open $\mathbb{A}^n_R$ in $P_{i,R}$. Let G_i be the pullback of G to $\widetilde{X}^m$ via the projection on the ith factor. Finally, we choose a non-empty open subset $U \subset X_k$ such that G is invertible on U and the $\partial_{i,j}$ generate the tangent space at every point of U (this last condition just means that U is disjoint from the $n+1$ coordinate hyperplanes). In particular, this implies that $\mathrm{pr} : X_k \to \mathbb{P}^n_k$ is etale on U. Everything up to here will be fixed during the rest of the proof.

The following two variables σ and b will be crucial; σ is a rational number with $0 < \sigma < \varepsilon$, and b is any rational number. In the end we will first have to take σ small enough, then b large enough. For any σ and b we can find a point $x = (x_1, \ldots, x_m)$ in $X_k^m(k)$ such that:

1. $h(x_1) \geq b$ (here h denotes the height function on $X_k(\bar{k})$ associated to $\mathcal{L}$; in particular we have $h(x_1) = [k : \mathbb{Q}]^{-1} \deg(x_1^*\mathcal{L})$)
2. $h(x_2)/h(x_1) \geq 2s^2, \ldots, h(x_m)/h(x_{m-1}) \geq 2s^2$
3. $\langle x_i, x_{i+1} \rangle \geq (1-\varepsilon/4)\|x_i\|\cdot\|x_{i+1}\|$ (here $\langle x, y \rangle$ denotes the Néron-Tate pairing in $A_k(\bar{k})$ and $\|x\|^2 = \langle x, x \rangle$)
4. $x_i \in U(k)$

(here one uses that, by the theorem of Mordell-Weil, the unit ball in $A(k) \otimes \mathbb{R}$ is compact). Let $h_i := h(x_i)$. We choose $s_i \in \mathbb{Q}$ close to $h_i^{-1/2}$: $|s_i^2 h_i - 1| < b^{-1}$ will do. Let d be a square integer which is big enough and sufficiently divisible, and put $d_i := 4ds_i^2$. Suppose that $b \geq 3$. Then $s_i/s_{i+1} \geq s$ for $1 \leq i < m$, so we can choose a section f in $\Gamma\left(Y^o, \mathcal{L}(\sigma - \varepsilon, s_1, \ldots, s_m)^d\right)$ as in Thm. 3.6: the index of f at x is less than σ and the norm of f at the infinite places is bounded by $\exp(c(\sigma)\sum_i d_i)$. Since the index of f at x is less than σ there exist non-negative integers $e_{i,j}$ such that $\left(\left(\prod_{i,j} \partial_{i,j}^{e_{i,j}}\right) f\right)(x) \neq 0$ and, if we write $e_i = \sum_j e_{i,j}$, $\sum_i e_i/d_i =$ (index of f at x) $<$ σ. By Lemma 4.2 we have a non-zero integral section

$$f' = \prod_{i,j} \left(\frac{(NG_i)^{e_{i,j}}}{e_{i,j}!}\right) \left(\prod_{i,j} (\partial_{i,j}^{e_{i,j}})\, f\right)(x)$$

of the metrized line bundle

$$\mathcal{M} = x^* \left(\mathcal{L}(\sigma - \varepsilon, s_1, \ldots, s_m)^d \otimes \bigotimes_{i=1}^m \mathcal{L}_i^{e_i g}\right)$$

on $\mathrm{Spec}(R)$. We will compute an upper bound on the degree of $\mathcal{M}$ using the properties of x and of $\mathcal{L}(\sigma - \varepsilon, s_1, \ldots, s_m)$, and a lower bound using f'. These bounds then give a contradiction.

5.2 Lemma. *There exists $c_2 \in \mathbb{R}$, not depending on b, σ and the x_i, such that*

$$[k:\mathbb{Q}]^{-1} \deg x^*\mathcal{L}(\sigma-\varepsilon, s_1, \ldots, s_m)^d \;\le\; md(\sigma-\varepsilon/2) + c_2 d b^{-1}$$

Proof. Let $\hat{h}$ denote the Néron-Tate height on $A_k(\bar{k})$ associated to $\mathcal{L}$. Then there exists $c_1 \in \mathbb{R}$ such that $|\hat{h}(y) - h(y)| < c_1$ and $|\langle y, z\rangle - [k:\mathbb{Q}]^{-1}\deg(y,z)^*\mathcal{P}| < c_1$ for all y, z in $A_k(k)$. Then we compute:

