Sous-variétés des variétés de Shimura

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Notations et conventions

1. On note $\widehat{\mathbb{Z}} := \mathrm{limproj}_n \mathbb{Z}/n\mathbb{Z} = \prod_p \mathbb{Z}_p$ le completé profini de $\mathbb{Z}$. On noté $\mathbb{A}_\mathrm{f} := \widehat{\mathbb{Z}} \otimes \mathbb{Q}$ l'anneau des adèles finis et $\mathbb{A} := \mathbb{R} \times \mathbb{A}_\mathrm{f}$ l'anneau des adèles.

2. Par corps de nombres on entend une extension finie de $\mathbb{Q}$.

3. On note $\overline{\mathbb{Q}}$ la cloture algébrique de $\mathbb{Q}$ dans $\mathbb{C}$.

4. On dit qu'un premier p décompose un corps de nombres F si $F \otimes \mathbb{Q}_p$ est un produit de copies de $\mathbb{Q}_p$.

5. On ne suppose pas les variétés analytiques lisses. Par exemple si X est une variété algébrique sur $\mathbb{C}$ alors $X(\mathbb{C})$ est une variété analytique.

6. Par groupe algébrique, on entend un groupe algébrique affine.

7. Les groupes reductifs sont supposés connexes.

8. Etant donné un groupe algébrique G sur $\mathbb{R}$, on note $G(\mathbb{R})^+$ la composante neutre de $G(\mathbb{R})$.

9. Soit G un groupe réductif sur $\mathbb{Q}$. Alors G^{ad} désigne le groupe adjoint et G^{der} le groupe dérivé correspondant.

10. Soit S un schéma et soient X et T deux S-schémas. On note $X_T := X \times_S T$ le T-schéma obtenu par changement de base de S à T de X. Si A est un anneau commutatif alors on écrira souvent A au lieu de $\mathrm{Spec}(A)$. Par exemple, si G est un groupe algébrique sur $\mathbb{Q}$, $G_\mathbb{R}$ désigne $G \times_{\mathrm{Spec}(\mathbb{Q})} \mathrm{Spec}(\mathbb{R})$, le groupe obténu par extension de scalaires de $\mathbb{Q}$ à $\mathbb{R}$.

11. Soit M un tore sur un corps k. Un tore est dit décomposé par une extension k' de k si $M_{k'}$ est un produit de copies de $\mathbb{G}_{\mathrm{m}k'}$. Si M est un tore sur $\mathbb{Q}$ et p est un premier alors M est dit décomposé par p si $M_{\mathbb{Q}_p}$ est un produit de copies de $\mathbb{G}_{\mathrm{m}\mathbb{Q}_p}$.

12. Une variété algébrique sur un corps k est un schéma réduit separé et de type fini sur k. Nous ne supposons pas une variété algébrique irréductible. Une variété algébrique X sur un corps k est dite géométriquement irréductible si pour toute cloture algébrique $\overline{k}$, le schéma $X_{\overline{k}}$ est irréductible.

13. Soit G un groupe algebrique sur $\mathbb{Q}$ avec une representation fidèle $G \hookrightarrow \mathrm{GL}_n$. On définit un $G_\mathbb{Z}$ comme l'adherence de Zariski de $G(\mathbb{Q})$ dans $\mathrm{GL}_{n,\mathbb{Z}_p}$. Le schéma $G_\mathbb{Z}$ n'est pas, en général lisse mais pour presque tout premier p, le schéma $G_{\mathbb{Z}_p} := G_\mathbb{Z} \times \mathbb{Z}_p$ est lisse.

14. Soit G un groupe algébrique sur $\mathbb{Q}$. Soit g un élément de $G(\mathbb{A}_\mathrm{f})$. Alors, par la p-composante de g, notée g_p, on entend l'image de g par la projection canonique $G(\mathbb{A}_\mathrm{f}) \to G(\mathbb{Q}_p)$.

15. On identifie systématiquement $\mathbb{Q}_p$ avec un sous-anneau de $\mathbb{A}_\mathrm{f}$. En particulier, si G est un $\mathbb{Q}$-groupe, on identifie $G(\mathbb{Q}_p)$ avec un sous-groupe normal de $G(\mathbb{A}_\mathrm{f})$.

16. Le mot "rationnel" veut dire défini sur $\mathbb{Q}$.

17. En général on ne differencie pas, dans les notations, les variétés algébriques sur $\mathbb{C}$ de l'ensemble de leurs points à valeurs dans $\mathbb{C}$. Par exemple, si Z est une variété sur $\mathbb{C}$, on dira souvent que x est un point Z pour dire que x est dans $Z(\mathbb{C})$.

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Chapter 1

Introduction

Les variétés de Shimura sont des généralisations naturelles des courbes modulaires dont l'étude a connu un développement extrêmmement important motivé par les applications aux problèmes de théorie des nombres tels que, par exemple, le dernier théorème de Fermat. Actuellement la théorie des variétés de Shimura constitue une branche importante de la géométrie arithmétique dont l'étude est liée, par exemple, à la théorie des motifs, au programme de Langlands, à la conjecture de Mumford-Tate, etc ...

La théorie arithmétique des variétés de Shimura a ses origines dans les travaux de Shimura qui a démontré l'existence de modèles sur des corps de nombres de certains quotients de domaines hermitiens symétriques par des groupes arithmétiques. Il a défini la notion de modèle canonique, qui est unique si elle existe, et a montré dans certains cas l'existence de telles modèles. A ce sujet on peut consulter ses articles [34] et [35].

La définition "moderne" des variétés de Shimura, et ce sera celle avec laquelle on va travailler, basée sur la théorie de Hodge est due à Deligne et est exposée dans les articles [12] et [13]. Grâce aux travaux de Borovoi, Deligne, Milne et certains autres on connait l'existence de modèles canoniques pour toutes les variétés de Shimura. Voir [23] pour les références précises.

Pour pouvoir expliciter le genre de problèmes que l'on étudie dans cette thèse, rappelons très rapidement la définition des variétés de Shimura au sens de Deligne. On considère un couple (G, X) formé d'un groupe algébrique reductif G et d'une classe de $G(\mathbb{R})$-conjugaison X dans $\mathrm{Hom}(\mathbb{S}, G_{\mathbb{R}})$ vérifiant les 3 conditions que l'on explicitera dans le chapitre suivant. Ces conditions impliquent que les composantes connexes de X sont des domaines hermitiens symétriques et que les familles de structures de Hodge paramétrées par les points de X obtenues naturellement à partir de representations de G sur des $\mathbb{Q}$-espaces vectoriels de dimension finie sont des variations de structure de Hodge polarisable.

Soit K un sous-groupe compact ouvert de $G(\mathbb{A}_{\mathrm{f}})$, l'ensemble

$$\mathrm{Sh}_K(G,X)(\mathbb{C}) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_{\mathrm{f}})/K$$

est l'ensemble de points complexes d'une variété algébrique complexe $\mathrm{Sh}_K(G,X)_{\mathbb{C}}$ lisse si K est suffisamment petit. Tout element $g \in G(\mathbb{A}_{\mathrm{f}})$ induit sur $\mathrm{Sh}_K(G,X)_{\mathbb{C}}$ une correspondance algébrique notée T_g appelée correspondance de Hecke. Pour un choix convenable de représentation de G sur un $\mathbb{Q}$-espace vectoriel V, la variation de structure de Hodge obtenue sur X passe au quotient et en induit une sur $\mathrm{Sh}_K(G,X)_{\mathbb{C}}$ et donc, par restriction, sur toute sous-variété lisse de $\mathrm{Sh}_K(G,X)_{\mathbb{C}}$. Etant donné un ensemble fini de tenseurs dans $\oplus_{r,s\geq 0} V^{\otimes r} \otimes V^{\vee\otimes s}$, on regarde l'ensemble de points de X tels que ces tenseurs sont de type $(0,0)$ (comme on le dira sont des classes de Hodge) pour le structure de Hodge définie par ces points. On s'intéresser à l'image de cet l'ensemble dans $\mathrm{Sh}_K(G,X)(\mathbb{C})$. C'est une réunion denombrable de sous-variétés algébriques. Ses composantes irréductibles sont précisement ce que l'on appelle les sous-variétés de type Hodge. La propriété d'une sous-variété d'être de type Hodge peut être characterisée en termes de données de Shimura uniquement (voir chapitre 2). En effet, une sousvariété de type Hodge de $\mathrm{Sh}_K(G,X)_{\mathbb{C}}$ est une composante irréducible de la transformée par un opérateur de Hecke d'une sous-variété définie par une sous-donnée de Shimura $(G',X') \subset (G,X)$. Tout ceci sera précisé dans le chapitre suivant. Les points spéciaux sont par définition des sous-variétés de type Hodge de dimension zéro. Il est à noter que dans certains cas les variétés de Shimura admettent une interprétation modulaire - elles paramétrisent les systèmes formés de variétés abéliennes et de quelques classes de leur cohomologie rationnelle. Dans de tels cas les points spéciaux correspondent aux variétés abéliennes à multiplications complexes au sens de Shimura-Taniyama.

