On Néron models, divisors and modular curves.

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March 26, 1999

1 Introduction.

For p a prime number, let $X_0(p)_{\mathbb{Q}}$ be the modular curve, over $\mathbb{Q}$, parametrizing isogenies of degree p between elliptic curves, and let $J_0(p)_{\mathbb{Q}}$ be its jacobian variety. Let $f\colon X_0(p)_{\mathbb{Q}} \to J_0(p)_{\mathbb{Q}}$ be the morphism of varieties over $\mathbb{Q}$ that sends a point P to the class of the divisor $P - \infty$, where ∞ is the $\mathbb{Q}$-valued point of $X_0(p)_{\mathbb{Q}}$ that corresponds to $\mathbb{P}^1_{\mathbb{Q}}$ with 0 and ∞ identified, equipped with the subgroup scheme $\mu_{p,\mathbb{Q}}$. We suppose that $X_0(p)_{\mathbb{Q}}$ has genus at least one, i.e., that $p = 11$ or $p > 13$. Then f is a closed immersion. Let 0 denote the other cusp of $X_0(p)_{\mathbb{Q}}$. The $\mathbb{Q}$-valued point $f(0)$ of $J_0(p)_{\mathbb{Q}}$ is well known to be of order n, the numerator of $(p-1)/12$, expressed in lowest terms. Let now $X_0(p)$ denote the model over $\mathbb{Z}$ of $X_0(p)_{\mathbb{Q}}$ as described by [8] and [15]: it is the compactified coarse moduli space for generalized elliptic curves with finite locally free subgroup schemes of rank p that meet all irreducible components of all geometric fibres. Let $X_0(p)^\sim$ denote the minimal regular model of $X_0(p)$, and let $J_0(p)$ be the Néron model over $\mathbb{Z}$ of $J_0(p)_{\mathbb{Q}}$. See [8] for a description of the semi-stable curve $X_0(p)^\sim$. By the defining properties of $J_0(p)$, the morphism f extends uniquely to a morphism $f\colon X_0(p)^{\sim,0} \to J_0(p)$, where $X_0(p)^{\sim,0}$ is the open part of $X_0(p)^\sim$ where the morphism to $\mathrm{Spec}(\mathbb{Z})$ is smooth; $X_0(p)^{\sim,0}$ is the complement of the set of double points in the fibre $X_0(p)^\sim_p$ over $\mathbb{F}_p$. Robert Coleman asked about how the image of $X_0(p)^{\sim,0}_p$ under f intersects C, the closure in $J_0(p)$ of the group generated by $f(0)$. We will prove, see Theorem 8.2, for all p for which $X_0(p)_{\mathbb{Q}}$ has genus at least two, that the intersection consists just of the two obvious elements $f(0)_p$ and $f(\infty)_p$. (Of course, when $X_0(p)_{\mathbb{Q}}$ has genus one, f is an isomorphism, hence the intersection is all of C_p.) This result is used by Coleman, Kaskel and Ribet in [4] to study the inverse image under f of the torsion subgroup of $J_0(p)(\mathbb{C})$.

To prove our result, we use Raynaud's description (see [25] or [2, 9.5]) of the Néron model of a jacobian variety of a curve in terms of a regular model of the curve. (A good reference for jacobians of modular curves is [26].) It turns out that in order to describe the special fibre of such a Néron model, and not just the connected component of the

*partially supported by the Institut Universitaire de France, and by the European TMR Network Contract ERB FMRX 960006 "arithmetic algebraic geometry".

identity element and the group of connected components, one needs to know a bit more than just the special fibre of the regular model of the curve. This extra information is related, in the semi-stable case, to the log structure induced on the special fibre of the curve (see Section 6), and also to the map from the space of global deformations to that of local deformations of a stable curve (see [7, p. 81]). For the curve $X_0(p)^\sim$ we describe this extra information in terms of a well known modular form of weight $p+1$ on the supersingular elliptic curves. We do not use rigid uniformization, and our methods can be used as well for curves whose special fibre is not totally degenerate. A rigid analytic description of $J_0(p)_p$ is given by de Shalit in [29].

In Sections 2 and 3 we consider a proper nodal curve X with smooth generic fibre X_K over a discrete valuation ring, and study the open part J of the Néron model of the jacobian variety given by the relative Picard functor of X. We give a description of the special fibre J_k of J in terms of divisors on X_k and some invariants $c(x)$ at the double points of X_k. Some properties of these $c(x)$ are discussed in Sections 4, 5 and 6. The $c(x)$ for the modular curve $X_0(p)$ are determined in Section 7. Coleman's question mentioned above is answered in Section 8. Section 9 answers the question as to what extent the morphism from the smooth locus X^0 of X to the Néron model of J_K is a closed immersion. For example, if X is regular, we show that this morphism is a closed immersion if and only if all double points of $X_{\bar{k}}$ are non-disconnecting (see Corollary 9.8). Some of the results in this last section can also be found in Coleman's preprint [3].

2 Raynaud's description in the nodal case.

Let D be a complete discrete valuation ring, let π be a uniformizer, K the fraction field of D and k the residue field. We suppose that k is separably closed. Let X_K be a smooth, proper and geometrically irreducible curve over K, and let X be a proper flat model of it over D. We suppose that X is nodal, i.e., the only singularities of the geometric special fibre are ordinary double points. The irreducible components of X_k are absolutely irreducible. The singular points of X_k are k-rational points. At such a point x the complete local ring $\mathcal{O}^\wedge_{X,x}$ is isomorphic to one of the form $D[[u,v]]/(uv-\pi^{e(x)})$, for a unique integer $e(x)\geq 1$. Let $R\colon X^\sim \to X$ be the minimal resolution of singularities of X; for x in X_k a singular point, $R^{-1}x$ is a chain of $e(x)-1$ projective lines.

Let $N\colon X_k^{\mathrm{nor}} \to X_k$ be the normalization morphism. We define S_0 to be the set of irreducible components of X_k^{nor}. Note that N induces a bijection between S_0 and the set of irreducible components of X_k. Let S_1 be the set of singular points of X_k, and let S_1^{nor} be $N^{-1}S_1$. We will view S_0 and S_1 as the sets of vertices and edges of a graph; the end points of x in S_1 are the elements of S_0 containing at least one of the two elements of $N^{-1}\{x\}$.

According to [25, Thm. 8.2.1] or [2, Thm. 8.2/2], the relative Picard functor $\mathrm{Pic}_{X/D}$ is representable, and is smooth over D (but not of finite type, and not separated if X_k has more than one irreducible component). As usual, let $\mathrm{Pic}^0_{X/D}$ be the connected component of the identity element in $\mathrm{Pic}_{X/D}$; its D-valued points correspond to line bundles on X whose restrictions to all irreducible components of X_k are of degree zero. As in [18], let $\mathrm{Pic}^{[0]}_{X/D}$ be the closure in $\mathrm{Pic}_{X/D}$ of $\mathrm{Pic}^0_{X_K/K}$; $\mathrm{Pic}^{[0]}_{X/D}(D)$ is the set of isomorphism classes of line bundles on X of which the sum of the degrees of the restrictions to the irreducible components is zero. Let E be the closure in $\mathrm{Pic}_{X/D}$ of the identity element of $\mathrm{Pic}_{X_K/K}$. Then $E(D)$ is the set of isomorphism classes of line bundles on X whose restriction to X_K is trivial. Such line bundles are given by Cartier divisors with support in X_k. The group of Weil divisors with support in X_k is the group $\mathbb{Z}^{S_0}$. Let F be the subgroup of $\mathbb{Z}^{S_0}$ consisting of those Weil divisors that define Cartier divisors. Of course, if X is regular, then F equals $\mathbb{Z}^{S_0}$. Let us consider a Weil divisor $a := \sum a(C)C$ in $\mathbb{Z}^{S_0}$. Then a defines a Cartier divisor if and only if a is principal at all x in S_1. Let x be in S_1, and let C and C' in S_0 be the endpoints of x. Then a is principal at x if and only if $a(C)$ and $a(C')$ are equal modulo $e(x)$ (see [26, p. 15]). It follows that we have an exact sequence:

$$0 \to F \to \mathbb{Z}^{S_0} \to \prod_{x\in S_1} (\mathbb{Z}/e(x)\mathbb{Z})^{N^{-1}\{x\}}/(\mathbb{Z}/e(x)\mathbb{Z}), \tag{2.1}$$

where the $\mathbb{Z}/e(x)\mathbb{Z}$ are diagonally embedded. By definition, we have a map from F to $\mathrm{Pic}^{[0]}_{X/D}(D)$, whose image is $E(D)$. This map sends a in F to the invertible $\mathcal{O}_X$-module $\mathcal{O}_X(a)$, whose sections on open $U \subset X$ are rational functions f on U with $\mathrm{div}(f)+a \geq 0$.

By [25], or the proof of [2, Prop. 9.5/3], E is the espace étalé of the sheaf $i_*E(D)$, with $i\colon \mathrm{Spec}(k) \to \mathrm{Spec}(D)$ the closed immersion given by the canonical projection $D \to k$. In particular, $E(k) = E(D)$, and $E(K) = 0$. It follows that $J := \mathrm{Pic}^{[0]}_{X/D}/E$ is representable and separated, and that J represents the open part of the Néron model over D of $\mathrm{Pic}^0_{X_K/K}$ that corresponds to the line bundles on X_K that admit an extension to a line bundle on X. Note that if X is regular, then J is the Néron model itself.

Let $\Phi := J_k/J_k^0$ be the group of connected components of J_k. By [2, Thm. 9.7/1], $J_k^0 = \mathrm{Pic}^0_{X_k/k}$. We have the following complex of $\mathbb{Z}$-modules:

$$0 \to \mathbb{Z} \to F \to \mathbb{Z}^{S_0} \to \mathbb{Z} \to 0. \tag{2.2}$$

The element 1 in $\mathbb{Z}$ is mapped to X_k, the principal divisor defined by π. The morphism from F to $\mathbb{Z}^{S_0}$ is the composite of $F \to \mathrm{Pic}(X)$ and the morphism "multidegree" from $\mathrm{Pic}(X)$ to $\mathbb{Z}^{S_0}$ that sends a line bundle to its degrees on the irreducible components of X_k (note that this map is not the same as the one in (2.1)). Explicitly, for a in F and C in S_0, one has:

$$\deg_C(\mathcal{O}_X(a)) = \sum_{x\in C\cap S_1^{\mathrm{nor}}} \frac{a(C'_x)-a(C)}{e(x)}, \tag{2.3}$$

where C'_x denotes the other branch passing through x (see (3.5)). The morphism $\mathbb{Z}^{S_0} \to \mathbb{Z}$ is simply the sum. The fact that the $\mathbb{Q}$-valued intersection pairing on $\mathbb{Z}^{S_0}$ is negative definite on $\mathbb{Z}^{S_0}/\mathbb{Z}X_k$ shows that (2.2) is exact at F. The definition of J, and the fact that $E(D)$ is the image of F, show that the homology of (2.2) at $\mathbb{Z}^{S_0}$ is Φ. See [2, Thm. 9.6/1] for this result in the case where X is regular.

3 A description of J_k in terms of divisors.

We keep the notation and hypotheses of the preceding section. The results in that section give us a description of J in terms of $\mathrm{Pic}_{X/D}$. Further on, we will have to do computations in the special fibre J_k of J, without knowing too much about the D-scheme X, but in terms of the k-scheme X_k with some additional data.