$$\begin{aligned}
&[k:\mathbb{Q}]^{-1} \deg x^*\mathcal{L}(\sigma-\varepsilon, s_1, \ldots, s_m)^d = \\
= \;& (1+\sigma-\varepsilon)ds_1^2h_1 + \sum_{i=2}^{m-1}(2+\sigma-\varepsilon)ds_i^2h_i + (1+\sigma-\varepsilon)ds_m^2h_m \\
& -\sum_{i=1}^{m-1} ds_is_{i+1}[k:\mathbb{Q}]^{-1}\deg(x_i,x_{i+1})^*\mathcal{P} \\
\le \;& (1+\sigma-\varepsilon)d(1+b^{-1}) + \sum_{i=2}^{m-1}(2+\sigma-\varepsilon)d(1+b^{-1}) + (1+\sigma-\varepsilon)d(1+b^{-1}) \\
& -\sum_{i=1}^{m-1} ds_is_{i+1}(\langle x_i, x_{i+1}\rangle - c_1) \\
\le \;& (m(\sigma-\varepsilon)d + 2(m-1)d)\,(1+b^{-1}) - \sum_{i=1}^{m-1} ds_is_{i+1}(1-\varepsilon/4)\|x_i\|\cdot\|x_{i+1}\| + 2c_1dmb^{-1} \\
\le \;& (m(\sigma-\varepsilon)d + 2(m-1)d)\,(1+b^{-1}) - d(1-\varepsilon/4)\sum_{i=1}^{m-1} s_is_{i+1}\sqrt{2}\hat{h}_i^{1/2}\sqrt{2}\hat{h}_{i+1}^{1/2} \\
& +2c_1dmb^{-1} \\
\le \;& (m(\sigma-\varepsilon)d + 2(m-1)d)\,(1+b^{-1}) \\
& -d(1-\varepsilon/4)2\sum_{i=1}^{m-1} s_is_{i+1}(h_i-c_1)^{1/2}(h_{i+1}-c_1)^{1/2} + 2c_1dmb^{-1} \\
\le \;& (m(\sigma-\varepsilon)d + 2(m-1)d)\,(1+b^{-1}) - 2d(1-\varepsilon/4)\sum_{i=1}^{m-1} s_ih_i^{1/2}s_{i+1}h_{i+1}^{1/2}(1-c_1b^{-1}) \\
& +2c_1dmb^{-1} \\
\le \;& (m(\sigma-\varepsilon)d + 2(m-1)d)\,(1+b^{-1}) - 2d(1-\varepsilon/4)(1-c_1b^{-1})(m-1)(1-b^{-1})^2 \\
& +2c_1dmb^{-1} \\
\le \;& (m(\sigma-\varepsilon)d + 2(m-1)d)\,(1+b^{-1}) - 2d(1-\varepsilon/4)(m-1)(1-(2+c_1)b^{-1}) + 2c_1dmb^{-1} \\
\le \;& md(\sigma-\varepsilon/2) + b^{-1}(2md + 2dm(2+c_1) + 2c_1dm) \\
= \;& md(\sigma-\varepsilon/2) + (6+4c_1)mdb^{-1}
\end{aligned}$$

This means that the lemma holds with $c_2 = (6+4c_1)m$. □

5.3 Lemma. *There exists $c_3 \in \mathbb{R}$, not depending on b, σ and the x_i, such that*

$$[k:\mathbb{Q}]^{-1}\deg \mathcal{M} \;\le\; md(c_3\sigma - \varepsilon/2) + c_2db^{-1}$$

Proof. We have:

$$\begin{aligned}
[k:\mathbb{Q}]^{-1}\deg\mathcal{M} &\leq md(\sigma-\varepsilon/2)+c_2db^{-1}+\sum_{i=1}^{m}(e_i/d_i)gd_ih_i \\
&= md(\sigma-\varepsilon/2)+c_2db^{-1}+\sum_{i=1}^{m}(e_i/d_i)g4ds_i^2h_i \\
&\leq md(\sigma-\varepsilon/2)+c_2db^{-1}+\sum_{i=1}^{m}(e_i/d_i)g4d(1+b^{-1}) \\
&\leq md((1+8g/m)\sigma-\varepsilon/2)+c_2db^{-1}
\end{aligned}$$

This means that the lemma holds with $c_3 = 1+8g/m$. $\square$

5.4 Lemma. *There exists $c_4(\sigma)$, not depending on b and the x_i, such that for each infinite place v of k we have* $\log\|f'\|_v \leq c_4(\sigma)db^{-1}$.