Une variété de Shimura a toujours beaucoup de points spéciaux. L'ensemble de points spéciaux est dense dans la variété pour la topologie de Zariski et même pour la topologie analytique. De même, étant donnée une sous-varété S de type Hodge, l'ensemble de points spéciaux de S est dense dans S. La question naturelle à se poser est la réciproque - est-ce que les composantes irréductibles de l'adhérence de Zariski d'un ensemble de points spéciaux est de type Hodge. Pour les composantes irréductibles de dimension 0, la réponse est positive car, par définition, ce sont les points spéciaux. Cette question a d'abord été posée par André pour les composantes irréductibles de dimension 1 (voir [1]) et ensuite par Oort pour les sous-variétés de l'espace de modules des variétés abéliennes principalement polarisées (voir [27]) et porte maintenant le nom de la conjecture d'André-Oort. Cette conjecture fait l'objet de cette thèse.

Conjecture 1.0.1 (André-Oort) *Soit (G,X) une donnée de Shimura et soit K un sous-groupe compact ouvert. Soit $S := \mathrm{Sh}_K(G,X)_{\mathbb{C}}$ la variété de Shimura complexe associée et $Z \subset S$ une sous-variété algébrique fermée irréductible contenant un sous-ensemble Zariski dense de points* *spéciaux. Alors Z est de type Hodge.*

Signalons quelques résultats et conjectures analogues à cette conjecture ainsi que quelques cas déjà connus de la conjecture d'André-Oort. L'une des motivations d'André et de Oort était une conjecture plus ancienne, celle de Manin et de Mumford, maintenant démontrée par Raynaud (voir [31]) dont on rappelle l'énoncé.

Theorem 1.1 (Raynaud) *Soit A une variété abélienne complexe et $Z \subset A$ une sous-variété algébrique complexe irréductible contenant un sous-ensemble Zariski dense de points de torsion. Alors Z est le translaté d'une sous-variété abélienne par un point de torsion.*

L'analogie entre la conjecture d'André-Oort et celle de Mumford-Manin (théorème de Raynaud) peut être résumée dans le tableau suivant.

André-Oort	Manin-Mumford
Variété de Shimura	Variété abélienne
Point spécial	Point de torsion
Sous-variété de type Hodge	Sous-variété abélienne translatée par un point de torsion

Le premier résultat sur la conjecture d'André-Oort est celui de Moonen (voir [24] et [25]) qui considère le cas de l'espace de modules des variétés abéliennes principalement polarisées. Il a montré qu'une sous-variété Z est de type Hodge si et seulement si Z vérifie la condition suivante. La variété Z contient un ensemble Zariski dense Σ de points spéciaux tel qu'il existe un premier p tel que les variétés abéliennes correspondantes à tous les s dans Σ ont une réduction ordinaire dont elles sont le relevement canonique. Il est relativement facile de donner des exemples d'ensembles infinis de points speciaux qui ne vérifient pas cette condition.

Les autres résultats obtenus traitent des cas plus particuliers. En 1996, Edixhoven et André ont (indépendamment, voir [19] et [2]) démontré la conjecture dans le cas des courbes dans $\mathbb{A}^1_{\mathbb{C}} \times \mathbb{A}^1_{\mathbb{C}}$ (vu comme espace de modules pour les couples de courbes elliptiques). La preuve d'Edixhoven suppose l'hypothèse de Riemann généralisée tandis que celle d'André est inconditionnelle mais semble être difficilement généralisable au cas des autres variétés de Shimura car elle utilise de façon essentielle un résultat de Masser d'approximation diophantienne sur la fonction modulaire j. En revanche, la méthode d'Edixhoven, bien que conditionnelle, donne une strategie qui semble être généralisable au cas vraiment le plus général. Je signale quand même qu'en essayant de généraliser la méthode d'Edixhoven on rencontre très rapidement des difficultés techniques importantes mais qui ne semblent pas être insurmontables. Récemment Edixhoven a généralisé sa méthode au cas des sous-variétés de dimension quelconque dans un produit $\mathbb{A}^1_{\mathbb{C}} \times \cdots \times \mathbb{A}^1_{\mathbb{C}}$ et aux courbes sur les surfaces modulaires de Hilbert (les surfaces modulaires de Hilbert sont les espaces de modules des surfaces abéliennes à multiplications par un corps quadratique réel). Ces résultats également supposent l'Hypothèse de Riemann Généralisée. En utilisant la méthode d'Edixhoven, j'ai prouvé la conjecture sous l'Hypothèse de Riemann Généralisée dans le cas du produit $S_1 \times S_2$ où S_1 et S_2 sont les courbes de Shimura associées aux algèbres de quaternions indéfinies sur $\mathbb{Q}$ (voir [40] ou chapitre 3 de cette thèse). En s'appuyant sur un résultat de Bas Edixhoven sur la minoration des orbites de Galois, on a démontré qu'une courbe conténant un ensemble infini de points spéciaux qui sont dans une orbite de Hecke est de type Hodge. C'est le résultat principal de cette thèse, sa démonstration occupe les chapitres $4-7$. Certains resultats en rapport avec la conjecture ont été obtenus par Breuer et Beljah Dahman. On peut aussi indiquer un problème qui est semblable à la conjecture d'André-Oort. Il s'agit de la conjecture de Coleman dont voici l'énoncé.

Conjecture 1.1.1 (Coleman) *Soit $g \geq 4$. Il n'existe qu'un nombre fini de courbes projectives lisses C sur $\mathbb{C}$ de genre g telles que la jacobienne $\mathrm{Jac}(C)$ soit de type CM.*

Pour $g = 4$ et $g = 6$, il y a des contre-exemples. En effet, de Jong et Noot ont construit (voir [11]) des familles de courbes $C \to S$ (S étant un $\mathbb{C}$-schéma) de genre 4 et 6 telles que l'image de la famille des variétés abéliennes correspondante $\mathrm{Jac}(C/S) \to S$ dans $\mathcal{A}_4$ (resp. $\mathcal{A}_6$) est une sous-variété de type Hodge et donc contient une infinité de points spéciaux. Pour $g = 5$ et $g \geq 7$, la conjecture de Coleman est complètement ouverte. Le rapport de cette conjecure avec la conjecture d'André-Oort est le suivant. Supposons que la conjecture d'André-Oort soit varie et supposons qu'il existe une infinite de courbes de genre donné $g > 4$ et $g \neq 6$ dont la Jacobienne est de type CM. Soit $\mathcal{M}_g$ l'espace de module de courbes de genre g. Alors, l'adherence de Zariski de l'image dans $\mathcal{A}_{g,\mathbb{C}}$ de l'ensemble de points de $\mathcal{M}_g$ correspondant à cette famille est une sous-variété de type Hodge. La question est alors de savoir s'il y a des ouverts de sousvariétés de type Hodge dans l'image de $\mathcal{M}_g$ dans $\mathcal{A}_g$. Il s'agit d'une question très interessante et completement ouverte.

Une autre question interessante est de savoir si une sous-variété d'une variété de Shimura qui contient un point special et un ensemble Zariski dense de points (non nécessairement spéciaux) qui sont dans une orbite de Hecke est de type Hodge. Cette question est énoncée comme problème dans [1, Problème 3]. Sans hypothèse d'existence d'un point spécial on peut s'attendre à avoir une sous-variété totalement géodésique. Dans l'analogie entre André-Oort et Manin-Mumford signalée avant, une variété totalement géodésique correspond au translaté d'une sous-variété abélienne par un point qui n'est pas nécessairement de torsion. Le cas d'une courbe contenue dans un éspace de modules de variétés abéliennes semble être faisable avec les méthodes de cette thèse et fait l'objet de travaux en cours.

Maintenant, expliquons les résultats obtenus dans cette thèse. Le resultat principal, comme on l'a déjà signalé, dit qu'une courbe algébrique fermée contenue dans une variété de Shimura et contenant un sous-ensemble infini de points spéciaux qui sont dans la même orbite de Hecke est de type Hodge.

L'une des motivations de notre recherche était le travail de Cohen, Wolfart et Wüstholz sur la transcendance de certaines fonctions hypergéométriques (voir [39] et [9]). En effet, le resulat principal de cette thèse implique leur conjecture sur la transcendence de valeurs de fonctions hypergéométriques. En effet ce résultat est la Conjecture 4 dans le preprint [9]. Le premier à avoir fait le lien entre la conjecture d'André-Oort et la transcendence des valeurs de fonctions hypergéométriques était Wolfart. En effet, cette conjecture apparait déjà implicitement dans la séction 9 de l'article de Wolfart [39] où l'auteur pretend le démontrer dans certains cas particuliers. Il y a quand même une serieuse lacune dans cette section. Les arguments de l'auteur ne suffisent pas pour conclure. Dans cette thèse on comble cette lacune (dans une beaucoup plus grande généralité).

Détaillons un peu de quel genre de question en trnscendence il s'agit. Considérons d'abord la fonction modulaire classique j (voir [33] pour les détails sur cette fonction). Rappelons qu'il s'agit d'une fonction holomorphe

$$j : \mathbb{H} \to \mathbb{C}$$

(où $\mathbb{H}$ désigne le demi-plan de Poincaré) qui est invariante sous l'action de $\mathrm{SL}_2(\mathbb{Z})$ par des homographies, c'est à dire qu'on a

$$j(\frac{az+b}{cz+d}) = j(z),\ z \in \mathbb{H},\ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}).$$

et dont le développement de Fourier est de la forme

$$j(z) = e^{-2\pi i z} + 744 + \sum_{n=1}^{\infty} a_n e^{2\pi i n z},\ a_n \in \mathbb{C},\ z \in \mathbb{H}$$

On a un résultat suivant concernant la transcendance des valeurs de la fonction j (voir [36]).