In Section 2 we have seen that J_k has the following presentation:

$$F \to \mathrm{Pic}^{[0]}_{X_k/k} \to J_k \to 0, \tag{3.1}$$

with $\mathrm{Pic}^{[0]}_{X_k/k}$ the open part of $\mathrm{Pic}_{X_k/k}$ corresponding to line bundles on X_k with total degree zero, and with F the subgroup of $\mathbb{Z}^{S_0}$ consisting of the Weil divisors that are locally principal (we view F as a constant group scheme over k). This subgroup F is described in terms of S_0 and the $e(x)$ in (2.1). The only ingredient in (3.1) that does not only involve X_k and the $e(x)$ is the map from F to $\mathrm{Pic}(X_k)$. The aim of this section is to describe $\mathrm{Pic}_{X_k/k}$ and the map from F to it, in terms of X_k, the $e(x)$, and some extra data related to the reduction of X modulo some power of π (recall that π is a uniformizer of D).

Let us first describe $\mathrm{Pic}_{X_k/k}$. The proof of Example 8 of Chapter 9 of [2] shows that we have an exact sequence:

$$0 \to \mathbb{G}_{\mathrm{m},k} \to \bigoplus_{C\in S_0} \mathbb{G}_{\mathrm{m},k} \to \bigoplus_{x\in S_1} \left(\mathbb{G}_{\mathrm{m},k}^{N^{-1}\{x\}}/\mathbb{G}_{\mathrm{m},k}\right) \to \mathrm{Pic}_{X_k/k} \to \mathrm{Pic}_{X_k^{\mathrm{nor}}/k} \to 0. \tag{3.2}$$

Let us assume that S_1 is not empty. Then (3.2) gives the following description of $\mathrm{Pic}_{X_k/k}(S)$ for all k-schemes S.

3.3 Proposition. *Under the assumptions above, $\mathrm{Pic}_{X_k/k}(S)$ is the set of isomorphism classes of line bundles on X_S that are obtained as follows. Choose a line bundle $\mathcal{L}$ on X_S^{nor}, i.e., for each C in S_0, a line bundle $\mathcal{L}_C$ on C_S, with a given isomorphism $\phi_x\colon \mathcal{O}_S \to x^*\mathcal{L}$ for each x in S_1^{nor}. Then glue, at each x in S_1, the line bundles $\mathcal{L}_{C_1}$ and $\mathcal{L}_{C_2}$ via ϕ_{x_1} and ϕ_{x_2}, where $\{x_1, x_2\}$ is $N^{-1}\{x\}$ and where C_1, C_2 are the elements of S_0 with $x_1 \in C_1$ and $x_2 \in C_2$. Let $(\mathcal{L}, \phi)$ denote the line bundle on X_S obtained in this way. Two such line bundles $(\mathcal{L}, \phi)$ and $(\mathcal{L}', \phi')$ are isomorphic if and only if there exists an isomorphism $\alpha\colon \mathcal{L} \to \mathcal{L}'$ that is compatible with the glueing data in the sense that for all x in S_1 we have*

$$\alpha_{x_2}\circ\phi_{x_2}\circ\phi_{x_1}^{-1} = \phi'_{x_2}\circ{\phi'_{x_1}}^{-1}\circ\alpha_{x_1}.$$

3.4 Remark. Another way to state the previous proposition is to say that $N\colon X_k^{\mathrm{nor}} \to X_k$ is a morphism of universal effective descent for line bundles (see [11, Déf. 1.7]). □

Our next aim is to describe the line bundles of the form $\mathcal{O}_X(a)|_{X_k}$, with a in F, in the form of Proposition 3.3. Note that the divisor of π is $X_k = \sum_C C$, so that multiplication by π induces an isomorphism from $\mathcal{O}_X$ to $\mathcal{O}_X(-X_k)$. Let a be in F. For C in S_0 we have:

$$(3.5)\quad \mathcal{O}_X(a)|_C \tilde{\to} \mathcal{O}_X\left(\sum_{C'\neq C}(a(C')-a(C))C'\right)\Bigg|_C = \mathcal{O}_C\left(\sum_{x\in C\cap S_1^{\mathrm{nor}}}\frac{a(C'_x)-a(C)}{e(x)}x\right),$$

where the first isomorphism is given by multiplication by $\pi^{a(C)}$, and where C'_x is the other branch passing through x. Now we have to see how these line bundles are glued at the singular points. So let x be in S_1, $N^{-1}\{x\}=\{x_1,x_2\}$, and let C_1 and C_2 be the elements of S_0 containing x_1 and x_2, respectively. Let $a_1 := a(C_1)$ and $a_2 := a(C_2)$. Then we have:

$$(3.6)\qquad x_1^*\mathcal{O}_{C_1}\left(\frac{a_2-a_1}{e(x)}x_1\right) = \Omega_{C_1}(x_1)^{\otimes\frac{a_1-a_2}{e(x)}},$$

where Ω_{C_1} is the sheaf of relative 1-forms of C_1 over k. Of course, we have a similar formula at x_2. It follows that multiplication by $\pi^{a_1-a_2}$ defines an isomorphism:

$$(3.7)\qquad \Omega_{C_2}(x_2)^{\otimes\frac{a_2-a_1}{e(x)}} \to \Omega_{C_1}(x_1)^{\otimes\frac{a_1-a_2}{e(x)}},$$

or, equivalently, a non-zero element of $(\Omega_{C_1}(x_1)\otimes\Omega_{C_2}(x_2))^{\otimes(a_2-a_1)/e(x)}$. A simple computation shows that if we choose an isomorphism $D[[u,v]]/(uv-\pi^{e(x)})\to\mathcal{O}^{\wedge}_{X,x}$, such that u is zero on C_1 and v is zero on C_2, then this non-zero element is simply the $(a_2-a_1)/e(x)$th power of

$$(3.8)\qquad c(x) := dv\otimes du \text{ in } \Omega_{C_1}(x_1)\otimes\Omega_{C_2}(x_2)$$

(we will see in the next section that $c(x)$ does not depend on the choice of the isomorphism). This finishes our description of the map from F to $\mathrm{Pic}(X_k)$, and hence of the description of J_k. For the sake of notation, let us say that for a in F, we call $\mathcal{L}_a$ the line bundle on X_k that we just constructed (so $\mathcal{L}_a$ is isomorphic to $\mathcal{O}_X(a)|_{X_k}$, but its description in terms of Proposition 3.3 depends on π). Some properties of the $c(x)$ will be discussed in the next section. To finish this section, we describe J_k in terms of degree zero divisors on X_k.

Let $\mathrm{Div}(X_k)$ be the $\mathbb{Z}$-module of Weil divisors on X_k with support outside the singular locus S_1. Since such divisors are locally principal, we have a morphism $\mathrm{Div}(X_k)\to\mathrm{Pic}(X_k)$, sending E in $\mathrm{Div}(X_k)$ to isomorphism class of the line bundle $\mathcal{O}_{X_k}(E)$. This morphism is easily seen to be surjective. Let $\mathrm{Div}^{[0]}(X_k)$ be the subgroup of $\mathrm{Div}(X_k)$ consisting of the divisors of degree zero. Recall that J_k is the quotient of $\mathrm{Pic}^{[0]}_{X_k/k}$ by the image of F. We get a morphism $\mathrm{Div}^{[0]}(X_k)\to J_k(k)$, sending E to P_E, say. It remains to describe the kernel of $E\mapsto P_E$.

So let E be in $\mathrm{Div}^{[0]}(X_k)$. Of course, P_E is zero if and only if there exists an a in F such that $\mathcal{L}_a$ is isomorphic to $\mathcal{O}_{X_k}(E)$. A necessary condition for P_E to be zero is that the image of P_E in Φ be zero. This last condition is equivalent to the multidegree of E in $\mathbb{Z}^{S_0}$ being in the image of F (see (2.2)); let us suppose now that that is the case.

Let $a = \sum_C a(C)C$ be in F such that $\mathcal{L}_a$ and E have the same multidegree; a is unique up to adding multiples of X_k. Then P_E is zero if and only if $\mathcal{L}_a \otimes \mathcal{O}_{X_k}(-E)$ is trivial, or, equivalently, if $\mathcal{L}_a \otimes \mathcal{O}_{X_k}(-E)$ has a non-zero global section (note that it has degree zero on each C). Using the description in Proposition 3.3 of line bundles on X_k, the last condition amounts to saying that the $\mathcal{L}_{a,C} \otimes \mathcal{O}_C(-E|_C)$ have non-zero global sections f_C that are compatible at all x in S_1. By construction, see (3.5), each such f_C has to be a non-zero rational function on C, whose divisor is:

$$E|_C - \sum_{x \in C \cap S_1^{\mathrm{nor}}} \frac{a(C'_x) - a(C)}{e(x)} x. \tag{3.9}$$

Suppose now that there exist f_C with these divisors, or, equivalently, suppose that the $\mathcal{L}_{a,C} \otimes \mathcal{O}_C(-E|_C)$ are trivial. Let f_C be such functions. In order to make the compatibility conditions for the f_C explicit, we introduce some terminology. For f a non-zero rational function on C in S_0, and x in $C(k)$, we define the leading term $f(x)$ of f at x to be the non-zero element of $\Omega_C(x)^{\otimes n} = m_x^n / m_x^{n+1}$ given by f, where n is the order of f at x, and m_x the maximal ideal in $\mathcal{O}_{C,x}$. For x in S_1, C_1 and C_2 the elements of S_0 containing the preimages x_1 and x_2 of x under $N\colon X_k^{\mathrm{nor}} \to X_k$, the two rational functions f_{C_1} and f_{C_2} are compatible at x if and only if:

$$f_{C_1}(x_1)\, f_{C_2}(x_2)^{-1} c(x)^{(a(C_2)-a(C_1))/e(x)} = 1, \tag{3.10}$$

as can be seen from the definitions earlier in this section. The f_C are unique up to multiplication by elements from k^*. The problem of whether or not one can scale the f_C such that they satisfy the compatibility conditions at all x in S_1 is a problem concerning the structure of the graph (S_0, S_1) of X_k. The following proposition gives the final conclusion of the preceding discussion.

3.11 Proposition. *Let E be in $\mathrm{Div}^{[0]}(X_k)$. Then P_E in $J_k(k)$ is zero if and only if the following conditions are satisfied. First of all, the multidegree of E in $\mathbb{Z}^{S_0}$ has to be in the image of F; suppose this is so and let a be in F having the same image in $\mathbb{Z}^{S_0}$ as E. Secondly, for each C in S_0, the divisor in (3.9) has to be a principal divisor; suppose that this is so, and let, for each C, f_C be a rational function on C with that divisor. Thirdly, for every cycle $(C_0, x_0, C_1, x_1, \ldots, x_{m-1}, C_m = C_0)$ in the graph (S_0, S_1) we must have:*

$$\prod_{i=0}^{m-1} f_{C_i}(x'_i)\, f_{C_{i+1}}(x''_i)^{-1} c(x_i)^{(a(C_{i+1})-a(C_i))/e(x_i)} = 1, \tag{3.12}$$

with x'_i and x''_i in $N^{-1}\{x\}$ defined by $x'_i \in C_i$ and $x''_i \in C_{i+1}$.