Proof. Let v be an infinite place of k. To simplify notation, we denote by $\|\cdot\|$ a norm at v. We have:

$$\|f'\| = \left(\prod_i \|NG_i(x)\|^{e_i}\right)\cdot\left\|\left(\left(\prod_{i,j}\frac{\partial_{i,j}^{e_{i,j}}}{e_{i,j}!}\right)f\right)(x)\right\|$$

Note that $e_i < \sigma d_i < d_i$ as $\sum_i e_i/d_i < \sigma$, and that $\sum_i d_i < 8mdb^{-1}$. The first factor of the right hand side of the formula above is easy to bound, but when it is small we will need that to bound the whole right hand side. There exists a j such that $\rho_j\colon \mathcal{L}(\sigma-\varepsilon, s_1,\ldots,s_m)^d \to \otimes_i\mathcal{L}_i^{d_i}$ has norm $\geq \exp(-c_5\sum_i d_i)$ at x (this c_5 is the c_2 from Prop. 3.3). Let $h := \rho_j(f)$; then $h \in \Gamma(X_k^m, p^*\mathcal{O}(d_1,\ldots,d_m))$. The metric on $\otimes_i\mathcal{L}_i^{d_i}$ we fixed a long time ago and the pullback of the product of the standard metrics on the $\mathcal{O}(d_i)$ differ by a factor bounded by $c_6^{\sum_i d_i}$. Hence it suffices to show that there exists $c_4'(\sigma)$, such that we have:

$$\left(\prod_i \|NG_i(x)\|^{e_i}\right)\cdot\left\|\left(\left(\prod_{i,j}\frac{\partial_{i,j}^{e_{i,j}}}{e_{i,j}!}\right)h\right)(x)\right\| \leq \exp(c_4'(\sigma)db^{-1})$$

where the metrics are now the standard metrics. Recall that the standard metric on $\mathcal{O}(n)$ on $\mathbb{P}^m(\mathbb{C})$ is given by:

$$\|F\|(P_0:\cdots:P_m) = |F(P_0,\ldots,P_m)|/(\sum_i |P_i|^2)^{n/2}$$

Let $y_i = \mathrm{pr}_i(x_i)$; then $y_i \in \mathbb{P}^n(\mathbb{C})$. Let $Z_0,\ldots,Z_n$ in $\Gamma(\mathbb{P}_k^n, \mathcal{O}(1))$ be the homogeneous coordinates of $\mathbb{P}_k^n$, and for $1 \leq j \leq n$ let $D_+(Z_j) \subset \mathbb{P}_k^n$ be the affine open subscheme given by $Z_j \neq 0$; then $D_+(Z_j) \cong \mathbb{A}_k^n$ and the Z_l/Z_j $(l \neq j)$ are coordinates on $D_+(Z_j)$. For r in $\mathbb{R}$ let $D_+(Z_j)_r$ denote the standard polydisc of radius r in $D_+(Z_j)(\mathbb{C})$ given by $|Z_l/Z_j| \leq r$ $(l \neq j)$. For each i in $\{1,\ldots,m\}$ we choose j_i in $\{0,\ldots,n\}$ such that $y_i \in D_+(Z_j)_1$, i.e., the coordinates $y_{i,j}$ of y_i in $D_+(Z_{j_i})$ satisfy $|y_{i,j}| \leq 1$. We can bound the derivative, on all standard polydiscs $D_+(Z_j)_2$ of radius 2, of G in some trivialization of $\mathcal{O}(g)$. The conclusion is that there exists $c_7 > 0$, $c_7 < 1/\|G\|_{\sup,\mathbb{P}^n(\mathbb{C})}$, such that G has no zeros in the polydiscs $\Delta_i \subset D_+(Z_{j_i})_2$ of radius $r_i := c_7\|G\|(y_i)$ and center y_i. It follows that $\mathrm{pr}_i\colon X_k \to P_i$ is etale over Δ_i. Let $\Delta = \prod_{i=1}^m \Delta_i$. Then Δ is simply connected, hence $p^{-1}\Delta \subset X_k^m(\mathbb{C})$ is a finite disjoint union of copies of Δ; let Δ' denote the copy that contains x.

We trivialize $\mathcal{O}(d_i)$ on Δ_i using its section $Z_{j_i}^{d_i}$. In other words, on Δ' we write $h = h_1 \prod_i Z_{j_i}^{d_i}$. The standard norm of $Z_{j_i}^{d_i}$ on Δ_i satisfies: $(n+1)^{-d_i/2} \leq \|Z_{j_i}^{d_i}\| \leq 1$ on Δ_i. By definition we have:

$$\left\| \left(\left(\prod_{i,j} \frac{\partial_{i,j}^{e_{i,j}}}{e_{i,j}!} \right) h \right)(x) \right\| = \left| \left(\left(\prod_{i,j} \frac{\partial_{i,j}^{e_{i,j}}}{e_{i,j}!} \right) h_1 \right)(x) \right| \cdot \left(\prod_i \|Z_{j_i}^{d_i}\| \right)(x)$$

so it remains to estimate the first factor of the right hand side.

