On the André–Oort conjecture.

Conjecture. Let (G, X) be a Shimura datum. Let K be a compact open subgroup of $G(\mathbb{A}_{\mathrm{f}})$ and let S be a set of special points in $\mathrm{Sh}_K(G, X)(\mathbb{C})$. Then every irreducible component of the Zariski closure of S in $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$ is a subvariety of Hodge type.

Shimura varieties.

$\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{\mathrm{m},\mathbb{C}}$, algebraic group over $\mathbb{R}$. Equivalence: representations of $\mathbb{S}$ and $\mathbb{R}$-Hodge structures.

$G :=$ a reductive affine algebraic group over $\mathbb{Q}$.

$X :=$ a $G(\mathbb{R})$-conjugacy class in $\mathrm{Hom}(\mathbb{S}, G_{\mathbb{R}})$, satisfying Deligne's three conditions.

$\mathrm{Sh}_K(G,X)(\mathbb{C}) := G(\mathbb{Q})\backslash(X \times G(\mathbb{A}_{\mathrm{f}})/K)$, for $K \subset G(\mathbb{A}_{\mathrm{f}})$ compact open subgroup.

$\mathrm{Sh}_K(G,X)_{\mathbb{C}} :=$ the corresponding complex algebraic variety.

$\mathrm{Sh}(G,X)_{\mathbb{C}} := \lim_K \mathrm{Sh}_K(G,X)_{\mathbb{C}}$, has $G(\mathbb{A}_{\mathrm{f}})$-action.

SUBVARIETIES OF HODGE TYPE.

A closed subvariety $Z \subset \mathrm{Sh}_K(G, X)_{\mathbb{C}}$ is of Hodge type if it is an irreducible component of the image of some:

$$\begin{array}{ccccc} \mathrm{Sh}(G', X')_{\mathbb{C}} & \xrightarrow{f} & \mathrm{Sh}(G, X)_{\mathbb{C}} & \xrightarrow{g} & \mathrm{Sh}(G, X)_{\mathbb{C}} \\ & & & & \downarrow \\ & & & & \mathrm{Sh}_K(G, X)_{\mathbb{C}} \end{array}$$

with $f\colon G' \to G$ and g in $G(\mathbb{A}_{\mathrm{f}})$.

SPECIAL POINTS.

For h in X, $\mathrm{MT}(h) :=$ the smallest $H \subset G$ such that $h(\mathbb{S}) \subset H_{\mathbb{R}}$.

h in X is special iff $\mathrm{MT}(h)$ is commutative.

x in $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$ is special iff x is of Hodge type.

Conjecture. Let (G, X) be a Shimura datum. Let K be a compact open subgroup of $G(\mathbb{A}_{\mathrm{f}})$ and let S be a set of special points in $\mathrm{Sh}_K(G, X)(\mathbb{C})$. Then every irreducible component of the Zariski closure of S in $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$ is a subvariety of Hodge type.

RESULTS.

MOONEN: moduli space of abelian varieties; true for S for which there exists a prime number p at which all s in S have an ordinary reduction of which they are the canonical lift.

EDIXHOVEN: moduli space of pairs of elliptic curves, i.e., products of two modular curves, assuming GRH.

ANDRÉ: products of two modular curves.

YAFAEV: products of two Shimura curves associated to quaternion algebras over $\mathbb{Q}$ (GRH).

BELHAJ DAHMAN: work in progress on jacobians of $y^n = x(x-1)(x-\lambda)$.

EDIXHOVEN: Hilbert modular surfaces, products of modular curves (GRH).

TWO PRINCIPLES.

1. Level structures don't matter: for $K \subset K'$, $Z \subset \mathrm{Sh}_K(G,X)_{\mathbb{C}}$ is of Hodge type if and only if its image in $\mathrm{Sh}_{K'}(G,X)_{\mathbb{C}}$ is.

2. Irreducible components of intersections of subvarieties of Hodge type are again of Hodge type.

THE CASE OF $(\mathbb{P}^1)^n$.