3.13 Remark. Of course, in the third condition in Proposition 3.11, it is sufficient to consider a set of cycles that generates the first homology group of the graph. □

4 Some properties of the $c(x)$.

Let us briefly recall the definition, given in (3.8), of the $c(x)$. In the situation of (3.8), D is a complete discrete valuation ring with separably closed residue field k and fraction field K, π in D is a uniformizer and X/D is a proper nodal curve over D with X_K smooth and geometrically irreducible. Suppose that x is a singular point of X_k. Then the D-algebra $\mathcal{O}_{X,x}^\wedge$ is of the type $D[[u,v]]/(uv-\pi^{e(x)})$ for some unique integer $e(x)>0$. Let x_1 and x_2 be the two points of X_k^{nor} lying over x, and let C_1 and C_2 be the irreducible components of X_k^{nor} that they lie on. Let u and v be elements in $\mathcal{O}_{X,x}^\wedge$ that vanish on C_1 and C_2, respectively, that induce parameters, still denoted u and v, of $\mathcal{O}_{C_2,x_2}^\wedge$ and $\mathcal{O}_{C_1,x_1}^\wedge$, respectively, and such that $uv=\pi^{e(x)}$. Then we claim that the non-zero element $dv\otimes du$ in $\Omega_{C_1}(x_1)\otimes\Omega_{C_2}(x_2)$ does not depend on the choice of u and v. To see this, suppose that u', v' also satisfy the conditions. Then $u'=\lambda u$ and $v'=\lambda^{-1}v$ with λ in $\mathcal{O}_{X,x}^\wedge$ a unit. Let $\overline{\lambda}$ be the image in k^* of λ. Then $du'=\overline{\lambda}du$, and $dv'=\overline{\lambda}^{-1}dv$, proving our claim.

Hence the elements $c(x)$ of (3.8) are independent of the choice of the isomorphism. Let us now see how they depend on the choice of π. A simple computation shows that if $\pi'=\lambda\pi$, then $c'(x)=\overline{\lambda}^{e(x)}c(x)$, where $c'(x)$ is defined using π'. It follows that the construction of $c(x)$ defines an isomorphism of k-vector spaces between $\Omega_{C_1}(x_1)\otimes\Omega_{C_2}(x_2)$ and $(m_D/m_D^2)^{\otimes e(x)}$, where m_D is the maximal ideal of D.

The construction of $c(x)$ makes it clear that $c(x)$ depends only on the reduction of X mod $\pi^{e(x)+1}$. It follows that J_k, as defined in Section 2, depends only on X mod π^{e+1}, where e is the maximum of all the $e(x)$. In particular, if X is regular, the $c(x)$ and J_k depend only on X mod π^2.

The construction of the $c(x)$ also makes sense in more general situations (which we won't use in this article). For example, consider a semi-stable curve $f\colon C\to S$ in the sense of [6, 2.21], i.e., S is an arbitrary scheme and f is of finite presentation, flat, proper and all its geometric fibres are connected curves whose singularities are ordinary double points. Let $Z:=\mathrm{Sing}(f)$ be the closed subscheme of C defined by the first Fitting ideal of $\Omega_{C/S}$, and let $g\colon Z\to S$ be the inclusion, followed by f. As explained in [6, 2.21–2.23], g is of finite presentation, finite and unramified, and the kernel I of $g^{-1}\mathcal{O}_S\to\mathcal{O}_Z$ is locally principal. We assume that I/I^2 is a faithful $g^{-1}\mathcal{O}_S/I$-module (this is the case, for example, if S is integral and f generically smooth). Then $\mathcal{O}_Z\otimes_{g^{-1}\mathcal{O}_S}I$ is an invertible $\mathcal{O}_Z$-module. Let P be the tautological section of $f_Z\colon C_Z\to Z$, and let $b\colon C_Z^\sim\to C_Z$ be the blow up of C_Z in the ideal sheaf defined by P. Let Z' be the fibered product of P and b, and denote P' the morphism from Z' to $C_Z^\sim$. Then $Z'\to Z$ is the etale $\mathbb{Z}/2\mathbb{Z}$-torsor given by the two branches of C_Z at P, and $C_Z^\sim\to Z$ is smooth at P'. The global version of the construction of the $c(x)$ is then an isomorphism of invertible $\mathcal{O}_Z$-modules:

$$\mathrm{Norm}_{Z'/Z}\left(P'^*\Omega_{C_Z^\sim/Z}\right)\longrightarrow\mathcal{O}_Z\otimes_{g^{-1}\mathcal{O}_S}I. \tag{4.1}$$

To prove all this, one reduces to the noetherian case and uses the description of [6, 2.23].

We give a more geometric description of the global version of the $c(x)$ for the universal stable curve of some genus $g \geq 2$. So let $\overline{\mathcal{M}}$ be the algebraic stack over $\mathbb{Z}$ classifying such curves, and let $f: \overline{\mathcal{C}} \to \overline{\mathcal{M}}$ be the universal curve (see [7] and [22]). Put $\mathcal{Z} := \mathrm{Sing}(f)$. Then $\mathcal{Z}$ is the disjoint union of irreducible smooth closed substacks Δ_i^*, $0 \leq i \leq \lfloor g/2 \rfloor$, of $\overline{\mathcal{C}}$, of codimension two, and the morphism $\mathcal{Z} \to \overline{\mathcal{M}}$ is locally a closed immersion, defined by a locally principal ideal I in $f^{-1}\mathcal{O}_{\overline{\mathcal{M}}}$. Then then conormal bundle I/I^2 of the locally closed immersion $f: \mathcal{Z} \to \mathcal{M}$ is a line bundle on $\mathcal{Z}$. There is another line bundle, say $\mathcal{L}$, on $\mathcal{Z}$, with the property that for x in $\mathcal{Z}(k)$ corresponding to a pair (C, P) with C a stable curve of genus g over the algebraically closed field k and P a singular point of C, $\mathcal{L}(x) = \Omega_{C_1}(x) \otimes \Omega_{C_2}(x)$, with C_1 and C_2 the two branches at x. Then the global version of the $c(x)$ gives an isomorphism between $\mathcal{L}$ and I/I^2. The existence of this isomorphism was already noted, over $\mathbb{C}$, in [17, p. 143] and in [24, p. 477].

5 The $c(x)$ and certain morphisms.

Let X, D, etc., be as in the beginning of Section 2. In particular, $R: X^\sim \to X$ is the minimal resolution of the singularities of X. We want to compare the $c(x)$ on X and $X^\sim$. So let x be singular in $X(k)$, and suppose we have an isomorphism between $\mathcal{O}_{X,x}^\wedge$ and $D[[u, v]]/(uv - \pi^e)$. Let C_1 and C_2 be the irreducible components of X^{nor} on which u and v vanish, respectively. Then $R^{-1}\{x\}$ is the union of $e - 1$ smooth, proper and geometrically irreducible curves of genus zero, say $L_1, \ldots, L_{e-1}$, arranged in a chain: if we let $L_0 := C_1$ and $L_e := C_2$, then $L_i \cap L_j$ is empty if $|i - j| > 1$, and $L_{i-1} \cap L_i$ is a unique point P_i for $1 \leq i \leq e$. Let $u_i := \pi^{i+1-e}u$ and $v_i := \pi^{-i}v$. Then $\mathcal{O}_{X^\sim,P_i}^\wedge$ is $D[[u_i, v_i]]/(u_i v_i - \pi)$, v_i is a parameter of L_{i-1} at P_i and u_i is a parameter of L_i at P_i. So we have $c(P_i) = dv_i \otimes du_i$. In order to relate the $c(P_i)$ to $c(x)$ we use the following lemma, the proof of which is obvious.

5.1 Lemma. *On a two-pointed smooth projective curve of genus zero (L, P_1, P_2) over a field k, there is a unique non-zero element x in $\Omega_L(P_1) \otimes \Omega_L(P_2)$ with the following property: let f be a rational function on L with divisor $P_1 - P_2$, then $x = (df)P_1 \otimes (df^{-1})P_2$, with $(df)P_1$ and $df^{-1}P_2$ the values of df and df^{-1} at P_1 and P_2, respectively. This element does not depend on the numbering of the two points.*

This lemma gives us an isomorphism

$$\Omega_{C_1}(x) \otimes \Omega_{C_2}(x) \to \Omega_{L_0}(P_1) \otimes \Omega_{L_1}(P_1) \otimes \cdots \otimes \Omega_{L_{e-1}}(P_e) \otimes \Omega_{L_e}(P_e), \tag{5.2}$$

sending $c(x)$ to $c(P_0) \cdots c(P_e)$. It follows that we have the following proposition.

5.3 Proposition. *Let x_0 and x_e be parameters of $\mathcal{O}_{C_1,x}^\wedge$ and $\mathcal{O}_{C_2,x}^\wedge$, respectively. Then $c(x) = dx_0 \otimes dx_e$ if and only if there exist coordinates x_i on the L_i, for $1 \leq i < e$ such that $c(P_i) = d(x_{i-1}^{-1}) \otimes dx_i$ for $1 \leq i \leq e$. If such coordinates exist, they are unique.*

This proposition implies that knowing $X_k^\sim$ and the $c(x)$ for the singular x on it, up to isomorphism, is equivalent to knowing X_k and the $c(x)$ and $e(x)$ in that case, up to isomorphism.

The last topic of this section is the behavior of the $c(x)$ with respect to finite morphisms. Suppose that $f: X \to Y$ is a finite morphism of proper nodal curves over D. Let y be a singular point of Y_k, and let x be a point of X_k that lies over y. In [20, Prop. 5] it is explained that $e(x)$ divides $e(y)$, and that there exist isomorphisms from $D[[u,v]]/(uv-\pi^{e(x)})$ to $\mathcal{O}^\wedge_{X,x}$ and from $D[[s,t]]/(st-\pi^{e(y)})$ to $\mathcal{O}^\wedge_{Y,y}$, such that, via f, s corresponds to u^n and t to v^n, with $n := e(y)/e(x)$. This proves the following proposition.

5.4 Proposition. *Let $f: X \to Y$ be a finite morphism of nodal curves over D. Let y in Y_k be a singular point, let x be in $f^{-1}y$, let C_1 and C_2 the two branches at x, and D_1 and D_2 the two branches at y. Let n be $e(y)/e(x)$. Then f gives isomorphisms between $\Omega_{D_i}(y)$ and $\Omega_{C_i}(x)^{\otimes n}$. Under these isomorphisms, $c(y)$ is mapped to $c(x)^n$.*

6 The $c(x)$ and log structures.

(This section will not be used in the rest of the article.) The elements $c(x)$ as described in Sections 3–5 have a simple description in terms of log structures. The notations are as before. Recall from [14] that a log structure on a scheme Y consists of a morphism of sheaves of multiplicative monoids $\alpha: M_Y \to \mathcal{O}_Y$ on the small etale site of Y such that α induces an isomorphism from $\alpha^{-1}\mathcal{O}_Y^*$ to $\mathcal{O}_Y^*$. For our scheme X, let M_X be the log structure induced by its special fibre. This means that M_X is the subsheaf of $\mathcal{O}_X$ consisting of functions whose restriction to the generic fibre are invertible. Let Q_X be the sheaf of monoids $M_X/\mathcal{O}_X^*$. For x non-singular in $X_k(k)$, the stalk $Q_{X,x}$ is just $\mathbb{N}$, with generator the class of the uniformizer π of D. Let x in $X_k(k)$ be singular. Then, locally at x in the etale topology, X is isomorphic to the spectrum of $D[u,v]/(uv-\pi^{e(x)})$. It follows that $Q_{X,x}$ is the submonoid $(e(x),0)\mathbb{N}+(0,e(x))\mathbb{N}+(1,1)\mathbb{N}$ of $\mathbb{N}^2$, generated by the classes of u, v and π. We give D the log structure induced by its closed point: $M_D = D - \{0\}$. Then we have a morphism of log schemes from (X, M_X) to (D, M_D). We will now base change to k, equipped with its log structure M_k induced from M_D: M_k is the fibred sum of M_D and k^* over D^*. Let (X_k, M_{X_k}) be the pull back of (X, M_X) to (k, M_k). Then M_{X_k} is the fibred sum of M_X and $\mathcal{O}_{X_k}^*$ over $\mathcal{O}_X^*|_{X_k}$.