On Δ' we write $h_1 = \sum a_{(n)} z^{(n)}$, with $(n) = (n_{i,j})$ (here the $z_{i,j}$ are the standard coordinates on Δ': $z_{i,j} = \mathrm{pr}_i^*(Z_j/Z_{j_i} - y_{i,j})$). Then we have:

$$\left(\prod_{i,j} \frac{(\partial/\partial_{z_{i,j}})^{n_{i,j}}}{n_{i,j}!} \right)(h_1)(x) \ = \ a_{(n_{i,j})} \ = \ \frac{1}{(2\pi\sqrt{-1})^{mn}} \int \cdots \int_{|z_{i,j}|=r_i} h_1 \prod_{i,j} z_{i,j}^{-n_{i,j}-1} \prod_{i,j} dz_{i,j}$$

Hence for (n) with $\sum_j n_{i,j} = e_i$ for all i we have $|a_{(n_{i,j})}| \leq \|h_1\|_{\mathrm{sup},\Delta'} \prod_i r_i^{-e_i}$. Now note that $\|h_1\|_{\mathrm{sup},\Delta'}$ is bounded above by some $\exp(c_4''(\sigma)db^{-1})$ by the arguments above plus the fact that the norm of f is bounded by $\exp(c(\sigma)\sum_i d_i)$, hence we have:

$$\left| \left(\prod_{i,j} \frac{(\partial/\partial_{z_{i,j}})^{n_{i,j}}}{n_{i,j}!} \right)(h_1)(x) \right| \leq \exp(c_4''(\sigma)db^{-1}) \prod_i r_i^{-e_i}$$

A standard computation shows that on Δ_i one can write $\partial_{i,j} = \sum_l b_{i,j,l}(\partial/\partial_{z_{i,l}})$ with $|b_{i,j,l}(y_i)| \leq 1$ (in fact, $b_{i,j,l}(y_i) \in \{0, \pm y_{i,l}\}$). Using this, we can write:

$$\begin{aligned} \left| \left(\left(\prod_{i,j} \frac{\partial_{i,j}^{e_{i,j}}}{e_{i,j}!} \right) h_1 \right)(x) \right| &= \left| \sum_{(n_{i,j})} C_{(n_{i,j})}(x) \left(\prod_{i,j} \frac{(\partial/\partial_{z_{i,j}})^{n_{i,j}}}{n_{i,j}!} \right)(h_1)(x) \right| \\ &= \left| \sum_{(n_{i,j})} C_{(n_{i,j})}(x) \cdot a_{(n_{i,j})} \right| \end{aligned}$$

where the $C_{(n_{i,j})}$ are certain polynomials in the $b_{i,j,l}$. Note that $C_{(n_{i,j})}(x) = 0$ unless for all i one has $\sum_j n_{i,j} = e_i$. From $|b_{i,j,l}(x)| \leq 1$, and some combinatorics, it follows that:

$$|C_{(n_{i,j})}(x)| \leq \prod_i \frac{e_i!}{e_{i,1}! \cdots e_{i,n}!}$$

Note that the right hand side does not depend on $(n_{i,j})$. Using this one gets:

$$\sum_{(n_{i,j})} |C_{(n_{i,j})}(x)| \leq \prod_i \frac{e_i!}{e_{i,1}! \cdots e_{i,n}!} \frac{(e_i+n-1)!}{(n-1)!e_i!} \leq \prod_i (2^{e_i+n-1} n^{e_i}) \leq 2^{m(n-1)} (2n)^{\sum_i \sigma d_i}$$

This means that finally we have:

$$\left| \left(\left(\prod_{i,j} \frac{\partial_{i,j}^{e_{i,j}}}{e_{i,j}!} \right) h_1 \right)(x) \right| \leq \exp(c_4''(\sigma)db^{-1}) \cdot \left(\prod_i r_i^{-e_i} \right) \cdot 2^{m(n-1)} (2n)^{4md/b}$$

which finishes the proof. $\square$

5.5 Lemma. *There exists $c_8(\sigma)$, not depending on b and the x_i, such that*

$$[k:\mathbb{Q}]^{-1}\deg\mathcal{M} \;\geq\; -c_8(\sigma)db^{-1}$$

Proof. By definition we have:

$$\deg\mathcal{M} \;=\; \log\#\left(\frac{\mathcal{M}}{R\cdot f'}\right) - \sum_{v|\infty}\epsilon(v)\log\|f'\|_v \;\geq\; -[k:\mathbb{Q}]c_4(\sigma)db^{-1}$$

□

We can now finish the proof of Thm. 5.1. Take $\sigma < \varepsilon/(2c_3)$ (c_3 as in Lemma 5.3). Then for b big enough, Lemma 5.3 and Lemma 5.5 give contradicting estimates for $[k:\mathbb{Q}]^{-1}\deg\mathcal{M}$. □