Theorem 1.2 (Schneider) *On a $z \in \mathbb{H} \cap \overline{\mathbb{Q}}$ et $j(z) \in \overline{\mathbb{Q}}$ si et seulement si z est un point de multiplication complexe (i.e si le corps $\mathbb{Q}(z)$ est un corps quadratique imaginaire).*

Un résultat analogue pour les variétés modulaires de Siegel (c'est à dire celles qui paramétrisent les variétés abéliennes principalement polarisées) a été plus récemment démontré par Cohen, Shiga et Wolfart (voir [8] et [37]).

Dans leur preprint [9], Cohen et Wüstholz considerent les groupes définis de manière suivante: soient p, q, r trois entiers tels que $2 \leq p, q, r \leq \infty$ tels que $1/p + 1/q + 1/r \leq 1$, alors on considère le groupe $\Delta := \Delta(p, q, r)$ défini, à conjugaison près dans $\mathrm{PSL}_2(\mathbb{R})$, par générateurs et relations par:

$$\{M_1, M_2, M_3 : M_1^p = M_2^q = M_3^r = M_1 M_2 M_3 = 1\}$$

Ces groupes sont des groupes de monodromie de certaines équations différentielles hypergéométriques. En effet, posons

$$\mu_1 = \frac{1}{2}(1 - \frac{1}{p} + \frac{1}{q} + \frac{1}{r}),\ \mu_2 = \frac{1}{2}(1 + \frac{1}{p} - \frac{1}{q} + \frac{1}{r})$$

$$\mu_3 = \frac{1}{2}(1 + \frac{1}{p} + \frac{1}{q} - \frac{1}{r}),\ \mu_4 = \frac{1}{2}(1 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r})$$

$$a = 1 - \mu_4,\ b = \mu_3,\ c = \mu_1 + \mu_3$$

$$\mu := (\mu_1, \mu_2, \mu_3, \mu_4)$$

Alors le groupe Δ est conjugué au groupe de monodromie Γ_μ de la fonction multivaluée donnée par une intégrale hypergéométrique:

$$\int_g^h u^{-\mu_1}(u-1)^{-\mu_2}(u-x)^{-\mu_3} du,\ x \notin [g, h]$$

où $g, h \in \{0, 1, \infty, x\}$ et $g \neq h.$ Ces intégrales engendrent un espace vectoriel de dimension $2.$ La fonction hypergéométrique de Gauss est donnée par l'intégrale normalisé

$$F(x) = \frac{\int_0^\infty u^{-\mu_1}(u-1)^{-\mu_2}(u-x)^{-\mu_3} du}{\int_1^\infty u^{-\mu_1-\mu_3}(u-1)^{-\mu_2} du}$$

On montre alors que

$$F(x) = F(a, b, c; x) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)} \cdot \frac{x^n}{n!}$$

avec, pour $w \in \mathbb{C}$, $(w, n) := \prod_{j=1}^{n}(w + n - j)$. Une telle fonction s'appelle fonction hypergéométrique. Dans cette thèse on démontre qu'une courbe dans une variété de Shimura contenant un ensemble infini de points spéciaux qui sont dans une orbite de Hecke est de type Hodge. Ce résultat implique, d'après Cohen et Wüstholz ([9]) le résultat suivant sur la transcendance des valeurs de $F(x)$ pour $x \in \overline{\mathbb{Q}}$.

Theorem 1.2.1 (Cohen, Edixhoven, Wolfart, Wüstholz, Yafaev) *Soit Δ un groupe du type défini ci-dessus. Soit*

$$E = \{x \in \overline{\mathbb{Q}} \backslash ([1, \infty[\cap \overline{\mathbb{Q}}) : F(x) \in \overline{\mathbb{Q}}\}$$

Alors l'ensemble E est fini si et seulement si Δ n'est pas un groupe arithmétique.

Disons quelques mots sur notre stratégie. Comme on l'a déjà dit, cette stratégie est due à Edixhoven et utilisé par lui pour traiter certains cas de la conjecture ([18], [19] et [20]). La stratégie se décompose en deux parties, dans la première partie on utilise l'action de Galois sur les points spéciaux d'une variété de Shimura pour prouver que la sousvariété que l'on considère est contenue dans son image par certaines correspondances de Hecke bien choisies. La deuxième partie montre qu'une variété conténue dans son image par certaines correspondances de Hecke est de type Hodge.

Signalons enfin qu'il y a d'autres approches du problème. Récemment, Clozel et Ullmo ont démontré certains résultats d'équidistribution des images des points de certaines variétés de Shimura (précisément, celles associées au groupe symplectique GSp_{2g}) par des opérateurs de Hecke. Bien sur, pour les applications à la conjecture d'André-Oort, il faudrait avoir des résultats analogues dans le cas des orbites de Galois des points spéciaux ce qui est un problème difficile. Le seul résultat d'équidistribution des orbites de Galois connu est dans le cas de $\mathbb{A}^1_{\mathbb{C}}$ (espace de modules de courbes elliptiques), voir [17].

Donnons enfin le plan de cette thèse:

Chapitre 2 Contient les rappels sur les structures de Hodge, les variations de structures de Hodge sur les variétés analytiques complexes lisses et les variétés de Shimura.

Chapitre 3 Reproduit l'article accepté à “Manuscripta mathematica”. Dans cet article on démontre, sous l'hypothèse de Riemann généralisée la conjecture d'André-Oort pour les sous-variétés du produit $S_1 \times S_2$ de deux courbes de Shimura associées aux algèbres de quaternions indéfinies sur $\mathbb{Q}$.

Chapitre 4 Ici on énonce les résultats principaux de la thèse et on fait certaines étapes préliminaires. On expose aussi la strategie des preuves des resultats principaux.

Chapitre 5 Contient quelques résultats qui permettent de caractériser les sous-variétés de type Hodge des variétés de Shimura par la propriété qu'elles sont contenues dans leurs images par certains opérateurs de Hecke.

Chapitre 6 Ecrit par Bas Edixhoven. On donne une borne inferieure pour les orbites de Galois des points dans une orbite de Hecke.

Chapitre 7 Dans ce chapitre on démontre le résultat principal de la thèse: une courbe dans une variété de Shimura contenant un sous-ensemble infini de points spéciaux dans une orbite de Hecke est de type Hodge. On démontre également un resultat auxiliaire énoncé dans le chapitre 4 pour les souvariétés de dimension quelconque mais avec une hypothèse supplementaire sur les points.

Chapter 2

Théorie de Hodge et les variétés de Shimura

Dans ce chapitre on rapelle les définitions et les résultats principaux sur les structures de Hodge et leur groupes de Mumford-Tate. On donne également une définition d'une variation de structure de Hodge sur une variété analytique complexe lisse et enfin on définit les variétés de Shimura. Le contenu de ce chapitre n'est pas nouveau, pour les details, on peut consulter (par exemple) [12], [13], [23], [24] ainsi que [22]. Signalons aussi qu'on ne présente que des aspects de la thérie dont on se sert effectivement dans la suite. Pour un exposé beaucoup plus complet, on renvoie aux références citées ci-dessus.

2.1 Structures de Hodge

Soit $\mathbb{S}$ le groupe algébrique sur $\mathbb{R}$ défini par $\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{\mathrm{m}\mathbb{C}}$.

Definition 2.1.1 Soit V un $\mathbb{R}$-espace vectoriel de dimension finie. Une structure de Hodge sur V peut être définie de façons équivalentes:

- Une morphisme $h : \mathbb{S} \to \mathrm{GL}(V)$.
- Une décomposition $V \otimes \mathbb{C} = \oplus_{p,q \in \mathbb{Z}} V^{p,q}$, $\overline{V^{p,q}} = V^{q,p}$. La conjugaison complexe sur $V \otimes \mathbb{C}$ étant définie par $\overline{v \otimes z} := v \otimes \overline{z}$ pour tout $v \in V$ et $z \in \mathbb{C}$.
- Une graduation $V = \oplus_m V_m$ et une filtration descendante de $V \otimes \mathbb{C}$

$$\cdots \supset F^p \supset F^{p+1} \supset \cdots$$

telle que $V_m = (V_m \cap F^p) \oplus (V_m \cap \overline{F^q})$ pour tous les m, p, q avec $p + q = m + 1$.

Pour passer d'une définition à une autre on utilise les regles suivantes:

$$v \in V^{p,q} \Longleftrightarrow \forall z \in \mathbb{C}^\times, h(z)v = z^{-p}\overline{z}^{-q}v$$

$$V_m \otimes \mathbb{C} = \bigoplus_{p+q=m} V^{p,q},\ F^p = \bigoplus_{p' \geq p} V^{p',q}$$

$$V^{p,q} = V_{p+q} \cap F^p \cap \overline{F^q}$$

Soit T un sous-ensemble de $\mathbb{Z} \times \mathbb{Z}$. Une structure de Hodge sur un espace vectoriel V est dite de type T si $V^{p,q} = 0$ pour tous les $(p,q) \notin T$. Une structure de Hodge est dite (pure) de poids r si pour tout (p,q) tels que $p+q \neq r$, $V^{p,q} = 0$. Toute structure de Hodge est somme directe de structures pures, en effet V_m est pure de poids m.