Let now x in $X_k(k)$ be singular, and let $M_{X_k}(x)$ be the fibred sum of $M_{X_k,x}$ and k^* over $\mathcal{O}_{X_k,x}^*$. Let C_1 and C_2 be the two branches of X_k at x, with $u=0$ on C_1 and $v=0$ on C_2. We have an extension:

$$1 \to k^* \to M_{X_k}(x) \to Q_x \to 0, \tag{6.1}$$

with Q_x as described above. That description shows that the k^*-torsors induced on the inverse images of $(e(x),0)$ and $(0,e(x))$ correspond to the one dimensional k-vector spaces $\Omega_{C_2}(x)$ and $\Omega_{C_1}(x)$, respectively. Because of the monoid structure on $M_{X_k}(x)$, the inverse image of $(e(x),e(x))$ corresponds to $\Omega_{C_2}(x)\otimes_k\Omega_{C_1}(x)$. But via the morphism from (X_k, M_{X_k}) to (k, M_k) it also corresponds to $(m/m^2)^{\otimes e(x)}$. The element $c(x)$ is then just the image of $\pi^{e(x)}$. In the case that X is regular, the argument above shows that, conversely, the k-scheme X_k with the $c(x)$ determine the log-scheme (X_k, M_{X_k}) over (k, M_k) up to unique isomorphism. Compare with [30].

7 The $c(x)$ for $X_0(p)$.

Let p be a prime number and let $X_0(p)$ and $X_0(p)^\sim$ be the models over $\mathbb{Z}$ of modular curve $X_0(p)_\mathbb{Q}$ mentioned in the introduction. Let D be the ring of Witt vectors of an algebraic closure k of $\mathbb{F}_p$, and let $X := X_0(p)_D$ and $X^\sim := X_0(p)^\sim_D$. The aim of this section is to describe the $c(x)$ for the singular x in X_k. We will first give such a description in terms of the Kodaira-Spencer map and a well known mod p modular form B on supersingular elliptic curves. After that we give an expression of some small power of $c(x)$ in terms of the supersingular j-invariants mod p, using the fact that the point $0-\infty$ of $J_0(p)_\mathbb{Q}$ has order n, the numerator of $(p-1)/12$. Note that knowing the $c(x)$ for X is equivalent to knowing them for $X^\sim$, by Section 5. The results of this section will not be used in the explicit computations of the next section; instead, when we really need the $c(x)$, we will compute them using explicit formulas for the Hecke correspondence T_2.

Let $n \geq 3$ be an integer prime to p, and let $Y := \overline{\mathcal{M}}(\Gamma(n), \Gamma_0(p))_D$. Then Y is a regular projective nodal curve over D, and $Y \to X$ is the quotient for the faithful action of the group $G := \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})/\{1,-1\}$. The special fibre X_k of X consists of two irreducible components, X_∞ and X_0, containing the cusps ∞ and 0, respectively. We define Y_∞ to be the union of the irreducible components of Y_k that are mapped to X_∞, and Y_0 the union of those that are mapped to X_0. Let Z denote the j-line over D, i.e., $Z := X_0(1)_D$. The constructions that associate, to an elliptic curve $E/S/k$, the isogenies $F\colon E \to E^{(p)}$ and $V\colon E^{(p)} \to E$, induce isomorphisms $Z_k \to X_\infty$ and $Z_k \to X_0$, respectively, and isomorphisms from $X(n)_k$ to Y_∞ and Y_0. The singular points of X_k correspond to the $F\colon E \to E^{(p)}$ with E over k supersingular. Indeed, such an isogeny is isomorphic to $V\colon E = E^{(p^2)} \to E^{(p)}$. It follows that the singular point $F\colon E \to E^{(p)}$ on X_k corresponds to the pair $(j(E), j(E)^p)$ of points on X_∞ and X_0. A similar statement is of course true for Y_k.

Let P be a singular point of X_k, corresponding to a supersingular elliptic curve E over k, and let Q be in Y_k, lying over P. The Kodaira-Spencer map gives an isomorphism of line bundles on $X(n)_k$ from $\Omega(\text{cusps})$ to $\underline{\omega}^{\otimes 2}$, where $\underline{\omega}$ is the pullback along the zero section of the bundle of relative differentials of the universal elliptic curve. This map, and the isomorphisms from Y_∞ and Y_0 to $X(n)_k$, induce isomorphisms of k-vector spaces:

$$(7.1) \qquad \Omega_{Y_\infty}(Q) \to \underline{\omega}_E^{\otimes 2}, \quad \Omega_{Y_0}(Q) \to \underline{\omega}_{E^{(p)}}^{\otimes 2} = \underline{\omega}_E^{\otimes 2p}.$$

Via these isomorphisms, $c(Q)$ can be viewed as an element of $\underline{\omega}_E^{\otimes 2(p+1)}$, and then $c(P) = c(Q)^{e(P)}$ in $\underline{\omega}_E^{\otimes 2(p+1)e(P)}$. Now there does exist, for each supersingular elliptic curve E over k, a non-zero element $B(E)$ in $\underline{\omega}_E^{\otimes (p+1)}$, with the property that for $\phi\colon E \to E'$ any isogeny, one has $\deg(\phi)B(E) = \phi^* B(E')$. This property determines the $B(E)$ up to a common scalar, since all supersingular elliptic curves over k are isogenous via isogenies of degree prime to p. Such $B(E)$ were defined by Robert in [27, Thm. B] and used by Serre in [28]; see [9, Prop. 7.2] and [5, 5] for several constructions. We will prove below that, up to a common scalar, the $c(Q)$ are the squares of the $B(E)$. But before we do that, we give yet another construction of B.

So let E be a supersingular elliptic curve over k. Since $V\colon E^{(p)} \to E$ is the dual of $F\colon E \to E^{(p)}$, we have a natural isomorphism between the kernel $E^{(p)}[V]$ of V and the Cartier dual $E[F]^{\mathrm{D}}$ of the kernel of F ([15, 2.8]). But $E^{(p)}[V] = E^{(p)}[F] = E[F]^{(p)}$, hence we have a natural isomorphism $\lambda_E\colon E[F]^{(p)} \to E[F]^D$. Now recall that the category of finite commutative group schemes over k that are annihilated by F and V is anti-equivalent to the category of k-vector spaces by sending such a group scheme to its cotangent space at the origin, and that this converts Cartier duality into ordinary duality ([12, 3,Thm. 7.4]). Hence we get an isomorphism between $\underline{\omega}_E^{\otimes p}$ and $\underline{\omega}_E^{\otimes -1}$, i.e., a non-zero element $B(E)$ of $\underline{\omega}_E^{\otimes (p+1)}$. For $\phi\colon E \to E'$ an isogeny, one has $\lambda_E \circ \phi^{(p)t} = \phi^D \circ \lambda_{E'}$, hence $\phi^D \circ \lambda_{E'} \circ \phi^{(p)} = \deg(\phi)\lambda_E$, establishing $\phi^* B(E') = \deg(\phi) B(E)$.

7.2 Theorem. *There exists an element μ in k^* such that for all singular Q in Y_k we have $c(Q) = \mu B(E)^{\otimes 2}$.*

Proof. Let us first note that, to define the $c(Q)$, we have to choose a uniformizer of D, for which we take p, but that the choice does not matter for the existence of μ. To prove the proposition, it suffices to show that the $c(Q)$ behave in the right way with respect to isogenies of degree prime to p.

So let Q be a singular point of Y_k, corresponding to a pair (E, α), with E a supersingular elliptic curve over k and α an isomorphism from $(\mathbb{Z}/n\mathbb{Z})^2$ to $E[n](k)$. Let $\phi\colon E \to E'$ be an isogeny of some degree d that is not divisible by p. Let Q' be the singular point of Y_k corresponding to $(E', \phi \circ \alpha)$. Let C be the modular curve $\mathcal{M}(\Gamma_0(d), \Gamma(n), \Gamma_0(p))_D$, and s and t the usual morphisms from C to Y (s just forgets the subgroup of rank d, t takes the quotient by it). Let R be the point of C_k corresponding to (E, ϕ, α). The morphisms s and t are etale, hence induce isomorphisms between $\mathcal{O}^\wedge_{Y,Q}$, $\mathcal{O}^\wedge_{C,R}$ and $\mathcal{O}^\wedge_{Y,Q'}$, preserving the labeling of the branches by ∞ and 0. It follows that $c(Q)$ and $c(Q')$ correspond, via s and t repectively, to $c(R)$.

For $S \to T$ be any morphism of schemes, and $f\colon F \to F'$ any morphism of elliptic curves over S, the following diagram is commutative by [9, Lemma 7.4]:

$$\begin{tikzcd} \underline{\omega}^{\otimes 2}_{F/S} \arrow[dr, "\mathrm{KS}_{F/S}"'] & & \underline{\omega}^{\otimes 2}_{F'/S} \arrow[ll, "f^*"'] \arrow[dl, "(\deg f)\mathrm{KS}_{F'/S}"] \\ & \Omega_{S/T} & \end{tikzcd} \tag{7.3}$$

Applying this to our isogeny ϕ at the point R of C_∞ and of C_0 shows that the following diagram, in which the vertical arrows are given by (7.1), is commutative:

$$
\begin{array}{ccc}
\Omega_{Y_\infty}(Q') \otimes \Omega_{Y_0}(Q') & \stackrel{(t\circ s^{-1})^*}{\longrightarrow} & \Omega_{Y_\infty}(Q') \otimes \Omega_{Y_0}(Q') \\
\downarrow & & \downarrow \\
\underline{\omega}_{E'}^{\otimes 2(p+1)} & \stackrel{\phi^*/d^2}{\longrightarrow} & \underline{\omega}_{E}^{\otimes 2(p+1)}
\end{array}
\tag{7.4}
$$

This establishes that the $c(Q)$ behave the same as the $B(E)$ with respect to isogenies of degree prime to p. □

To finish this section, we will investigate the consequences of the fact that the point $0-\infty$ in $J_0(p)(\mathbb{Q})$ has order n, the numerator of the fraction $(p-1)/12$, expressed in lowest terms (see [18, II, Prop. 11.1]). These consequences will not be used in the following sections.

Let J denote $\mathrm{Pic}^{[0]}_{X/D}/E$, as in the notation of Section 2. Then J is the image in the Néron model $J_0(p)_D$ of $\mathrm{Pic}^{[0]}_X$. Let P denote the element of $J(D)$ given by $0-\infty$ (every line bundle of degree zero on X_K that admits an extension to a line bundle over X determines a unique element of $J(D)$). Let e be the least common multiple of the $e(x)$, x ranging through the singular points of X_k. We recall from [8, VI, Thm. 6.9] that $e(x)$ equals half the number of automorphisms of either of the two elliptic curves E and $E^{(p)}$ over k corresponding to x. Hence all $e(x)$ are equal to one, except when x has j-invariant 0 or 1728, in which case $e(x)$ equals three or two, respectively, assuming p is at least five. We also recall that the elliptic curves over k with j-invariant equal to 0 or 1728 are supersingular if and only if p is congruent to -1 modulo three or four, respectively. It follows that $n=(p-1)e/12$. The exact sequence (2.1) shows that $E(D)$ is generated by the isomorphism class of the invertible $\mathcal{O}_X$-module $\mathcal{O}_X(eX_\infty)$. The "mass formula" of [15, Cor. 12.4.6] or [8, VI, 4.9.1] gives:

$$
\deg\left(\mathcal{O}_X(eX_\infty)|_{X_0}\right) = \sum_{x\in S_1} e/e(x) = e\sum_{x\in S_1} 1/e(x) = e(p-1)/12 = n. \tag{7.5}
$$

We apply Proposition 3.11 to the divisor $n(0-\infty)$ on X_k. It follows that there exist rational functions f_∞ and f_0 on X_∞ and X_0 with divisors $-n\infty + \sum_x (e/e(x))x$ and $-n0+\sum_x (e/e(x))x$, respectively, such that at every x one has $c(x)^{e/e(x)} = f_\infty(x) f_0(x)$, where $f_\infty(x)$ and $f_0(x)$ are to be interpreted as leading terms as in Section 3. This proves the following theorem.