The subvarieties of Hodge type of $(\mathbb{P}^1)^n$ are products of:

1. a CM-point in $\mathbb{P}^1$;

2. closure of the image of $\mathbb{H}^\pm \to (\mathbb{P}^1)^m$ under $\tau \mapsto (j(g_1\tau), \ldots, j(g_m\tau))$, with the g_i in $\mathrm{GL}_2(\mathbb{Q})$.

Theorem. *Assume GRH. Let $n \geq 0$, let $d \geq 0$ and $X \subset (\mathbb{P}^1_{\mathbb{C}})^n$ a subvariety of dimension d, containing a dense subset of CM points. Then X is of Hodge type.*

The proof is by induction on n and d.

X is an irreducible component of: $\displaystyle\bigcap_{\substack{I \subset \{1,\ldots,n\} \\ |I|=d+1}} \mathrm{p}_I^{-1}\mathrm{p}_I X$

This reduces to the case: $d = n-1$ and all p_I with $|I| = d$ are surjective.

THE CASE $n = 2$.

$X_{\mathbb{Q}} :=$ the image of X in $(\mathbb{P}^1_{\mathbb{Q}})^2$.

$d_1, d_2 :=$ the degrees of the projections $X_{\mathbb{Q}} \to \mathbb{P}^1_{\mathbb{Q}}$.

Take $x = (x_1, x_2)$ in X CM with $\max(|\mathrm{discr}(\mathrm{End}(x_i))|)$ big with respect to d_1 and d_2.

Take (GRH) a prime number p such that:

1. $p \geq 5, p \geq d_i$;
2. p split in $\mathrm{End}(x_i)$;
3. p small w.r.t. $\max(|\mathrm{discr}(\mathrm{End}(x_i))|)$.

Then $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot x \subset X_{\mathbb{Q}} \cap T_p X_{\mathbb{Q}}$, and $|\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot x| > X_{\mathbb{Q}} \cdot T_p X_{\mathbb{Q}}$, hence $X_{\mathbb{Q}} \subset T_p X_{\mathbb{Q}}$.

Then one shows that X is of Hodge type.

The case $n = 3$.

We have $d = 2$, all p_I with $|I| = 2$ surjective. We should get a contradiction.

As before: $X_{\mathbb{Q}}$, d_1, d_2, d_3.

Take x in X CM with $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot x$ big w.r.t. the d_i.

Take (GRH) p_1 prime such that:

1. $p_1 \geq 5, p_1 \geq d_i$;
2. $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot x \subset X_{\mathbb{Q}} \cap T_{p_1} X_{\mathbb{Q}}$;
3. p_1 small w.r.t. $|\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot x|$.

If $X_{\mathbb{Q}} \subset T_{p_1} X_{\mathbb{Q}}$, get a contradiction (use $T_{p_1} X_{\mathbb{Q}}$ irreducible, and density of Hecke orbits). Hence suppose that $X_{\mathbb{Q}} \not\subset T_{p_1} X_{\mathbb{Q}}$.

THE CASE $n = 3$, CONTINUED.

Take X_1 an irreducible component of $X_{\mathbb{Q}} \cap T_{p_1} X_{\mathbb{Q}}$ containing x (X_1 is a curve).

For a suitable p_2 (GRH) one has $X_1 \subset T_{p_2} X_1$, hence X_1 is of Hodge type.

This gives a Zariski dense set of curves of Hodge type in X.

One finishes by classifying such curves. Those which project surjectively under all p_i are of the form:

$$C_{(n_1,n_2,n_3)} := \{(E_1, E_2, E_3) \mid \exists\, E, f_i \colon E \to E_i\}$$

with $\ker(f_i) \cong \mathbb{Z}/n_i\mathbb{Z}$ for a fixed triplet (n_1, n_2, n_3).

$\deg(\mathrm{p}_i \colon C_{(n_1,n_2,n_3)} \to \mathrm{p}_i C_{(n_1,n_2,n_3)}) \geq \phi(n_i)$ gives the desired contradiction.

THE CASE $n > 3$.

More successive intersections, the end is more difficult.

As for $n = 3$, get a dense set of curves of Hodge type. Then consider $X \cap \{x\} \times (\mathbb{P}^1)^{n-1}$ with x CM.

THE CASE OF HILBERT MODULAR SURFACES.

