HECKE MODULES, AND SUITABLE DEFORMATION PROBLEMS AT l.

"Table of contents":

1. technicalities on Hecke modules;
2. a conjecture on acceptability;
3. deformation problems at l and group schemes;
4. some new things (if time).

HECKE MODULES.

Input data:

- $\overline{\rho}\colon G_{\mathbb{Q}} \to \mathrm{GL}(V)$, V a k-vect. sp. of dim. 2, $\mathbb{F}_l \to k$ finite, $l \neq 2$, with $\overline{\rho}$ abs. irr. over the quadratic subfield of $\mathbb{Q}(\mu_l)$;
- $\tau\colon I_l \to \mathrm{GL}_2(\overline{\mathbb{Q}}_l)$ an l-type such that there exists a modular lift ρ of $\overline{\rho}$ of type τ
- S: a finite set of primes not containing l.

Formula for the associated Hecke module:

$$
\begin{aligned}
L :=& L_{(S,\tau)} := \mathrm{Hom}_U(M, \lim_{V} \mathrm{H}^1(Y_V, O))_m \\
&= \mathrm{H}^1(Y_U, \mathcal{F})_m
\end{aligned}
$$

SOME EXPLANATION.

U, V open subgroups of $\mathrm{GL}_2(\hat{\mathbb{Z}})$. $U = \prod_p U_p$, $U_p \subset \mathrm{GL}_2(\mathbb{Z}_p)$ open.

$Y_V = \mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})/V\cdot\mathbb{C}^*$, the modular curve associated to V.

$M = \otimes_p M_p$, M_p free O-module finite rank, with U_p-action, irreducible.

$\mathcal{F} = \mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A}) \times M^\vee)/U\cdot\mathbb{C}^*$, a locally constant sheaf of free O-modules on Y_V.

$\mathcal{H} := \lim_V \mathrm{H}^1(Y_V, O)$ has $\mathrm{GL}_2(\mathbb{A}_\mathrm{f})$-action, hence $\mathrm{Hom}_U(M, \mathcal{H})$ has Hecke operators at all p with $U_p = \mathrm{GL}_2(\mathbb{Z}_p)$ and $M_p = O$.

$\mathbb{T}_{(S,\tau)}$ the sub-O-algebra of $\mathrm{End}_O(\mathrm{Hom}_U(M, \mathcal{H}))$ generated by the Hecke operators.

m is the maximal ideal of $\mathbb{T}_{(S,\tau)}$ corresponding to $\overline{\rho}$.

$\mathbb{T}$ is the image of $\mathbb{T}_{(S,\tau)}$ in $\mathrm{End}_O(L)$.

Some properties.

L is free of finite rank as O-module, $\mathbb{Q} \otimes L$ is free of rank 2 over $\mathbb{Q} \otimes \mathbb{T}$.

L has a perfect pairing over O. This allows to control the congruence modules for $S \subset S'$.

L has a $G_{\mathbb{Q}}$-action, and is a $R_{(S,\tau)}$-module.

One can choose the U_p and M_p such that:

$$\overline{\mathbb{Q}}_l \otimes_O L \sim \oplus_{f \in \mathcal{N}_{(S,\tau)}} \rho_f$$

(the integral structure is very important).

Ingredients to choose the right U_p, M_p:

- Local Langlands (also at l);
- type theory: describes smooth rep's of p-adic groups in terms of an irreducible part of their restriction to suitable compact open subgroups.

SOME EXAMPLES.

1. $l \neq p \notin S$ and $\overline{\rho}$ unramified at p. Then $U_p = \mathrm{GL}_2(\mathbb{Z}_p)$ and $M_p = O$.
2. $l \neq p \in S$ and $\overline{\rho}$ unramified at p. Then

   $$U_p = \{(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \mathrm{GL}_2(\mathbb{Z}_p) \mid p^2|c,\ \overline{d} \in (\mathbb{Z}/p^2\mathbb{Z})_l^*\}$$

   and $M_p = O$.
3. Slightly more general when $l \neq p$ and p not vexing.
4. $l \neq p \notin S$, p vexing ($\overline{\rho}|_{I_p} = \mathrm{Ind}_{\mathbb{Z}_{p^2}^*}^{\mathbb{Z}_p^*} \chi_p$). Then $U_p = \mathrm{GL}_2(\mathbb{Z}_p)$ and $M_p = \Theta(\chi_p)$.
5. $p \in S$, p vexing. Then

   $$U_p = \{(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \mathrm{GL}_2(\mathbb{Z}_p) \mid p|c,\ \overline{d} \in (\mathbb{Z}/p^2\mathbb{Z})_l^*\}$$

   and $M_p = \Theta(\chi_p)$.
6. l itself. Then $U_l = \mathrm{GL}_2(\mathbb{Z}_l)$, and M_l is what τ Local Langlands and type theory suggest.

A CONJECTURE ON ACCEPTABILITY.

Input data:

- $\overline{\rho}\colon G_l \to \mathrm{GL}_2(k)$ continuous, with $\mathrm{End}_{G_l}(V) = k$.
- $\tau\colon I_l \to \mathrm{GL}_2(\overline{\mathbb{Q}}_l)$ a type.

We assume that there is a modular l-adic lift of $\overline{\rho}$ that is of type τ.

Let R^l_O be the universal deformation ring of $\overline{\rho}$.

Then $R^l_{O,\tau}$ is the smallest quotient of R^l_O from which all l-adic lifts ρ of $\overline{\rho}$ of type τ arise.

Definition. τ is *weakly acceptable* for $\overline{\rho}$ if there exists a surjection from $O[[X]]$ to $R^l_{O,\tau}$.