7.6 Theorem. *There exists a unique α in $\mathbb{F}_p^*$ such that for every supersingular x in X_k one has:*

$$
\begin{aligned}
c(x)^{e/e(x)} &= \alpha \prod_{y\neq x} (j(y)-j(x))^{e/e(y)} \prod_{y\neq x^{(p)}} (j(y)-j(x)^p)^{e/e(y)} \cdot (dj\otimes dj)^{e/e(x)} \\
&= \alpha \prod_{y\neq x} (j(y)-j(x))^{(p+1)e/e(y)} \cdot (dj\otimes dj)^{e/e(x)}.
\end{aligned}
$$

7.7 Remark. Theorem 7.6 only gives us the $c(x)^{e/e(x)}$, but Proposition 3.11 says that these powers are sufficient to describe the special fibre $J_0(p)_p$ of the Néron model of $J_0(p)_{\mathbb{Q}}$ completely. Hence it should be possible to improve the main theorem of [29] in the sense that one should be able to get rid of the factors ± 1 in it, because of [29, 0.5]. $\square$

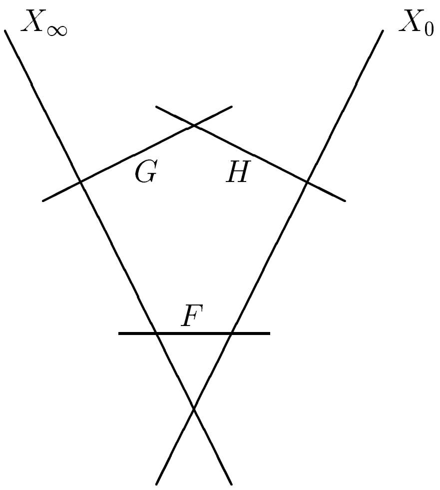
8 The intersection mod p of $X_0(p)$ and the cuspidal group.

Let p, k, X, $X^\sim$ be as in the previous section. As in the introduction, let 0 and ∞ denote the two cusps of X (we view them as D-valued points of X). Let J denote the Néron model over D of $J_0(p)_K$, and let $f\colon X^{\sim,0} \to J$ be the morphism from the smooth locus of $X^\sim$ to J that extends the morphism from X_K to J_K sending a point P to the class of the divisor $P-\infty$. Let c be $f(0)$. Then c is of order n, the numerator of $(p-1)/12$, expressed in lowest terms. Let C be the closed subgroup scheme of J generated by c; it is called the cuspidal subgroup because its points correspond to classes of divisors supported on the cusps. In this section, we want to determine $f^{-1}C_k$, the inverse image under f of the fibre over k of C, in order to answer the question of Coleman mentioned in the introduction. There are two obvious elements in $f^{-1}C_k$: ∞_k and 0_k; the idea is that there should be nothing more. Let us note that Mazur's theorem [19] on rational isogenies of prime degree implies that, for p bigger than 163, $f^{-1}C_K$ consists just of ∞ and 0, because those are the only two $\mathbb{Q}$-rational points of $X_0(p)$. But a priori $f^{-1}C_k$ could be bigger than the specialization of $f^{-1}C_K$.

We will now recall some facts about the geometry of $X^\sim$, the group $\Phi := J_k/J_k^0$ and the map from C_k to Φ, that can be found in [18, III, Prop. 4.2]. As in the previous section, let X_∞ and X_0 denote the two irreducible components of X_k. The $e(x)$ for X_k were described at the end of the previous section. It follows that, for p at least five, $X_k^\sim$ has at most five irreducible components: X_∞ and X_0, one component F of genus zero (if p is -1 mod 4) originating from the singularity at x with $j(x) = 1728$, and two components G and H of genus zero (if p is -1 mod 3) originating from the singularity at x with $j(x) = 0$. (The cases $p = 2$ or 3 will be of no interest, because then $X_0(p)$ has genus zero.) In order to fix the notation regarding G and H, we demand that G intersects X_∞. Figure 8.1 gives a picture in the case $p = 23$, where all five components are present. The intersection numbers of distinct components are as follows: $X_\infty \cdot X_0$ equals the number of supersingular j-invariants in $k - \{0, 1728\}$; $X_\infty \cdot F = X_0 \cdot F = 1$; $X_\infty \cdot G = 1$; $X_0 \cdot G = 0$; $X_\infty \cdot H = 0$; $X_0 \cdot H = 1$; $F \cdot G = F \cdot H = 0$, $G \cdot H = 1$. The self intersections can be computed from the fact that $X_k^\sim$ has trivial intersection with all components. From the resulting intersection matrices one can compute, as described in Section 2, that Φ is cyclic of order n, and generated by the image ϕ of $X_0 - X_\infty$ (i.e., Φ is generated by the class of any invertible $\mathcal{O}_{X_k^\sim}$-module that has degree one on X_0, degree minus one on X_∞, and degree zero on the other components). It follows that the composition $C_k \to J_k \to \Phi$ is an isomorphism, hence every connected component of J_k contains exactly one element of C_k.

Of course, if X_K is of genus zero, J is just the trivial group scheme, and $f^{-1}C_k$ is all

Figure 8.1: A picture of $X_k^\sim$ when $p = 23$.



of $X_k^{\sim,0}$. If X_K is of genus one, then f is an isomorphism ($X^{\sim,0}$ and J are both Néron models of X_K), hence $f^{-1}C_k$ is isomorphic, via f, to C_k. So this solves the problem in these two cases, i.e., for $p < 23$. From now on we assume that X_K has genus at least two, i.e., that $p \geq 23$.

8.2 Theorem. *Let $p \geq 23$ be prime. Then we have $f^{-1}C_k = \{\infty_k, 0_k\}$.*

The proof of this result will take some time. In the next section we will see that the morphism $f: X_k^{\sim,0} \to J_k$ is a closed immersion, so that $f^{-1}C_k$, as a closed subscheme, is even reduced. We will first prove Theorem 8.2 for $p > 71$, the remaining cases will then be dealt with by hand or by simple computer computations.

8.3 First step of the proof of Theorem 8.2.

For the moment, let $p \geq 23$ be prime. First we compute the map from the set of irreducible components of $X_k^\sim$ to Φ. This computation is done in [18, III, Prop. 4.2], but since we need the details of it later, we repeat it.

8.3.1 Lemma. *Under f, X_∞ is mapped to 0 in Φ, X_0 is mapped to ϕ, F is mapped to $(1/2)\phi$, G to $(1/3)\phi$ and H to $(2/3)\phi$, (in each case the fraction makes sense mod n).*

Proof. Of course, X_∞ is mapped to 0 in Φ, because X_∞ contains ∞_k and $f(\infty) = 0$. Likewise, X_0 is mapped to our generator ϕ of Φ, because it contains 0_k, and $f(0)$ is the class of the divisor $0 - \infty$, which has multidegree $X_0 - X_\infty$.

Suppose now that p is -1 mod 4. We want to compute the image of the component F. Note that n is then odd, so that we can invert 2 in order to compute in Φ using (2.2). Let P be in $F(k)$. Then $P - \infty$ has multidegree $F - X_\infty$. Note that $\mathcal{O}_{X^\sim}(F)$ has multidegree

$X_\infty + X_0 - 2F$. By the results of Section 2, the image of $f(P)$ in Φ is the class of the multidegree $F - X_\infty + (1/2)(X_\infty + X_0 - 2F) = (1/2)(X_0 - X_\infty)$.

Finally, suppose that p is -1 mod 3. Then n is -1 mod 3. The intersection numbers above tell us that $\mathcal{O}_{X^\sim}(G)$ and $\mathcal{O}_{X^\sim}(H)$ have multidegrees $X_\infty - 2G + H$ and $X_0 + G - 2H$, respectively. Let P be in $G(k)$. It follows that the image of $f(P)$ in Φ is the class of the multidegree

$$-X_\infty + G + (1/3)\left(2(X_\infty - 2G + H) + (X_0 + G - 2H)\right) = (1/3)(X_0 - X_\infty).$$

A similar computation can be done for H. □

For Z an irreducible component of $X_k^\sim$, let Z^0 be Z minus the double points of $X_k^\sim$ lying on it.

8.3.2 Lemma. *The only P in $X_\infty^0(k)$ with $f(P)$ in $C(k)$ is ∞_k.*

Proof. Suppose P is in $X_\infty^0(k)$, with $f(P)$ in $C(k)$. Lemma 8.3.1 tells us that $f(P) = 0$ in $J(k)$. Then Proposition 3.11 gives a rational function g on X_∞ with divisor $P - \infty$, and with a constant value at all supersingular points. Since there are at least two distinct supersingular points, g is constant, and $P = \infty$. □

8.3.3 Lemma. *The only P in $X_0^0(k)$ with $f(P)$ in $C(k)$ is 0_k.*

Proof. The proof is almost the same as the previous one. In this case one must have $f(P) = c$ in $C(k)$, and one applies Proposition 3.11 to the divisor $P - 0$ on $X_k^\sim$. □

8.3.4 Lemma. *Let $p \geq 23$ be -1 mod 4 and suppose that there exists a supersingular j-invariant in $\mathbb{F}_p - \{1728\}$ and one in $\mathbb{F}_{p^2} - \mathbb{F}_p$. Then there is no point P in $F^0(k)$ such that $2f(P)$ is in $C(k)$.*

Proof. Suppose that p is as in the Lemma, and that P in $F^0(k)$ has $2f(P)$ in $C(k)$. Then $2f(P) = c$ by Lemma 8.3.1. We apply Proposition 3.11 to the divisor $E := 2P - 0 - \infty$. It follows that there exist λ_∞ and λ_0 in k^*, such that the functions $f_\infty = \lambda_\infty(j - 1728)$ and $f_0 = \lambda_0(j - 1728)$ on X_∞ and X_0 satisfy $f_\infty(x) = f_0(x^p)$ for every supersingular j-invariant x in k other than 1728. The existence of such an x in $\mathbb{F}_p$ shows that $\lambda_\infty = \lambda_0$. But the existence of such an x in $\mathbb{F}_{p^2} - \mathbb{F}_p$ implies that $\lambda_\infty = -\lambda_0$. □

8.3.5 Lemma. *Let $p \geq 23$ be -1 mod 3 and suppose that there exist two supersingular j-invariants x and y in $k - \{0\}$ such that $x^{p-2} \neq y^{p-2}$. Then there is no point P in $G^0(k)$ or in $H^0(k)$ such that $3f(P)$ is in $C(k)$.*

Proof. Suppose that p is as in the Lemma, and that P in $G^0(k)$ has $3f(P)$ in $C(k)$. Then $3f(P) = c$ by Lemma 8.3.1. We apply Proposition 3.11 to the divisor $E := 3P - 2\infty - 0$. It follows that $\mathcal{O}_{X^\sim}(-2G - H)|_{X_k^\sim}$ has a rational section with divisor E. The restrictions to X_∞ and X_0 of this section are of the form $\lambda_\infty j^2$ and $\lambda_0 j$, respectively. It follows that for every supersingular x in $k - \{0\}$ we have $\lambda_\infty x^2 = \lambda_0 x^p$. This contradicts the hypotheses of the Lemma. The same result holds for H because of the w_p-operator. □

8.3.6 Lemma. *For every prime $p \geq 23$ with $p = -1$ mod 3 there do exist supersingular j-invariants x and y in $k - \{0\}$ with $x^{p-2} \neq y^{p-2}$.*