Si A est un sousanneau de $\mathbb{R}$, alors une A-structure de Hodge est un module V_A de type fini muni d'une structure de Hodge au sens de la définition ci-dessus sur $V_A \otimes_A \mathbb{R}$. Les seuls cas que l'on va rencontrer sont avec $A = \mathbb{Z}, \mathbb{Q}$ ou $\mathbb{R}$.

Definition 2.1.2 Soient V_1 et V_2 deux $\mathbb{R}$-espaces vectoriels et $h_1 : \mathbb{S} \to \mathrm{GL}(V_1)$, $h_2 : \mathbb{S} \to \mathrm{GL}(V_2)$ des structures de Hodge sur V_1 et V_2 respectivement. Un morphisme de structures de Hodge de (V_1, h_1) vers (V_2, h_2) est une application $\mathbb{R}$-lineaire $V_1 \to V_2$ telle que f est équivariante vis-à-vis de l'action de $\mathbb{S}$.

On généralise cette définition de manière évidente aux morphismes de A-structures de Hodge. La structure de Hodge de Tate $\mathbb{Z}(m)$ est obtenue en prenant pour $\mathbb{Z}$-module sousjacent $(2\pi i)^m\mathbb{Z}$ avec $\mathbb{S}$ agissant via le caractère $(z\overline{z})^m$. Bien sur, $\mathbb{Q}(m)$ ou $\mathbb{R}(m)$ sont définis comme $\mathbb{Z}(m) \otimes \mathbb{Q}$ et $\mathbb{Z}(m) \otimes \mathbb{R}$. Une polarisation d'une structure de Hodge réelle (V, h) est une famille (ψ_m) de morphismes de structures de Hodge

$$\psi_m : V_m \otimes V_m \to \mathbb{R}(-m),\ m \in \mathbb{Z}$$

tels que la forme bilinéaire $(x,y) \mapsto (2\pi i)^m \psi_m(x, h(i)y)$ est symétrique définie positive. Une polarisation d'une $\mathbb{Z}$-structure de Hodge est une famille de morphismes $\psi_m : V_m \otimes V_m \to \mathbb{Z}(-m),\ m \in \mathbb{Z}$ tels que la famille $(\psi_m \otimes \mathbb{R})_m$ est une polarization de structures de Hodge réelles. Une structure de Hodge (rationnelle ou réelle) est dite polarisable si elle admet une polarisation définie comme ci-dessus. On appelle structure de Hodge polarisée un triplet $(V = \oplus M_m, (\psi_m), F)$ où (V, F) est une structure de Hodge et (ψ_m) est une polarisation.

2.2 Groupe de Mumford-Tate d'une structure de Hodge

Dans cette section on définit les notions fondamentales de groupe de Mumford-Tate et de classe de Hodge. On caracterise le groupe de Mumford-Tate d'une structure de Hodge en termes de classes de Hodge. La définition du groupe de Mumford-Tate est comme suit.

Definition 2.2.1 (Groupe de Mumford-Tate) Soit (V, h) une structure de Hodge rationelle polarisable. Le groupe de Mumford-Tate $\mathrm{MT}(V)$ de (V, h) est défini comme étant le plus petit sous-groupe M de $\mathrm{GL}(V)$ tel que h se factorise par $M_{\mathbb{R}}$.

Remark 2.2.2 On peut bien sur définir le groupe de Mumford-Tate pour une structure de Hodge qui n'est pas forcement polarisable. Néanmoins, les structures de Hodge que l'on va considèrer dans cette thèse seront toujours polarisables. L'hypothèse de polarisabilité implique que le groupe de Mumford-Tate est un groupe algébrique réductif.

Ensuite on introduit la notion de classe de Hodge.

Definition 2.2.3 (Classe de Hodge) Soit (V, h) une structure de Hodge rationnelle polarisable. On appelle un élément v de V une classe de Hodge si v est du type $(0, 0)$ pour un certain entier p dans la décomposition de Hodge.

On peut caracteriser le groupe de Mumford-Tate comme le stabilisateur des droites engendrées par les classes de Hodge dans les constructions tensorielles obtenues à partir de (V, h). Plus précisement, on a la proposition suivante.

Proposition 2.2.4 *Soit (V, h) une structure de Hodge rationnelle polarisable. Alors $\mathrm{MT}(V)$ est le stabilisateur de toutes les droites dans les structures de Hodge $\oplus_i V^{n_i} \otimes V^{\vee \otimes m_i}$ engenrées par les classes de Hodge.*

Une démonstration de cette proposition peut être trouvée dans [18], Appendix K1. Elle repose sur le théorème de Chevalley énoncé ci-dessous, dont la preuve se trouve également dans l'appendice K1.

Theorem 2.2.5 (Chevalley) *Soit G un groupe algébrique affine sur un corps $k \supset \mathbb{Q}$. Soit H un sous-groupe algébrique de G et V representation de G de dimension finie. Alors il existe une droite L dans une certaine representation de G de la forme $\oplus_i V^{\otimes n_i} \otimes V^{\vee \otimes m_i}$ telle que H est le stabilisateur de L.*

2.3 Variation de structure de Hodge

Definition 2.3.1 Soit S une variété analytique complexe lisse et connexe. Une variation de $\mathbb{Z}$-structure de Hodge polarisée sur S est un triplet triplet $\mathcal{V} := (\mathcal{V}_{\mathbb{Z}}, F, Q)$ où $\mathcal{V}_{\mathbb{Z}}$ est un faisceau localement constant de $\mathbb{Z}$-modules libres de rang fini, F est une filtration de

$\mathcal{V}_{\mathcal{O}} := \mathcal{O}_S \otimes_{\mathbb{Z}} \mathcal{V}_{\mathbb{Z}}$ par des sous-fibrés holomorphes et Q est la collection de formes de polarization $Q_m : \mathcal{V}_{\mathbb{Z},m} \times_S \mathcal{V}_{\mathbb{Z},m} \to \mathbb{Z}_S(-m)$ telles que

- F satisfait $\nabla F^p \subset \Omega^1_S \otimes_{\mathcal{O}_S} F^{p-1}$ par rapport à la connexion de Gauss-Manin ∇.
- $\mathcal{V}$ induit une $\mathbb{Z}$-structure de Hodge polarisée $(\mathcal{V}_{\mathbb{Z},s}, F_s, Q_s)$ sur chaque fibre.

Bien sur, on définit de manière analogue les variations de $\mathbb{Q}$ ou $\mathbb{R}$-structures (et plus généralement des A-structures) de Hodge polarisées.

Example 2.3.2 Les variations de structures de Hodge polarisables apparaissent naturellement en géométrie algébrique de manière suivante. Soit $\pi : Y \to S$ un morphisme projectif et lisse de varietés algébriques complexes avec $S(\mathbb{C})$ connexe. Alors, pour tout entier $r \geq 0$, $R^r\pi_*\mathbb{Q}$ est une variation de $\mathbb{Q}$-structures de Hodge polarisables sur S de poids r. Ceci est prouvé dans [14].

Soit S une variété analytique complexe lisse et connexe munie d'une variation de $\mathbb{Q}$-structure de Hodge polarisable $(\mathcal{V}, F, Q)$. Soit $\pi : \widetilde{S} \to S$ le revêtement universel de S, alors le fibré $\pi^*\mathcal{V}$ est trivial. Choisissons une trivialisation $\pi^*\mathcal{V} \cong \widetilde{S} \times V$. Pour tout $s \in S$, soit $\mathrm{MT}_s \subset \mathrm{GL}(\mathcal{V}_s)$ le groupe de Mumford-Tate de la fibre $\mathcal{V}_s$ de $\mathcal{V}$ en s. Le choix d'un un point $\widetilde{s}$ de $\widetilde{S}$ dans la fibre $\pi^{-1}(s)$ donne un isomorphisme $\mathcal{V}_s \cong V$, ce qui permet de voir le groupe MT_s comme un sous-groupe de $\mathrm{GL}(V)$. On montre (voir par exemple [10]) qu'il existe un ensemble analytique Σ de S, union dénombrable de sous-variétés analytiques propres de S tel que

- Pour tout $s \notin \Sigma$, MT_s, vu comme sous-groupe de $\mathrm{GL}(V)$ par le procedé expliqué ci-dessus, est constant. Pour $s \notin \Sigma$, le groupe MT_s est appelé le groupe de Mumford-Tate générique de $\mathcal{V}$, noté $\mathrm{MT}^{\mathrm{gen}}(\mathcal{V})$.
- Pour tous les s et $\widetilde{s}$ comme ci-dessus avec $s \in \Sigma$, le groupe MT_s est un sous-groupe propre de $\mathrm{MT}^{\mathrm{gen}}(\mathcal{V})$.

L'ensemble Σ s'appelle le lieu de Hodge exceptionnel et son complement $S - \Sigma$ dans S le lieu de Hodge générique. Dans [10], il est montré que si S est l'espace analytique associé à une variété algébrique complexe alors les composantes irréductibles de Σ sont des sous-variétés algébriques de S.