A QUESTION.

If f is a newform of l-type τ, then what can one say about $\overline{\rho}_{f,l} : G_l \to \mathrm{GL}_2(\overline{\mathbb{F}}_l)$?

AN ANSWER.

$\overline{\rho}$ occurs in $\mathrm{H}^1(Y_U, k \otimes_O \mathcal{F}_\tau)$ with $U = U_l U^{(l)}$, $\mathcal{F}_\tau$ coming from M_l, and $U^{(l)}$ sufficiently small.

But $\overline{\mathcal{F}_\tau} := k \otimes_O \mathcal{F}_\tau$ has a Jordan-Hölder filtration with successive quotients of the form $\mathcal{F}_{n,m} := \mathrm{Sym}^n(\overline{\mathcal{F}}) \otimes \det(\overline{\mathcal{F}})^{\otimes m}$, with $\overline{\mathcal{F}}$ corresponding to the standard representation of $\mathrm{GL}_2(\mathbb{F}_l)$ over $\overline{\mathbb{F}}_l$, and $0 \leq n < l$ and m in $\mathbb{Z}/(l-1)\mathbb{Z}$.

Modular representation theory gives the multiplicities of the $\mathcal{F}_{n,m}$.

The weight part of Serre's conjectures does the rest.

THE CONJECTURE.

Conjecture (BCDT). τ is weakly acceptable for $\overline{\rho}$ if and only if the multiplicities of the $\mathcal{F}_{n,m}$ that lead to $\overline{\rho}|_{I_l}$ add up to one.

Remark: the "only if" part is true, and follows from Diamond's arguments that if the whole method works, then L is free of rank 2 over $\mathbb{T}$.

DEFORMATION PROBLEMS AT l AND GROUP SCHEMES.

Notation and assumptions as before.

Remember: $R^l_O \to R^l_{O,\tau}$ is the smallest quotient through which all lifts of $\overline{\rho}$ of type τ factor.

To prove, in enough cases: $R^l_{O,\tau}$ is a quotient of $O[[X]]$.

Strategy: define a suitable deformation problem with universal ring R^{univ}, such that

1. $R^l_O \to R^l_{O,\tau}$ factors through $R^l_O \to R^{\mathrm{univ}}$.
2. R^{univ} is a quotient of $O[[X]]$.

Shape of the deformation problem.

Let $\mathbb{Q}_l \to F$ be a finite Galois extension (group Γ), such that τ is unramified over F.

A *"descent datum"* for a finite free group scheme $\mathcal{G}$ over O_F a right action of Γ on $\mathcal{G}$, compatible with its action on $\mathrm{Spec}(O_F)$, i.e., for each γ in Γ, one has a commutative diagram:

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow[\sim]{[\gamma]} & \mathcal{G} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(O_F) & \xrightarrow[\mathrm{Spec}(\gamma)]{\sim} & \mathrm{Spec}(O_F) \end{array}$$

Base change via $O_F \to k_F$ gives:

$$\begin{array}{ccc} \mathcal{G}_{k_F} & \xrightarrow[\sim]{[\gamma]} & \mathcal{G}_{k_F} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(k_F) & \xrightarrow[\mathrm{Spec}(\gamma)]{\sim} & \mathrm{Spec}(k_F) \end{array}$$

This gives a morphism $W_l^+ \to \mathrm{End}_{k_F}(\mathcal{G}_{k_F})$ by:

$$\sigma \mapsto [\sigma^{-1}] \cdot \mathrm{Frob}_{\mathrm{abs}}^{d(\sigma)}.$$

For $\mathcal{G}$ l-divisible, this morphism should give the type τ. This gives a set S relations in $\mathbb{Z}[\mathbf{F}, \mathbf{V}][\Gamma]$ that have to be satisfied.

SHAPE OF THE DEFORMATION PROBLEM.

One considers: (R, ρ), with R an artinian O-algebra with residue field k, and ρ a deformation of $\overline{\rho}$, such that there exists $(\mathcal{G}, [\cdot])$ with $\rho \sim \mathcal{G}(\overline{\mathbb{Q}}_l)$ and $\mathcal{G}_{k_F}$ annihilated by S.

In the worst case (conductor 3^5), one also has to fix a model $(\mathcal{G}_0, [\cdot])$ of $\overline{\rho}$, and demand that $(\mathcal{G}, [\cdot])$ is a successive extension of copies of $(\mathcal{G}_0, [\cdot])$.

THE WORST Γ.

$$\{1\} \to I_\Gamma \to \Gamma \to \mathbb{Z}/2\mathbb{Z} \to \{0\},$$

with I_Γ the semidirect product of $\mu_4(\mathbb{F}_9)$ by $\mathbb{Z}/3\mathbb{Z}$.

Some new things.

Theorem (Taylor, Skinner-Wiles , . . .) Let l be an odd prime, and $\rho\colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_l)$ a continuous irreducible representation such that:

1. ρ is unramified at all but finitely many primes;
2. ρ is odd (i.e., $\det \rho(c) = -1$);
3. $\rho|_{G_l}$ is an extension of χ_2 by $\varepsilon^n\chi_1$, with $\chi_1|_{I_l}$ and $\chi_2|_{I_l}$ of finite order and n a non-zero positive integer, such that $\varepsilon^n\chi_1\chi_2^{-1}(I_l)$ is not pro-l.

Then there is a totally real number field E, a regular algebraic cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_E)$ and a place λ of the field of coefficients of π above l such that $\rho_{\pi,\lambda}$ (the λ-adic representation associated to π) is equivalent to $\rho|_{G_E}$.