Proof. Note that $p-2$ and p^2-1 have greatest common divisor 3, hence the required x and y exist if there are at least four distinct non-zero supersingular j-invariants in k. The mass formula of [15, Cor. 12.4.6] shows that the number of supersingular j-invariants is at least $(p-1)/12$, which proves the Lemma for $p > 49$. The primes that are left are 23, 29, 41 and 47. Table 6 in [1] shows that in each of these cases there are at least two non-zero supersingular j-invariants x and y in $\mathbb{F}_p$. For these we have $x^{p-2} = x^{-1} \neq y^{-1} = y^{p-2}$. $\square$

8.3.7 Lemma. *For every prime $p > 7$ with $p = -1$ mod 4 there exists a supersingular j-invariant in $\mathbb{F}_p - \{1728\}$.*

Proof. There is a unique elliptic curve over $\mathbb{C}$ (up to isomorphism) with endomorphism ring $\mathbb{Z}[2i]$. Its j-invariant is $2^3 3^3 11^3$, and we have $2^3 3^3 11^3 - 1728 = 2^3 3^6 7^2$. (An easy way to compute this is to use the explicit description of the Hecke correspondence T_2 given below in Section 8.4.) $\square$

8.3.8 Lemma. *For every prime $p > 71$ there exists a supersingular j-invariant in $k - \mathbb{F}_p$.*

Proof. This is a result of Ogg, which can be found as a combination of the identities (21) and (24) in [23]. We recall the argument in a few lines. The $\mathbb{F}_p$-rational supersingular points of X_k are precisely the fixed points of the Fricke involution w of X. It follows that all supersingular j-invariants in k are in $\mathbb{F}_p$ if and only if $X_k^+ := X_k/\{1, w\}$ has genus zero, hence if and only if $X_0(p)/\{1, w\}$ is isomorphic to $\mathbb{P}^1_{\mathbb{Z}}$. Suppose that this is the case. Then the inequalities:

$$10 = 2\#\mathbb{P}^1(\mathbb{F}_4) \geq \#X_0(p)(\mathbb{F}_4) \geq (p+1)/12 + 2$$

show that p is at most 95. Table 6 of [1] then finishes the proof. $\square$

Combining the previous lemmas in this section proves Theorem 8.2 for all primes $p > 71$: Lemmas 8.3.5 and 8.3.6 take care of points on G and H, and Lemmas 8.3.4, 8.3.7 and 8.3.8 take care of points on F. For the remaining primes p with $23 \leq p \leq 71$ with $p = -1$ mod 4 we still have to show that no point P in $F^0(k)$ has $f(P)$ in $C(k)$. Another inspection of Table 6 of [1] shows that the hypothesis of Lemma 8.3.4 is satisfied for 43 and 67. The only primes with which we still have to deal are 23, 31, 47, 59 and 71.

8.4 Second step in the proof of Theorem 8.2.

In this section we will deal with the remaining p for which we still have to prove Theorem 8.2: 23, 31, 47, 59 and 71. We recall that in these cases we only have to show that no P in $F^0(k)$ has $f(P)$ in $C(k)$. The proof will use the full strength of Proposition 3.11 and the behavior of the $c(x)$ under the Hecke correspondence T_2. Until now we only applied Proposition 3.11 in rather simple cases, not involving the $c(x)$.

In order to apply Proposition 3.11, it suffices to know the $c(x)$ up to a common scalar, for example, since the choice of a uniformizer of D is irrelevant. To compute the $c(x)$ up to a common scalar, we just pick a supersingular point x_0 with $e(x_0) = 1$ and choose $c(x_0) = dj \otimes dj$. Then we use the Hecke correspondence T_2 on X to find the $c(x)$ at the other supersingular x. The following explicit description of the Hecke correspondence T_2 on the affine j-line (over $\mathbb{Z}$) is given in [21]: $Y_0(2)_{\mathbb{Z}}$ is the curve in $\mathbb{A}^2_{\mathbb{Z}}$ given by the equation $uv = 2^{12}$, the morphisms s and t from $Y_0(2)_{\mathbb{Z}}$ to the j-line are given by $s^*j = (u+16)^3/u$ and $t^*j = (v+16)^3/v$. Now consider the correspondence T_2 on X; it is given by the morphisms s and t from $X_0(2p)_D$ to X. In the beginning of Section 7 we gave isomorphisms between the j-line $\mathbb{P}^1_k$ and the two irreducible components X_∞ and X_0 of X_k. Likewise, $X_0(2p)_k$ is the union of $X_0(2p)_\infty$ and $X_0(2p)_0$, both isomorphic to $X_0(2)_k$ via similar isomorphisms, and the induced correspondences on X_∞ and X_0 are given by the explicit formulas above.

8.4.1 Lemma. *Let y in $X_0(2p)(k)$ be supersingular, and put $x := s(y)$, $x' := t(y)$. Then $j - j(x)$ is a local coordinate on X_∞ at x, $j - j(x)^p$ is one on X_0 at x, $u - u(y)$ and $v - v(y)$ are coordinates on $X_0(2p)_\infty$ at y, $u - u(y)^p$ and $v - v(y)^p$ are coordinates on $X_0(2p)_0$ at y, and $j - j(x')$ and $j - j(x')^p$ are local coordinates at x' on X_∞ and X_0, respectively. Recall from Proposition 5.4 that $e(y)$ divides $e(x)$ and $e(x')$. Let α in k^* be such that $c(y) = \alpha(du \otimes du)$. Let β and γ in k^* be given by:*

$$\begin{aligned} s^*(j - j(x)) &= \beta(u - u(y))^{e(x)/e(y)} + \textit{higher order terms}, \\ t^*(j - j(x')) &= \gamma(v - v(y))^{e(x')/e(y)} + \textit{higher order terms}. \end{aligned}$$

Then we have:

$$\begin{aligned} c(x) &= \alpha^{e(x)/e(y)} \beta^{-(p+1)} dj \otimes dj, \\ c(x') &= (-\alpha u(y)/v(y))^{e(x')/e(y)} \gamma^{-(p+1)} dj \otimes dj. \end{aligned}$$

Proof. The local coordinates are obtained using the isomorphisms from the j-line to X_∞ and X_0 and from $X_0(2)_k$ to $X_0(2p)_\infty$ and $X_0(2p)_0$ that we discussed above. The rest is a direct application of Proposition 5.4. □

8.4.2 Remark. In practice, we will start with a supersingular x in $X(k)$ with $e(x) = 1$. In order to find the various y's in $s^{-1}x$, we factor the polynomial $(T+16)^3 - j(x)T$ modulo p. (Actually, we will always have $j(x)$ in $\mathbb{F}_p$.) The factor β of Lemma 8.4.1 is then just the derivative $(3(u(y)+16)^2 - j(x))/u(y)$. Let us write $c(x) = \delta\, dj \otimes dj$. Suppose that $e(x) = 1 = e(x')$. Then we simply have:

$$c(x') = \left(\frac{j(x') - 3(v(y)+16)^2}{3(u(y)+16)^2 - j(x)} \right)^{p+1} \delta\, dj \otimes dj.$$

In all our computations, $j(x)$, $j(x')$, $u(y)$ and $v(y)$ will be in $\mathbb{F}_p$, so that we can replace the exponent $p+1$ by 2. □

8.4.3 The case $p = 31$.

The supersingular j-values in k are $1728 = -8$, 2 and 4 (see for example Table 6 in [1]). We have to check that no non-singular point on the component F is mapped to the cuspidal group. Suppose that P is in $F^0(k)$ with $f(P)$ in $C(k)$. Lemma 8.3.1 shows that $f(P) = 3c$. Hence we apply Proposition 3.11 to the divisor $-2\infty + 3{\cdot}0 - P$. Note that $\mathcal{O}_{X^\sim}(X_0)$ has multidegree $2X_\infty + F - 2X_0$. Let x_2 and x_4 denote the supersingular points of $X(k)$ with j-invariants 2 and 4, respectively. It follows that the functions $f_\infty = (j-2)(j-4)$ on X_∞, $f_0 = 1/(j+8)(j-2)(j-4)$ on X_0 satisfy equation 3.12 for the cycle $(X_\infty, x_2, X_0, x_4, X_\infty)$. As we remarked above, we only need to know $c(x_2)$ and $c(x_4)$ up to a common scalar, hence we simply put $c(x_2) = dj \otimes dj$. We apply the method described in Remark 8.4.2. The y's in $s^{-1}x_2$ are given by $u(y) \in \{-15, 2, -4\}$. We take y with $u(y) = -4$; then $j(t(y)) = 4$. It follows that $c(x_4) = -6dj \otimes dj$. The factor at x_2 in equation 3.12 is:

$$((2-4)dj \otimes ((2+8)(2-4))dj)c(x_2)^{-1} = 9(dj \otimes dj)c(x_2)^{-1} = 9.$$

The analogous factor at x_4 is:

$$((4+8)(4-2)dj \otimes (4-2)dj)^{-1}c(x_4) = -6{\cdot}17^{-1}(dj \otimes dj)^{-1}(dj \otimes dj) = -4.$$

Since $-4{\cdot}9 = -5 \neq 1$ in $\mathbb{F}_{31}$, we have found a contradiction, so we have proved that no P in $F^0(k)$ has image in $C(k)$.

8.4.4 The case $p = 47$.

The supersingular j-invariants are 0, $1728 = -11$, 9, 10 and -3. Suppose that P is in $F^0(k)$ with $f(P)$ in $C(k)$. Then $f(P) = 12c$ by Lemma 8.3.1. We apply Proposition 3.11 to the divisor $E := -11\infty + 12{\cdot}0 - P$. The divisor $X_\infty - 2X_0 - H$ has the same multidegree as E. Let x_9 and x_{10} in $X(k)$ be the points with j-invariants 9 and 10. The functions:

$$f_\infty = j(j+11)(j-9)^3(j-10)^3(j+3)^3 \text{ and } f_0 = (j(j+11)^2(j-9)^3(j-10)^3(j+3)^3)^{-1}$$

then satisfy equation 3.12 for the cycle $(X_\infty, x_9, X_0, x_{10}, X_\infty)$. We normalize the $c(x)$ such that $c(x_9) = dj \otimes dj$. In order to compute $c(x_{10})$, consider the point y on $X_0(2)(k)$ with $u(y) = -1$. Then we have $j(s(y)) = 9$ and $j(t(y)) = 10$. One computes that $c(x_{10}) = 3dj \otimes dj$, and that the factors at x_9 and x_{10} in equation 3.12 equal 15 and -11, respectively. Since $-11{\cdot}15 = 23 \neq 1$, we have proved that no P in $F^0(k)$ has $f(P)$ in $C(k)$.

8.4.5 The case $p = 59$.

The supersingular j-invariants are 0, $1728 = 17$, 15, 28, -12 and -11. Suppose that P is in $F^0(k)$ with $f(P)$ in $C(k)$. Then computations as in the previous two cases show that the functions:

$$\begin{aligned} f_\infty &= j(j-17)(j-15)^3(j-28)^3(j+12)^3(j+11)^3 \\ f_0 &= (j(j-17)^2(j-15)^3(j-28)^3(j+12)^3(j+11)^3)^{-1} \end{aligned}$$

satisfy equation 3.12 for the cycle $(X_\infty, x_{15}, X_0, x_{28}, X_\infty)$. We normalize the $c(x)$ such that $c(x_{15}) = dj \otimes dj$. In order to compute $c(x_{28})$, consider the point y on $X_0(2)(k)$ with $u(y) = -3$. One finds that $c(x_{28}) = 21dj \otimes dj$, and that the factors at x_{15} and x_{28} in equation 3.12 equal 35 and 10, respectively. Since $35{\cdot}10 = -4 \neq 1$, we have proved that no P in $F^0(k)$ has $f(P)$ in $C(k)$.