Maintenant on considère une variation de $\mathbb{Z}$-structure de Hodge $\mathcal{V}_{\mathbb{Z}}$ sur une variété analytique S qui est l'espace analytique associé à une variété algébrique complexe. Soit comme avant le revêtement universel $\pi : \widetilde{S} \to S$ et choisissons une trivialisation $\pi^*\mathcal{V}_{\mathbb{Z}} \cong \widetilde{S} \times V_{\mathbb{Z}}$. Soit $s \in S$ et soit $\widetilde{s}$ un point de $\widetilde{S}$ au dessus de s, ceci donne un isomorphisme $\mathcal{V}_{\mathbb{Z},s} \cong V_{\mathbb{Z}}$. Le faisceau localement constant $\mathcal{V}_{\mathbb{Z}}$ correspond alors à une representation $\rho : \pi_1(S, s) \to \mathrm{GL}(V_{\mathbb{Z}})$ du groupe fondamental $\pi_1(S,s)$ de S. L'adherence de Zariski de l'image de cette representation est appelé le groupe de monodromie algébrique. La composante connexe neutre de ce groupe est appelé le groupe de monodromie algébrique connexe. On a le théorème suivant du à André et démontré dans [2].

Theorem 2.3.3 (André) *Avec les notations ci-dessus, on a les resultats suivants:*

- *Pour tout $s \in S\backslash\Sigma$ le groupe de monodromie algébrique connexe est un sous-groupe normal du groupe de Mumford-Tate générique $\mathrm{MT}_s^{\mathrm{der}}$.*
- *Supposons qu'il y ait un point $s_0 \in S$ tel que MT_{s_0} soit abélien, alors le groupe de monodromie algébrique connexe est égal à $\mathrm{MT}_s^{\mathrm{der}}$.*

2.4 Données de Shimura

Definition 2.4.1 Une donnée de Shimura est un couple (G,X) où G est un $\mathbb{Q}$-groupe réductif connexe et $X \subset \mathrm{Hom}(\mathbb{S}, G_{\mathbb{R}})$ est une classe de $G(\mathbb{R})$-conjugaison tel que pour tout $h \in X$ on ait

1. La structure de Hodge sur $\mathrm{Lie}(G)$ induite par h est de type $\{(-1,1),(0,0),(1,-1)\}$.
2. L'involution $\mathrm{Int}(h(i))$ est une involution de Cartan sur $G_{\mathbb{R}}^{\mathrm{ad}}$.
3. Le groupe adjoint G^{ad} n'a pas de facteurs simples sur $\mathbb{Q}$ sur lesquels h se projette trivialement.

Definition 2.4.2 Soit (G_1,X_1) et (G_2,X_2) deux données de Shimura. Un morphisme de données de Shimura de (G_1,X_1) vers (G_2,X_2) est un morphisme $f : G_1 \to G_2$ de groupes algébriques qui envoie X_1 sur X_2.

2.5 Variétés de Shimura en tant que variétés analytiques complexes

Soit (G,X) une donnée de Shimura. Soit K un sousgroupe compact ouvert de $G(\mathbb{A}_{\mathrm{f}})$. Considérons d'abord l'ensemble

$$\mathrm{Sh}_K(G,X)(\mathbb{C}) := G(\mathbb{Q})\backslash X \times G(\mathbb{A}_{\mathrm{f}})/K$$

où un élément (q,k) de $G(\mathbb{Q}) \times K$ agit sur $X \times G(\mathbb{A}_{\mathrm{f}})$ en envoyant un élément (h,g) sur $(q,k)\cdot(h,g) := (qh, qgk)$. On va noter $[h,g]$ l'image de $(h,g) \in X \times G(\mathbb{A}_{\mathrm{f}})$ dans $\mathrm{Sh}_K(G,X)(\mathbb{C})$.

Soit X^+ une composante connexe de X. Alors X^+ est une classe de $G^{\mathrm{ad}}(\mathbb{R})^+$-conjugaison d'un morphisme $\mathbb{S} \to G^{\mathrm{ad}}_{\mathbb{R}}$ (voir [24, 1.2]) et X est l'orbite de X^+ pour l'action de $G(\mathbb{R})$ sur X. Le stabilizateur de X^+ pour l'action de $G(\mathbb{R})$ sur X est $G(\mathbb{R})_+$, image inverse de $G^{\mathrm{ad}}(\mathbb{R})^+$ dans $G(\mathbb{R})$ (voir [13, prop. 1.2.7]). La varieté X^+ est un domaine hérmitien symétrique et X est une union de copies de X^+ indexée par $G(\mathbb{R})/G(\mathbb{R})_+$. Soit $G(\mathbb{Q})_+ := G(\mathbb{R})_+ \cap G(\mathbb{Q})$. On a alors (voir [13, 2.1.1]):

$$G(\mathbb{Q})\backslash X \times G(\mathbb{A}_{\mathrm{f}})/K = G(\mathbb{Q})_+\backslash X^+ \times G(\mathbb{A}_{\mathrm{f}})/K = \bigcup_g \Gamma_g\backslash X^+$$

où g parcourt un ensemble de representants dans $G(\mathbb{A}_{\mathrm{f}})$ pour l'ensemble fini de doubles classes $G(\mathbb{Q})\backslash G(\mathbb{A}_{\mathrm{f}})/K$ et $\Gamma_g := G(\mathbb{Q})_+ \cap gKg^{-1}$. Chaque composante $\Gamma_g\backslash X^+$ admet une structure de variété algébrique complexe quasi-projective d'après un résultat de Baily et de Borel ([4]).

2.6 Variations de structures de Hodge sur les variétés de Shimura et sous-variétés de type Hodge

On conserve les notations de la section précedente. Dans ce paragraphe on explique comment certains choix de representations de G sur un $\mathbb{Q}$-espace vectoriel induisent une variation de structure de Hodge sur $\mathrm{Sh}_K(G,X)_{\mathbb{C}}$.

Soit ρ une representation de G sur un $\mathbb{Q}$-espace vectoriel V qui se factorise par une representation fidèle de G^{ad}. Cette representation définit naturellement une famille de structures de Hodge rationnelles paramétrée par les points de X. En effet, soit x un element de X, c'est un morphisme $\mathbb{S} \to G_{\mathbb{R}}$. Le composé $\rho_{\mathbb{R}} \circ x$ est un morphisme $\mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$ donc c'est une structure de Hodge sur $V_{\mathbb{R}}$. Les conditions de 2.4.1 impliquent que cette famille est une variation de structure de Hodge polarisable avec le fibré vectoriel sous-jacent $X \times V$ (voir [13, 1.13-1.17] pour une démonstration de ce fait). On introduit la notation suivante.

Definition 2.6.1 Soit x un point de X. On appelle le groupe de Mumford-Tate de x le plus petit sous-groupe H de G tel que x se factorise par $H_{\mathbb{R}}$. On le note $\mathrm{MT}(x)$.

Remark 2.6.2 Le lien entre ce groupe et le groupe de Mumford-Tate d'une structure de Hodge défini précedemment est le suivant. Soit ρ une representation de G sur un $\mathbb{Q}$-espace vectoriel V. Alors, $\rho(\mathrm{MT}(x))$ est le groupe de Mumford-Tate de la structure de Hodge sur V définie par $\rho_{\mathbb{R}} \circ x$.

Considérons une composante $\Gamma_g\backslash X^+$ de $\mathrm{Sh}_K(G,X)_{\mathbb{C}}$ avec X^+ et Γ_g comme avant. Sur X^+ on a une variation structure de Hodge polarisable avec le fibré vectoriel sousjacent $X^+ \times V$. Ce fibré est muni d'une action de Γ_g: un élément γ de Γ_g opère sur $X \times G(\mathbb{A}_{\mathrm{f}})$ de manière suivante $\gamma \cdot (x, v) = (\gamma \cdot x, \rho(\gamma) \cdot v)$. La condition pour avoir une variation structure de Hodge sur $\Gamma_g \backslash X^+$ est que le stabilizateur $\Gamma_{g,x}$ de tout point $x \in X^+$ pour l'action de Γ_g agit trivialement sur V. Ceci est réalisé si on prend le groupe K assez petit (voir [24, 2.3]). En fait, il suffit que K soit net au sens de [28]. On note $\mathcal{V}(\rho)$ cette variation de structure de Hodge. Un choix d'un reseau Γ_g invariant $V_{\mathbb{Z}}$ dans V (qui existe par [30, prop. 4.2]) donne une variation de $\mathbb{Z}$-structure de Hodge sur $\Gamma_g \backslash X^+$ que l'un note $\mathcal{V}(\rho)_{\mathbb{Z}}$. On recapitule ce qui vient d'être dit.