$K :=$ a real quadratic extension of $\mathbb{Q}$.

$G := \mathrm{Res}_{O_K/\mathbb{Z}}(\mathrm{GL}_2(O_K))$.

$G(\mathbb{R}) = \mathrm{GL}_2(\mathbb{R} \otimes O_K) = \mathrm{GL}_2(\mathbb{R})^2$.

$S_{\mathbb{C}} := G(\mathbb{Q})\backslash((\mathbb{H}^{\pm})^2 \times G(\mathbb{A}_{\mathrm{f}})/G(\hat{\mathbb{Z}}))$

$S_{\mathbb{Q}} :=$ the canonical model over $\mathbb{Q}$; it is the moduli space for abelian surfaces with O_K-action.

Theorem. *Assume GRH. Let $C \subset S_{\mathbb{C}}$ be an irreducible curve containing infinitely many CM points. Then C is of Hodge type.*

GENERIC MUMFORD-TATE GROUPS.

The generic Mumford-Tate group of $S_{\mathbb{C}}$ is the group G' given by:

$$\begin{array}{ccc} G' & \hookrightarrow & G_{\mathbb{Q}} \\ \downarrow & \square & \downarrow_{\det} \\ \mathbb{G}_{\mathrm{m}\,\mathbb{Q}} & \hookrightarrow & \mathrm{Res}_{K/\mathbb{Q}}\mathbb{G}_{\mathrm{m}\,K} \end{array}$$

$M :=$ the generic Mumford-Tate group of C.

If $M \neq G'$, then C is of Hodge type, because of dimensions.

Hence suppose that $M = G'$. Then C is not of Hodge type, and we have to get a contradiction.

Algebraic monodromy.

$H :=$ the connected component of the Zariski closure of the image of $\pi_1(C')$ (C' a suitable finite cover of C), under the monodromy map.

Theorem (André). $H = M^{\mathrm{der}}$
$(= G^{\mathrm{der}} = \mathrm{Res}_{K/\mathbb{Q}}\mathrm{SL}_{2,K})$.

FINITE MONODROMY.

Theorem (Nori). The closure of the image of $\pi_1(C')$ in $H(\hat{\mathbb{Z}})$ is open.

Irreducibility.

For p prime, let T_p be the Hecke correspondence on $S_{\mathbb{Q}}$, induced by $\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$ in $G(\mathbb{Q}_p)$.

Corollary. *For p big enough, T_pC is irreducible.*

GALOIS ACTION.

Let $A/\mathbb{C}$ be an abelian surface with O_K-action, and with CM.

Then $R := \mathrm{End}_{O_K}(A)$ is an order in a CM field L of degree 2 over K.

The set of $A'/\mathbb{C}$ with $\mathrm{End}_{O_K}(A') = R$ and with the same CM type as A is a $\mathrm{Pic}(R)$-torsor.

$E :=$ the reflex field of A with its O_K action.

$\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ acts via the reflex type norm:
$E^*\backslash\mathbb{A}^*_{\mathrm{f},E} \longrightarrow L^*\backslash\mathbb{A}^*_{\mathrm{f},L}$

Analytic number theory.

Theorem (Siegel). $|\mathrm{Pic}(R)| = |\mathrm{discr}(R)|^{1/2+o(1)}$ *as* $|\mathrm{discr}(R)| \to \infty$.

Corollary. *There exists* $\varepsilon > 0$ *and* $c > 0$ *such that for* x *on* S *a CM point we have*

$$|\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot x| \geq c|\mathrm{discr}(R)|^{\varepsilon}.$$

Theorem (L.O.M.). *Assume GRH. For* x *with* $|\mathrm{discr}(R)| \gg 0$ *there are many* p *split in* R *with* p *at most polynomial in* $\log |\mathrm{discr}(R)|$.

End of the proof.

For x with $|\mathrm{discr}(R)|$ big enough, there is a p such that:

$$T_p x \ni \sigma x, \quad T_p C \text{ is irreducible}$$

for some σ in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Hence $T_p C_{\mathbb{Q}} = C_{\mathbb{Q}}$, which contradicts the density of the T_p-orbit of x in its connected component of $S(\mathbb{C})$.