8.4.6 The case $p = 71$.

The supersingular j-invariants are 0, $1728 = 24$, 17, -31, -30, -23 and -5. Suppose that P is in $F^0(k)$ with $f(P)$ in $C(k)$. Computations as in the previous cases show that the functions:

$$\begin{aligned}
f_\infty &= j(j-24)(j-17)^3(j+31)^3(j+30)^3(j+23)^3(j+5)^3 \\
f_0 &= (j(j-24)^2(j-17)^3(j+31)^3(j+30)^3(j+23)^3(j+5)^3)^{-1}
\end{aligned}$$

satisfy equation 3.12 for the cycle $(X_\infty, x_{-31}, X_0, x_{-23}, X_\infty)$. We normalize the $c(x)$ such that $c(x_{-31}) = dj \otimes dj$, and find that $c(x_{-23}) = -13dj \otimes dj$ using the point y with $u(y) = -14$. The factors at x_{-31} and x_{-23} in equation 3.12 equal 49 and 38, respectively. Since $49{\cdot}38 = 16 \neq 1$, we have proved that no P in $F^0(k)$ has $f(P)$ in $C(k)$.

8.4.7 The case $p = 23$.

The supersingular j-invariants are 0, $1728 = 3$ and -4. Suppose that P is in $F^0(k)$ with $f(P)$ in $C(k)$. In this case we work with the model X' of X obtained by blowing up X in its k-valued point with j-invariant 1728. We apply Proposition 3.11 to the divisor $-5\infty + 6{\cdot}0 - P$. For a in the proposition one can take $-3X_0 - F$. One finds that the functions:

$$f_\infty = j(j-3)(j+4)^3 \text{ and } f_0 = (j(j-3)^2(j+4)^3)^{-1}$$

satisfy equation 3.12 for the cycle $(X_\infty, x_0, X_0, x_{-4}, X_\infty)$. We normalize the $c(x)$ such that $c(x_{-4}) = dj \otimes dj$. Consider the point y with $u(y) = -3$. In the notation of Lemma 8.4.1 one finds: $\beta = 6$, $\alpha = \beta^2 = 13$ and $\gamma = 1/7$. It follows that $c(x_0) = -3dj \otimes dj$. The factors at x_0 and x_{-4} in equation 3.12 equal 13 and -5, respectively. Since $13 \cdot -5 = 4 \neq 1$, we have proved that no P in $F^0(k)$ has $f(P)$ in $C(k)$.

9 Closed immersions.

We go back to the situation of Section 2. So let D be a complete discrete valuation ring with fraction field K, separably closed residue field k, and uniformizer π. Let X_K be a proper smooth geometrically irreducible curve, and X a proper flat nodal model of it. Let $J := \mathrm{Pic}^{[0]}_{X/D}/E$ be the open part of the Néron model $J^\sim$ of the jacobian of X_K over D corresponding to line bundles on X_K that admit an extension to a line bundle on X. Let X^0 be the open part of X where the morphism to $\mathrm{Spec}(D)$ is smooth, i.e., X^0 is the complement in X of the set of singular points of X_k. Let P be in $X^0(D)$. Then we have a morphism $f\colon X^0 \to J$ that sends, for S any D-scheme, Q in $X^0(S)$ to the class of $\mathcal{O}_{X_S}(Q - P_S)$. Equivalently, f is the morphism one obtains by applying the Néron property to X^0 and $f_K\colon X_K \to J_K$, using that the image of X^0 in $J^\sim$ is in J. In this section we study to what extent f is a closed immersion. The first observation in doing this is that the disconnecting double points behave differently from the non-disconnecting ones. Recall that a singular point x of X_k is called disconnecting if and only if $X_k - \{x\}$ is not connected. We let X' be the complement in X of the set of non-disconnecting double points of X_k.

9.1 Proposition. *The morphism $f\colon X^0 \to J$ extends to a morphism $f\colon X' \to J$.*

Proof. Let $i\colon \Gamma \to X^0 \times_D J$ be the graph of f, and let $\overline{\Gamma}$ be its closure in $X' \times_D J$. We have to show that the projection p from $\overline{\Gamma}$ to X' is an isomorphism. We will show, using the valuative criterion of properness ([13, II, 7.3.8 and 7.3.9]), that p is proper. Then we show that p is bijective, hence finite. The normality of X then implies that p is an isomorphism.

In order to prove that p is proper, it suffices to show that for D' any complete discrete valuation ring with fraction field K' and algebraically closed residue field k', and any Q in $X'(D')$ with $Q(\mathrm{Spec}(K'))$ in X_K, the element $f(Q)$ in $J(K')$ extends uniquely to one in $J(D')$. (The fact that it is sufficient to consider Q in $X'(D')$ with $Q(\mathrm{Spec}(K'))$ in the open dense subset X_K of X' can be found for example, without proof, in [7, p. 103], and, with a proof, in the forthcoming book [16]. The proof of this fact starts with a reduction to the quasi-projective case, using Chow's Lemma; then one considers a compactification.) Let D' and Q in $X'(D')$ be as above. If Q is in $X^0(D')$, the condition is obviously satisfied (just compose Q with f). So we suppose that Q is not in $X^0(D')$. That means that $Q(\mathrm{Spec}(k'))$ is a disconnecting double point, say x, of X_k. Base change from D to D' reduces the whole problem to the case $D' = D$. By definition, $f_K(Q)$ extends in $J^\sim$; we have to show that that extension is actually in J. Equivalently, we have to show that the connected component of $J_k^\sim$ to which $f_K(Q)$ specializes is a connected component of J_k. Let $F_1, \ldots, F_{e(x)-1}$ be the chain of $\mathbb{P}^1_k$'s lying over x in the minimal resolution $R\colon X^\sim \to X$. Since x is disconnecting, $X_k^\sim$ can be written as $Y \cup Z \cup (\cup_i F_i)$, with Y and Z connected, disjoint, and $Y \cup Z$ the union of all irreducible components of $X_k^\sim$ other than the F_i. To fix who is who, we demand that P is in Y and that F_1 meets Y. In $X^\sim$, Q specializes to one of the F_i, say F_i. Let C be the irreducible component of X_k to which P specializes. Then the connected component of $J_k^\sim$ to which $f(Q)$ specializes is, in the description of Section 2, the class of the multidegree $F_i - C$. Let a be the divisor on $X^\sim$ which is given as the sum of all irreducible components of Y, and the F_j with $j < i$. The multidegree of a equals $F_i - F_{i-1}$. It follows that the class of the multidegree $F_i - C$ is the same as that of $F_0 - C$, if F_0 denotes the unique irreducible component of Y meeting F_1. Since F_0 maps onto an irreducible component of X_k, we have shown that the class of $F_i - C$

is represented by a multidegree with support in the set of irreducible components of X_k, hence that $f(Q)$ extends to an element of $J(D)$. This finishes the proof that p is proper.

Let us now show that p is bijective. So let x be a disconnecting double point of X_k. It suffices now to show that for every Q in $X(K)$ specializing to x, $f(Q)$ specializes to the same element in $J(k)$ (the argument is similar to the proof of the generalization in [7] of the valuative criterion of properness mentioned above). Recall that $J_k = \mathrm{Pic}^{[0]}_{X_k/k}/E_k$. Let Q and Q' be two elements of $X(K)$ specializing to x. The fact that the F_i are isomorphic to $\mathbb{P}^1_k$ implies that $f(Q)$ and $f(Q')$ specialize to the same element of $J(k)$ if Q and Q' specialize to the same F_i; one reduces to that situation using divisors of the type $Y + \sum_{j<i} F_j$ as above. □

Now that we know that f extends to $f\colon X' \to J$, we want to know when this extension is a closed immersion. We begin by analyzing what happens in the special fibres. We denote by G the graph of X_k as in Section 2, with S_0 as set of vertices and S_1 as set of edges. We use the analogous notation $G^\sim$, $S_0^\sim$ and $S_1^\sim$ for $X_k^\sim$.

9.2 Proposition. *Suppose that C_1 and C_2 are two distinct elements of S_0 such that the multidegree $C_1 - C_2$ has image zero in $\Phi = J_k/J_k^0$. Then there is a unique path in G from C_1 to C_2 such that every edge in that path is a disconnecting double point.*

Proof. The graph $G^\sim$ is obtained from G by replacing each edge x by a chain of $e(x)$ edges, hence it suffices to prove this proposition in the case $X = X^\sim$. So we assume that X is regular. We choose an orientation on G, i.e., maps s and t from S_1 to S_0 such that, for every x in S_1, $s(x)$ and $t(x)$ are the vertices of x. Then we have maps s_* and t_* from $\mathbb{Z}^{S_1}$ to $\mathbb{Z}^{S_0}$ and s^* and t^* from $\mathbb{Z}^{S_0}$ to $\mathbb{Z}^{S_1}$ given by $(s_*f)(C) = \sum_{s(x)=C} f(x)$ and $(s^*f)x = f(s(x))$, etc. We define the usual boundary and coboundary maps $d_* := t_* - s_*$ and $d^* := t^* - s^*$. We recall from [10, 1] that we have the exact sequence:

$$(9.3) \qquad 0 \longrightarrow \mathbb{Z} \xrightarrow{\mathrm{diag}} \mathbb{Z}^{S_0} \xrightarrow{d_* d^*} \mathbb{Z}^{S_0}[+] \longrightarrow \Phi \longrightarrow 0,$$

with $\mathbb{Z}^{S_0}[+]$ the kernel of $\mathbb{Z}^{S_0} \to \mathbb{Z}$, $f \mapsto \sum_C f(C)$. In fact, this exact sequence is just a simple reformulation of Raynaud's description of Φ from Section 2. (The hypothesis in [10] that the irreducible components of X_k are smooth is actually never used in that article, and the hypothesis that k be algebraically closed can be replaced with k separably closed.) All this is to indicate that Φ is the cokernel of a kind of Laplace operator. For g in $\mathbb{Z}^{S_0}[+]$, there exists h in $\mathbb{Q}^{S_0}$, unique up to adding a constant function, such that $g = d_*d^*h$. Such functions g and h have the following electrodynamical interpretation. We view G as an electric circuit in which each edge has a resistance of one ohm, and in which at each C in S_0 an electric current of $g(C)$ ampère enters the circuit; then h is a function that gives at each C the potential in volts.

In our situation, we know that for $g = C_1 - C_2$ there exists such an h with integer values. Since the total current leaving the circuit is one, the current in each edge is then

0, 1 or -1. This implies that if we delete an edge through which there goes a non-zero current, the resulting graph is disconnected, which gives the conclusion we wanted.