Proposition 2.6.3 *Soit (G, X) une donnée de Shimura et K un sous-groupe compact ouvert de $G(\mathbb{A}_{\mathrm{f}})$. Soit ρ une representation de G se factorisant par une representation fidèle de G^{ad}. Soit $S^0 := \Gamma_g \backslash X^+$ une composante connexe de $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$ avec $\Gamma_g := G(\mathbb{Q})_+ \cap gKg^{-1}$ pour un certain $g \in G(\mathbb{A}_{\mathrm{f}})$. Alors, si le groupe K est suffisamment petit, la variation de structure de Hodge induite par ρ sur X^+ avec le fibré vectoriel sousjacent $X^+ \times V$ induit une variation de structure de Hodge sur S^0 avec le fibré vectoriel sousjacent le quotient de $X^+ \times V$ par l'action de Γ_g donnée par $\gamma \cdot (x, v) = (\gamma \cdot x, \rho(\gamma) \cdot x)$. Un choix d'un reseau Γ_g-invariant $V_{\mathbb{Z}}$ dans V donne une variation de $\mathbb{Z}$-structure de Hodge.*

On considere une variation de structure de Hodge comme ci-dessus. Soit maintenant Σ le lieu de Hodge exceptionnel. Alors Σ est l'union de sous-variétés définies comme suit. Considerons un tenseur t dans $\oplus_{m,n \in W} V^{\otimes m} \otimes V^{\vee \otimes n}$ où W est un certain sous-ensemble dans $\mathbb{N} \times \mathbb{N}$. En utilisant la representation ρ, on peut considérer le lieu de points x de $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$ tel que t est une classe de Hodge pour la structure de Hodge sur la fibre de $\mathcal{V}(\rho)$ en x. On a déjà signalé que Σ est une reunion dénombrable de sous-variétés analytiques. Les composantes irréductibles de Σ sont, comme on l'a déjà signalé, des sous-variétés algébriques de $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$. On les appelle les sous-variétés de type Hodge. Avec cette définition il parait clair que la notion de sous-variété de type Hodge ne depend pas du choix du groupe compact ouvert K. Plus précisement, soit $K' \subset K$ un sousgroupe compact ouvert alors une sous-variété de $\mathrm{Sh}_{K'}(G, X)_{\mathbb{C}}$ est de type Hodge si et seulement si son image dans $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$ est de type Hodge. Aussi, l'intérprétation des sous-variétés de type Hodge comme les lieux de points où certaines classes sont des classes de Hodge montre que les composantes irréductibles des intersections des sous-variétś de type Hodge sont des sous-variétés de type Hodge. Pour un exposé detaillé, on renvoie à [24, Sect. 2-3]. On peut caractériser les sous-variétés de type Hodge en termes purement de la théorie de groupes.

Definition 2.6.4 Soit (G, X) une donnée de Shimura et K un sousgroupe compact ouvert de $G(\mathbb{A}_{\mathrm{f}})$. Soit Z une sous-variété irréductible fermée de $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$. Alors Z est une sous-variété de type Hodge si et seulement si il existe une donnée de Shimura (G', X'), un morphisme de données de Shimura $f : (G', X') \to (G, X)$ et un élément g de $G(\mathbb{A}_\mathrm{f})$ tel que Z est une composante irreductible de l'image du morphisme:

$$\mathrm{Sh}(G', X')_\mathbb{C} \stackrel{\mathrm{Sh}(f)}{\to} \mathrm{Sh}(G, X)_\mathbb{C} \stackrel{\cdot g}{\to} \mathrm{Sh}(G, X)_\mathbb{C} \to \mathrm{Sh}_K(G, X)_\mathbb{C}.$$

Pour voir que les deux caractérisations sont effectivement équivalentes, voir [24, sect. 2-3].

Définissons enfin la notion de correspondance de Hecke définies par des éléments de $G(\mathbb{A}_\mathrm{f})$. Soient K_1 et K_2 deux sous-groupes compacts ouverts de $G(\mathbb{A}_\mathrm{f})$ et soit $K' := K_1 \cap gK_2g^{-1}$. Soit g un élément de $G(\mathbb{A}_\mathrm{f})$. La correspondance de Hecke T_g de $\mathrm{Sh}_{K_1}(G, X)_\mathbb{C}$ dans $\mathrm{Sh}_{K_2}(G, X)_\mathbb{C}$ est définie par le diagramme

$$\mathrm{Sh}_{K_1}(G, X)_\mathbb{C} \stackrel{\pi_1}{\leftarrow} \mathrm{Sh}_{K'}(G, X)_\mathbb{C} \stackrel{\pi_2}{\to} \mathrm{Sh}_{K_2}(G, X)_\mathbb{C}$$

où l'application π_1 est donnée par l'inclusion naturelle $K' \subset K_1$ et π_2 envoie un point $[x, \theta]$ sur $[x, \theta \cdot g]$. Pour une sous-variété Z de $\mathrm{Sh}_{K_1}(G, X)_\mathbb{C}$, on note $T_g(Z) := \pi_2(\pi_1^{-1}(Z))$ et on l'appelle le transformé de Z par l'opérateur T_g.

2.7 Modèles canoniques

On a déjà signalé que les variétés de Shimura étaient des variétés algébriques complexes quasiprojectives. En fait, on sait qu'elles admettent des modèles sur des corps de nombres qui verifient certaines conditions que l'on va expliciter. Pour un exposé détaillé (avec preuves) du contenu de cette section, on renvoie à [13].

Considerons d'abord une donnée de Shimura définie par un tore. On se donne donc un couple (T, h) où T est un tore sur $\mathbb{Q}$ et h est un morphisme $\mathbb{S} \to T_\mathbb{R}$. Soit M le groupe de Mumford-Tate de h. Rappelons que M est le plus petit sous-groupe algébrique H de T tel que h se factorise par $H_\mathbb{R}$. Alors M est un sous-tore de T. Le morphisme $\mu := h_\mathbb{C}(z, 1)$ se factorise par $M_\mathbb{C}$ et, en fait, $M_\mathbb{C}$ est le plus petit sous-groupe algébrique H de T tel que $H_\mathbb{C}$ contient l'image de μ (voir [15, I, prop 3.4]). Soit $E(T, h)$ le plus petit corps de définition de ce morphisme μ. On l'appelle le corps reflexe de (T, h). Alors μ induit le morphisme $r : \mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_{\mathrm{m}E} \to M$ défini sur $\mathbb{Q}$ défini par

$$\mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_{\mathrm{m}E} \stackrel{\mathrm{Res}_{E/\mathbb{Q}}\mu}{\to} \mathrm{Res}_{E/\mathbb{Q}}M_E \stackrel{N_{E/\mathbb{Q}}}{\to} M$$

Proposition 2.7.1 *Le morphisme r défini ci-dessus est surjectif (en tant qu'un morphisme de tores algébriques).*

Proof. Supposons que l'image de r soit contenue dans un sous-tore M' de M. Fixons un plongement de E dans $\mathbb{C}$. L'extension de scalaires de r de $\mathbb{Q}$ à $\mathbb{C}$ via ce plongement redonne

$\mu := h_{\mathbb{C}}(z, 1)$. Alors, d'après la propriété de M d'être le plus petit sousgroupe algébrique H de T tel que $H_{\mathbb{C}}$ contient l'image de μ, on voit que $M' = M$. $\square$

Le lemme suivant est très important pour nos démonstrations futures.

Lemma 2.7.2 *When p ranges over the set of all primes, the index $[M(\mathbb{Q}_p) : r((E \otimes \mathbb{Q}_p)^\times)]$ is finite, bounded independently of p.*

Proof. Soit $T_E := \mathrm{Res}_{E/\mathbb{Q}} \mathbb{G}_{\mathrm{m}E}$. Le morphisme $r : T_E \to M$ est un morphisme surjectif de tores algébriques. Soit $r^* : X(M) \hookrightarrow X(T_E)$ l'injection au niveau de groupes de charactères induite par ce morphisme. Soit X le groupe de torsion $X(T_E)/r^*(X(M))$, alors, comme $X(T_E)/X$ est sans torsion, c'est le groupe de charactères d'un certain $\mathbb{Q}$-tore, appelons-le T. L'inclusion $X(T_E) \subset X$ donne une isogenie $\nu : T \to T_E$. Le noyau de ν est de la forme $\oplus_{k \in I} \mu_k$ où I est un ensemble fini d'entiers et μ_k des le groupe de racines $k$ièmes de l'unité dans E. Alors on a une suite exacte

$$0 \to \oplus_{k \in I} \mu_k \to T \to T_E \to M \to 0.$$

Comme p décompose E, il décompose T, T_E et M, donc cette suite induit une suite exacte

$$0 \to (\oplus_{k \in I} \mu_k)(\mathbb{Q}_p) \to T(\mathbb{Q}_p) \to (E \otimes \mathbb{Q}_p)^\times \to M(\mathbb{Q}_p) \to \oplus_{k \in I} H^1(\mu_k, \mathbb{Q}_p) \to 0.$$

Donc le conoyau $[M(\mathbb{Q}_p) : r((E \otimes \mathbb{Q}_p)^\times)]$ est l'ordre de $\oplus_{k \in I} H^1(\mu_k, \mathbb{Q}_p)$. La théorie de Kummer dit que l'ordre de $H^1(\mu_k, \mathbb{Q}_p)$ est l'ordre de $\mathbb{Q}_p^\times / \mathbb{Q}_p^{\times, k}$. L'ordre de ce dernier groupe est $k \cdot |\mu_k(\mathbb{Q}_p)| \cdot p^{v_p(k)}$. Bien sur $|\mu_k(\mathbb{Q}_p)| \leq k$ et pour p assez grand $p^{v_p(k)} = 1$. Ceci montre que $[M(\mathbb{Q}_p) : r((E \otimes \mathbb{Q}_p)^\times)]$ est borné par entier independant de p. $\square$

La variété de Shimura définie par la donnée (T, h) est de dimension zéro. Pour tout sousgroupe compact ouvert K de $T(\mathbb{A}_{\mathrm{f}})$, l'ensemble

$$\mathrm{Sh}_K(T, h)(\mathbb{C}) := T(\mathbb{Q}) \backslash \{h\} \times T(\mathbb{A}_{\mathrm{f}})/K$$

est fini. Pour définir une structure de $E(T, h)$-schema sur cet ensemble, il suffit donc spécifier l'action du groupe de Galois $\mathrm{Gal}(\overline{\mathbb{Q}}/E(T, h))$ là-dessus. La théorie de corps de classes global donne un morphisme continu et surjectif de $(E(T, h) \otimes \mathbb{A})^\times$ vers $\mathrm{Gal}(\overline{\mathbb{Q}}/E(T, h))$. Soit σ dans $\mathrm{Gal}(\overline{\mathbb{Q}}/E(T, h))$ et $[h, g]$ un élément de $T(\mathbb{Q}) \backslash \{h\} \times T(\mathbb{A}_{\mathrm{f}})/K$. Soit s un idèle dans $(E(T, h) \otimes \mathbb{A})^\times$ dont l'image dans $\mathrm{Gal}(\overline{\mathbb{Q}}/E(T, h))$ par le morphisme de corps de classes est σ. Ecrivons $s = s_\infty \cdot s_f$ où s_∞ (resp. s_f) sont des projections sur la partie infinie (resp. finie) de $(E(T, h) \otimes \mathbb{A})^\times$. Alors, on définit l'action de σ par

$$\sigma([h, g]) = [h, r(s_f) \cdot g]$$

Pour deux sousgroupes compacts ouverts $K' \subset K$ de $T(\mathbb{A}_{\mathrm{f}})$, le morphisme $\mathrm{Sh}_{K'}(T,h)_{\mathbb{C}} \to \mathrm{Sh}_K(T,h)$ descend à $E(T,h)$, on obtient donc un modèle de la variété pro-algébrique indexée par les sous-groupes compacts ouverts de $T(\mathbb{A}_{\mathrm{f}})$,

$$\mathrm{Sh}(T,h)_{\mathbb{C}} := \mathrm{limproj}_K \mathrm{Sh}_K(T,h)_{\mathbb{C}}$$

.

Considerons maintenant une donnée de Shimura quelconque (G,X). Rappelons que X est une classe de conjugaison d'un morphisme $h : \mathbb{S} \to G_{\mathbb{R}}$. On renvoie à [13] pour la définition du corps reflexe $E(G,X)$.

Un modèle de la limite projective indexée par les groupes compacts ouverts de $G(\mathbb{A}_{\mathrm{f}})$,

$$\mathrm{Sh}(G,X)_K := \mathrm{limproj}_K \mathrm{Sh}_K(G,X)_{\mathbb{C}}$$

sur un corps de nombres $E(G,X) \subset E \subset \mathbb{C}$ est dit canonique si pour tous les couples spéciaux $(T,\{h\})$ où T est un sous-tore de G et h est un morphisme $\mathbb{S} \to T_{\mathbb{R}}$ qui appartient à X et un élément g de $G(\mathbb{A}_{\mathrm{f}})$, le point $[h,g]$ est dans $\mathrm{Sh}(G,X)(E(T,h)^{\mathrm{ab}})$ et $\mathrm{Gal}(E(T,h)^{\mathrm{ab}}/E(T,h))$ conjugue $[h,g]$ suivant la regle donnée ci-dessus. Grace aux travaux de Borovoi, Deligne, Milne, Shih et autres, on sait maintenant que toutes les variétés de Shimura admettent des modèles canoniques sur leur corps reflexe (voir [23, 3]).

Chapter 3

Special points on products of two Shimura curves

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3.1 Introduction and results.

In [4], Edixhoven proves, assuming GRH for imaginary quadratic fields, that an irreducible closed algebraic curve in $\mathbb{C}^2$ containing infinitely many pairs of CM (complex multiplication) j-invariants and which is not a fiber of one of the projections is the Hecke correspondence $\{(j(\tau), j(n\tau)) | \tau \in \mathbb{H}\} \subset \mathbb{C}^2$ for some integer $n \geq 1$. The same result has been obtained inconditionally by André in [1]. His proof uses a result of Masser on Diophantine approximation and the j-function and hence it doesn't seem to generalize to the case of all Shimura curves that we propose ourselves to treat in this article. In this article we extend the result and the method of Edixhoven ([4]) to products of two Shimura curves attached to rational indefinite quaternion algebras. Let us consider two Shimura curves S_1 and S_2 associated to indefinite rational quaternion algebras $B_{1\mathbb{Q}}$ and $B_{2\mathbb{Q}}$ with maximal orders B_1 and B_2. We recall the necessary facts about these curves and their modular interpretation in section 2. A point (x_1, x_2) in $(S_1 \times S_2)(\mathbb{C})$ is called special if both x_1 and x_2 correspond to abelian varieties of CM type. Using the definition given in [7] of sub-varieties of Hodge type, one finds the following list of sub-varieties of Hodge type of $(S_1 \times S_2)_{\mathbb{C}}$ up to isomorphism.

1. If $B_{1\mathbb{Q}}$ and $B_{2\mathbb{Q}}$ are not isomorphic then the only sub-varieties of Hodge type are $(S_1 \times S_2)_{\mathbb{C}}$, $S_{1,\mathbb{C}} \times \{x\}$ (where x is special), $\{y\} \times S_{2,\mathbb{C}}$ (where y is special) and (x, y) a special point in $(S_1 \times S_2)(\mathbb{C})$.

2. If $B_{1\mathbb{Q}}$ and $B_{2\mathbb{Q}}$ are isomorphic, then there exists an isomorphism from $B_{1\mathbb{Q}}$ to $B_{2\mathbb{Q}}$ which sends B_1 to B_2 and hence induces an isomorphism from S_1 to S_2. Therefore, it suffices to classify the sub-varieties of Hodge type of $(S \times S)_{\mathbb{C}}$ where S is the Shimura curve associated to a quaternion algebra $B_{\mathbb{Q}}$ over $\mathbb{Q}$ with maximal order B. The sub-varieties of Hodge type of $(S \times S)_{\mathbb{C}}$ are $(S \times S)_{\mathbb{C}}$, $S_{\mathbb{C}} \times \{x\}$ (where x is special), $\{y\} \times S_{\mathbb{C}}$ (where y is special), a Hecke correspondence associated to an element $g \in B_{\mathbb{A}_f}{}^{\times} := (B \otimes \mathbb{A}_f)^{\times}$ embedded in $(S \times S)_{\mathbb{C}}$ via the map which on complex-valued points is induced by the map $(h, x) \mapsto ((h, x), (h, gx))$ from $\mathbb{H}^{\pm} \times B_{\mathbb{A}_f}{}^{\times}$ to $(\mathbb{H}^{\pm} \times B_{\mathbb{A}_f}{}^{\times}) \times (\mathbb{H}^{\pm} \times B_{\mathbb{A}_f}{}^{\times})$ and the special points.

Our main result is the following.

Theorem 3.1.1 *Assume the generalized Riemann hypothesis for imaginary quadratic fields. Let S_1 and S_2 be as above. Let $C \subset (S_1 \times S_2)_{\mathbb{C}}$ be a closed irreducible algebraic curve such that $C(\mathbb{C})$ contains infinitely many special points. Then C is of Hodge type.*

Note that this result include the one of Edixhoven ([4]) since for $B_{\mathbb{Q}} = M_2(\mathbb{Q})$ we get a modular curve. The idea of the proof is to consider the intersection of C with some well-chosen Hecke correspondences. Using the Galois action on special points, one proves that for some well chosen Hecke correspondences this intersection is not proper. Some topological arguments allow then to conclude that C is of Hodge type. The details of many proofs are omitted since they are completely similar to [4].

3.2 Shimura curves.

3.2.1 Modular interpretation.

For definitions and properties of quaternion algebras we refer to [13] and [9]. Let $B_{\mathbb{Q}}$ be a rational indefinite quaternion algebra with maximal order B and let d be its discriminant. Let us choose an isomorphism $f : B \otimes \mathbb{R} \to \mathrm{M}_2(\mathbb{R})$ and for any p prime to d, isomorphisms $f_p : B \otimes \mathbb{Z}_p \to \mathrm{M}_2(\mathbb{Z}_p)$. Let $x \mapsto x^*$ be the involution of $B_{\mathbb{Q}}$ characterized by the property that $\mathrm{tr}(x) = x + x^*$. The order B is then invariant by this involution. Consider the moduli problem of classifying the isomorphism classes of triples $(A/M/\mathbb{Q}, i : B \hookrightarrow \mathrm{End}(A/M), \lambda)$ where M is a $\mathbb{Q}$-scheme, A is an abelian scheme over M of relative dimension 2 with an action of B given by i and λ is a principal $*$-polarization in the sense of [3, 1.5]. It is well known that there is a coarse moduli scheme S over $\mathbb{Q}$ for this moduli problem. To prove this one can, for example, use the methods of [2]. Proposition 1.5 of [3, chap. III] shows that given $(A/M/\mathbb{Q}, i)$, a principal

$*$-polarization exists and is unique hence we don't bother to mention it. A pair $(A/M/\mathbb{Q}, i)$ will be called a QM-abelian surface. Let $\mathbb{H}^{\pm}$ be $\mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R})$ which is the union of the upper half-plane and its complex conjugate. We identify $\mathbb{H}^{\pm}$ with a quotient of $(B \otimes \mathbb{R})^{\times} = \mathrm{GL}_2(\mathbb{R})$ hence it has a natural left action of $B_{\mathbb{Q}}^{\times}$. The set of complex points of S is $S(\mathbb{C}) = B_{\mathbb{Q}}^{\times} \backslash (\mathbb{H}^{\pm} \times B_{\mathbb{A}_f}^{\times} / \widehat{B}^{\times})$. As an analytic variety, $S(\mathbb{C})$ is compact if and only if $B_{\mathbb{Q}}$ is a division algebra. The scheme S is in fact the canonical model of $B_{\mathbb{Q}}^{\times} \backslash (\mathbb{H}^{\pm} \times B_{\mathbb{A}_f}^{\times} / \widehat{B}^{\times})$ in the sense of Deligne, see [8].