For those readers (including the referee) who do not accept such physical arguments as mathematics, we give the following rigorous argument. Let V and E be the sets of vertices and edges. Say one ampère enters at x and leaves at y. Let $V_{\max}$ be the set of z in V such that $h(z) = h(x)$ and $z \neq x$. Let $V_{\min}$ be the set of z in V such that $h(z) = h(y)$ and $z \neq y$. Then $V - \{x, y\}$ is the disjoint union of $V_{\max}$, $V_{\min}$ and say V'. Since $d_* d^* h$ is zero at all z in $V - \{x, y\}$, it follows that $V_{\max}$, $V_{\min}$ and V' are disconnected from each other in $G - \{x, y\}$. Since every edge connecting x to V' carries a non-zero current, there is a unique such edge. Likewise for y. Induction on the number of elements of V (remove $V_{\max}$ and $V_{\min}$, and x and y) finishes the proof. □

9.4 Proposition. *The morphism $f_k \colon X'_k \to J_k$ is proper.*

Proof. We apply the valuative criterion of properness. It suffices to consider the local rings of X_k^{nor} at closed points x' whose image x in X_k is a non-disconnecting double point. Let x be such a point. We have to show that f_k does not extend over some neighborhood of x. Suppose first that x is an intersection point of two distinct irreducible components C and C' of X_k. Proposition 9.2 implies that the images of $C \cap X'_k$ and $C' \cap X'_k$ under f_k lie in two distinct connected components of J_k. It follows that f_k does not extend over x. To finish, suppose that x is a double point of an irreducible component C of X_k. Let C' be the partial normalization of C, in which all singular points are normalized except x. As before, we may and do suppose that P specializes to C. Then $f_k(C \cap X'_k)$ lies in $J_k^0 = \mathrm{Pic}^0_{X_k/k}$. Let $g \colon J_k^0 \to \mathrm{Pic}^0_{C'/k}$ be the morphism obtained by restriction of line bundles. We claim that $g \circ f_k$ does not extend over x. Suppose that it does. Then $\mathrm{Pic}^0_{C'/k}$ would be generated by the image of C' in it, hence would be a complete variety, which it isn't (it is an extension of $\mathrm{Pic}^0_{C^{\mathrm{nor}}/k}$ by $\mathbb{G}_{\mathrm{m}}$). □

9.5 Proposition. *The following two conditions are equivalent:*

1. *$f_k \colon X'_k \to J_k$ is injective;*
2. *every irreducible component of X_k isomorphic to $\mathbb{P}^1_k$ contains a non-disconnecting double point.*

If these conditions are satisfied, then f_k is a closed immersion.

Proof. Let us first show that the first condition implies the second. So assume that the second condition is not satisfied, and let C is an irreducible component of X_k, isomorphic to $\mathbb{P}^1_k$ and not containing any non-disconnecting double point. Let C^0 be denote the complement in C of the set of double points of X_k contained in C. Let Q and Q' be two distinct elements of $C^0(k)$. An easy application of Proposition 3.11 shows that $f(Q) = f(Q')$.

Assume now that the second condition is satisfied. We show that f_k is a closed immersion. Since we know that f_k is proper, it suffices to show that f_k is injective, and injective on tangent spaces. Let us first consider the restrictions of f_k to the irreducible components of X'_k, and prove that these are injective and injective on tangent spaces.

Let C be an irreducible component of X_k. Replacing P by an element of $X^0(D)$ that specializes to C changes f by a translation, hence we may and do assume that P specializes to C. Then $f_k(C \cap X'_k)$ is contained in $J_k^0 = \mathrm{Pic}^0_{X_k/k}$. Since the double points of X_k contained in $C \cap X'_k$ are disconnecting they are smooth points of C, hence the morphism $f_k: C \cap X'_k \to J_k^0$ is the unique extension of its restriction to C^0. If C is not isomorphic to $\mathbb{P}^1_k$ then the composition of $f_k: C \cap X'_k \to J_k^0$ with the morphism $J_k^0 \to \mathrm{Pic}^0_{C/k}$ induced by restriction of line bundles on X_k to C is injective and is injective on tangent spaces, hence the same is true for $f_k: C \cap X'_k \to J_k^0$. So suppose that C is isomorphic to $\mathbb{P}^1_k$. Then $f_k(C \cap X'_k)$ is contained in the maximal torus T of J_k^0. Since C does contain a non-disconnecting double point, there exists a connected component D of $X_k - C$ whose closure in X_k meets C in at least two points, say x and y. Let g be the regular function on $C - \{x, y\}$ with divisor $x - y$ and $g(P_k) = 1$. The standard description of T says that the character group of T is the homology group $\mathrm{H}_1(G, \mathbb{Z})$ of the graph G. It follows that there is a character $\chi: T \to \mathbb{G}_{\mathrm{m}}$ such that $f_k: C \cap X'_k \to T$ composed with χ is the function g. Hence also in this case the restriction of f is injective and injective on tangent spaces.

To finish the proof, it suffices to prove that f_k is injective, and injective on the tangent spaces at the double points of X'_k. Suppose that Q and Q' in $X'(k)$ have $f_k(Q) = f_k(Q')$. Proposition 9.2 then implies that Q and Q' lie on irreducible components C and C' of X_k that are connected in G by a path consisting of disconnecting double points. Let $C_1, \ldots, C_n$ be the irreducible components in this path, and let Y denote their union, as a reduced closed subscheme of X_k. After replacing P by a point that specializes to $Y \cap X^0$, f_k sends $Y \cap X'$ into J_k^0. Let $h: J_k^0 \to \mathrm{Pic}^0_{Y/k}$ be the morphism obtained by restriction of line bundles. Since the double points in the path above are disconnecting, the morphism from $\mathrm{Pic}^0_{Y/k}$ to the product of the $\mathrm{Pic}^0_{C_i/k}$ is an isomorphism. For each i, let p_i be the projection from $\mathrm{Pic}^0_{Y/k}$ to $\mathrm{Pic}^0_{C_i/k}$. Proposition 3.11 implies that the restriction of $p_i h f_k$ to $C_j \cap X'$ is constant if $i \neq j$, and an immersion if $i = j$. This proves that $Q = Q'$, and also that f_k is injective on the tangent spaces at the double points. $\square$

9.6 Proposition. *The following three conditions are equivalent:*

1. *the morphism $f: X' \to J$ is proper;*
2. *for every prime number p, every non-disconnecting double point x of X_k is contained in a loop in the graph G such that $v_p(e(x)) \leq v_p(e(y))$ for every y in that loop, where v_p denotes the p-adic valuation;*
3. *for every finite extension K' of K and every Q of $X(K')$ specializing to a non-disconnecting double point, $f(Q)$ specializes to a connected component of $J^{\sim}_{D'}$ that*

is not in $J_{D'}$, with D' denoting the integral closure of D in K', $J_{D'}$ the pullback of J to D', and $J^{\sim}_{D'}$ the Néron model over D' of $J_{D'}$.

In particular, the morphism $f: X' \to J$ is proper if X is regular at all non-disconnecting double points of X_k.

Proof. We apply the valuative criterion of properness. Arguments similar to those in the proof of Proposition 9.1 show that conditions 1 and 3 are equivalent. So it remains to show that conditions 2 and 3 are equivalent.

We begin by giving a useful description of $\Phi^{\sim}/\Phi$. Let M denote the first homology group $\mathrm{H}_1(G, \mathbb{Z})$ of the graph G; note that it is canonically isomorphic to $\mathrm{H}_1(G^{\sim}, \mathbb{Z})$. We have:

$$M = \ker(d_*: \mathbb{Z}^{S_1^{\sim}} \to \mathbb{Z}^{S_0^{\sim}}), \quad \text{and} \quad M^{\vee} = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z}) = \mathrm{coker}(d^*: \mathbb{Z}^{S_0^{\sim}} \to \mathbb{Z}^{S_1^{\sim}}). \tag{9.6.1}$$

Section 1 of [10] shows that:

$$\Phi^{\sim} = \mathbb{Z}^{S_1^{\sim}}/(\ker(d_*) + \mathrm{im}(d^*)) = \mathbb{Z}^{S_0^{\sim}}[+]/\mathrm{im}(d_* d^*). \tag{9.6.2}$$

Raynaud's description of Φ in Section 2 shows that Φ is the image in $\Phi^{\sim}$ of the subgroup $\mathbb{Z}^{S_0}[+]$ of $\mathbb{Z}^{S_0^{\sim}}[+]$. The inverse image under d_* of this subgroup is the relative homology group $\mathrm{H}_1(G^{\sim}, S_0, \mathbb{Z})$. Diagram 1.12 of [10] says that the map from $\mathbb{Z}^{S_1^{\sim}}$ to $\Phi^{\sim}$ factors through the map $\mathbb{Z}^{S_1^{\sim}} \to \mathbb{Z}^{S_1}$ that sends x in $S_1^{\sim}$ to $R(x)$ (recall that $R: X^{\sim} \to X$ is the resolution morphism). The image of $\mathrm{H}_1(G^{\sim}, S_0, \mathbb{Z})$ in $\mathbb{Z}^{S_1}$ is the set of elements a with $a(x)$ a multiple of $e(x)$ for all x in S_1. Let q denote the quotient map from $\mathbb{Z}^{S_1}$ to $\oplus_x \mathbb{Z}/e(x)\mathbb{Z}$. Then we have:

$$\Phi^{\sim}/\Phi = (\oplus_x \mathbb{Z}/e(x)\mathbb{Z})/q d^* \mathbb{Z}^{S_0}. \tag{9.6.3}$$

9.6.4 Lemma. *Let Q be in $X(K)$, specializing to a non-disconnecting double point y of X_k. Let i be the integer such that Q specializes to the ith irreducible component in the chain of projective lines replacing y in $X^{\sim}$, using the orientation of G as in [10, 2.2]. Then the image of $f(Q)$ in $\Phi^{\sim}/\Phi$ is represented by the element $\bar{i}$ in the factor at y of $\oplus_x \mathbb{Z}/e(x)\mathbb{Z}$.*

Proof. This follows directly from [10, 2.3]. □

Lemma 9.6.4 implies that condition 3 above is equivalent to the condition that, for every non-disconnecting double point y of X_k, the map from $\mathbb{Z}/e(y)\mathbb{Z}$ to $\Phi^{\sim}/\Phi$ is injective. This injectivity is equivalent to the injectivity after tensoring with $\mathbb{Z}_p$, for all p, hence also to surjectivity in the other direction after applying $\mathrm{Hom}(\cdot, \mathbb{Q}_p/\mathbb{Z}_p)$, for all p. It follows that the map from $\mathbb{Z}/e(y)\mathbb{Z}$ to $\Phi^{\sim}/\Phi$ is injective if and only if for every p there exists an element g in $\mathrm{H}_1(G, \mathbb{Q}_p/\mathbb{Z}_p)$ such that $g(y)$ is the class of $e(y)^{-1}$ and has $g(x)$ in $e(x)^{-1}\mathbb{Z}_p/\mathbb{Z}_p$ for all x in S_1. Using that for any abelian group A one has $\mathrm{H}_1(G, A) = A \otimes \mathrm{H}_1(G, \mathbb{Z})$, the existence of such a g is seen to be equivalent to the existence of a loop in G, containing y, such that $v_p(e(y)) \leq v_p(e(x))$ for all x in that loop. This finishes the proof of Proposition 9.6. □

9.7 Proposition. *Suppose that X_K has genus at least one. The morphism $f: X' \to J$ is a closed immersion if and only if the following two conditions are satisfied:*

1. *every irreducible component of X_k that is isomorphic to $\mathbb{P}^1_k$ contains a non-disconnecting double point;*

2. *for every prime number p, every non-disconnecting double point x of X_k is contained in a loop in the graph G such that $v_p(e(x)) \leq v_p(e(y))$ for every y in that loop.*

Proof. Apply Propositions 9.5 and 9.6. $\square$

9.8 Corollary. *Let X be a regular proper nodal curve over a discrete valuation ring D, with smooth geometrically irreducible generic fibre X_K of non-zero genus, with a given P in $X(K)$. Let $f_K: X_K \to J_K$ be the usual closed immersion sending P to zero. Let $f: X^0 \to J$ be the induced morphism from the complement in X of the set of singular points of the special fibre X_k of X to the Néron model over D of J_K. Then f is a closed immersion if and only if all double points of $X_{\bar{k}}$ are non-disconnecting, with $\bar{k}$ some algebraic closure of k.*

Acknowledgements. I would like to thank Torsten Ekedahl for pointing out to me the relation with log structures, as given in Section 6.

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