3.2.2 Galois action on special points.

One fixes an embedding of $\overline{\mathbb{Q}}$ into $\mathbb{C}$, then special points are in $S(\overline{\mathbb{Q}})$. Let x be a special point in $S(\overline{\mathbb{Q}})$ and let $(A_x/\overline{\mathbb{Q}}, i_x)$ be the corresponding QM-abelian surface. We denote $\mathrm{End}_B(A_x)$ the ring of endomorphisms commuting with the action of B given by i_x. One easily sees that this ring is an order in an imaginary quadratic field K. There exists a unique integer $c \geq 1$ called the conductor such that $O_{K,c} := \mathbb{Z} + cO_K$ where O_K is the ring of integers of K. The Picard group $\mathrm{Pic}(O_{K,c})$ acts on the set S_c of isomorphism classes of QM-abelian surfaces (A, i) such that $\mathrm{End}_B(A) = O_{K,c}$ in the way we are going to describe. Let (A, i) be an element of S_c and I an invertible $O_{K,c}$-module. We wish to define a QM-abelian surface $(I \otimes A, I \otimes i)$ whose isomorphism class depends only on the class $[I]$ of I in $\mathrm{Pic}(O_{K,c})$. For any scheme S over $\mathbb{C}$, $A(S)$ is a $O_{K,c}$-module and one associates to S the $O_{K,c}$-module $I \otimes_{O_{K,c}} A(S)$. This defines a functor from $\mathbb{C}$-schemes to $O_{K,c}$-modules sending S to $I \otimes_{O_{K,c}} A(S)$ which is easily checked to be representable by a QM-abelian surface $(I \otimes A, I \otimes i)$. One sees that, if we write $A(\mathbb{C}) = \mathbb{C}^2/\Lambda$, then $(I \otimes A)(\mathbb{C})$ is the quotient of $I \otimes \mathbb{C}^2$ by $I \otimes \Lambda$. Moreover, the isomorphism class of this pair depends only on the class $[I]$ of I in $\mathrm{Pic}(O_{K,c})$. This defines an action of $\mathrm{Pic}(O_{K,c})$ on S_c commuting with the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$. One can, moreover, show that S_c is a $\mathrm{Pic}(O_{K,c})$-torsor hence the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ factorizes through a morphism $\mathrm{Gal}(\overline{\mathbb{Q}}/K) \to \mathrm{Pic}(O_{K,c})$. This morphism is surjective. The orbit $\mathrm{Gal}(\overline{\mathbb{Q}}/K) \cdot x$ has exactly $\mathrm{Pic}(O_{K,c})$ elements. The Frobenius element at a maximal ideal m not containing c is sent to the element $[m]^{-1}$ of $\mathrm{Pic}(O_{K,c})$. The Galois extension $K(x)/K$ corresponds to this quotient of $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$.

Remark 3.2.3 If one wishes to generalize the methods of this article to the case of the moduli space of abelian varieties of some dimension $d > 1$, one has to deal with the problem of finding lower bounds for the size of the Galois orbit of an abelian variety of CM type in terms of the discriminant of the center of its endomorphism ring. This seems to be a hard open problem.

3.2.4 Hecke operators.

We note $\widehat{B}^\times := (B \otimes \widehat{\mathbb{Z}})^\times$. Let $g \in B_{\mathbb{A}_f}^\times$, the map $(h,x) \mapsto ((h,x),(h,gx))$ from $\mathbb{H}^\pm \times B_{\mathbb{A}_f}^\times$ to $(\mathbb{H}^\pm \times B_{\mathbb{A}_f}^\times) \times (\mathbb{H}^\pm \times B_{\mathbb{A}_f}^\times)$ induces a map

$$B_{\mathbb{Q}}^\times \backslash (\mathbb{H}^\pm \times (B_{\mathbb{A}_f}^\times / \widehat{B}^\times \cap g\widehat{B}^\times g^{-1})) \to B_{\mathbb{Q}}^\times \backslash (\mathbb{H}^\pm \times (B_{\mathbb{A}_f}^\times / \widehat{B}^\times)) \times B_{\mathbb{Q}}^\times \backslash (\mathbb{H}^\pm \times (B_{\mathbb{A}_f}^\times / \widehat{B}^\times))$$

The image of this map is the Hecke correspondence T_g. By the theory of canonical models of Shimura varieties, it is defined over $\mathbb{Q}$. For p prime to d, T_p is defined to be T_g for $g := \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$ at the place p and 1 elsewhere. We now give the modular interpretation of the Hecke correspondences T_p. Let $(A/\mathbb{C}, i)$ be a QM-abelian surface. We define $T_p[(A,i)] = \sum_G [(A/G, i)]$ where G ranges through the $B \otimes \mathbb{F}_p$-submodules of $A[p](\mathbb{C})$ isomorphic to $\mathbb{F}_p^2$. Since $A[p](\mathbb{C})$ is a $B \otimes \mathbb{F}_p$-module isomorphic to $B \otimes \mathbb{F}_p$ and $B \otimes \mathbb{F}_p$ is isomorphic to $\mathrm{M}_2(\mathbb{F}_p)$ because n is prime to d, there exists an isomorphism of $B \otimes \mathbb{F}_p$-modules between $A[p](\mathbb{C})$ and $\mathrm{M}_2(\mathbb{F}_p)$. Through this isomorphism, the subgroups G correspond to $\mathrm{M}_2(\mathbb{F}_p)$-submodules of $\mathrm{M}_2(\mathbb{F}_p)$ isomorphic to $\mathbb{F}_p^2$ as $\mathbb{F}_p$-modules. Under the Morita equivalence, such submodules correspond to $\mathbb{F}_p$-submodules of $\mathbb{F}_p^2$ isomorphic to $\mathbb{F}_p$ hence, in particular, there are $p+1$ such submodules. Let us describe the action of Hecke operators on $S(\mathbb{C})$. So, let x be a point of $S(\mathbb{C})$ and (A,i) the corresponding abelian surface. The lattice $\mathrm{H}_1(A(\mathbb{C}), \mathbb{Z})$ defines a class g in $B_{\mathbb{A}_f}^\times / \widehat{B}^\times$. The complete set of lattices defining the subgroups G as above is $\left(\begin{smallmatrix} p & i \\ 0 & 1 \end{smallmatrix}\right) g$ where $0 \leq i \leq p-1$ and $\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right) g$. For n square free, prime to the discriminant and composed out of primes $p_1, \cdots, p_r$, one defines $T_n := T_{p_1} \cdots T_{p_r}$.

3.3 The curve is contained in its images by infinitely many Hecke correspondences.

Let C be as in the main theorem. Since C contains infinitely many points in $(S_1 \times S_2)(\overline{\mathbb{Q}})$, the set $C(\mathbb{C}) \cap (S_1 \times S_2)(\overline{\mathbb{Q}})$ is Zariski-dense in C hence there exists a unique closed curve $C_{\overline{\mathbb{Q}}}$ in $(S_1 \times S_2)_{\overline{\mathbb{Q}}}$ that gives C after base change from $\overline{\mathbb{Q}}$ to $\mathbb{C}$. Let $C_{\mathbb{Q}}$ be the image of C under the morphism $(S_1 \times S_2)_{\mathbb{C}} \to (S_1 \times S_2)_{\mathbb{Q}}$, then $C_{\mathbb{Q}} \times \overline{\mathbb{Q}}$ is the finite set of conjugates of $C_{\overline{\mathbb{Q}}}$ under $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. For a special point $x \in S_i(\overline{\mathbb{Q}})$, we denote by $\mathrm{End}_B(x)$ the ring $\mathrm{End}_B(A_x)$ where (A_x, i_x) is the corresponding QM-abelian surface. The imaginary quadratic field $\mathrm{End}_B(A_x) \otimes \mathbb{Q}$ is called the CM field of x.

Proposition 3.3.1 *Let $C_{\mathbb{Q}}$ be as above. For all but finitely many special points (x_1, x_2) in $C_{\mathbb{Q}}(\overline{\mathbb{Q}})$, the CM fields of x_1 and x_2 are isomorphic.*

Proof. One just has to follow the proof of [4, prop. 3.1]. $\square$

Let (x_1, x_2) be a special point in $C_{\mathbb{Q}}(\overline{\mathbb{Q}})$ such that the CM fields of x_1 and x_2 are isomorphic to the quadratic imaginary field K and let c_i be the conductors of the orders $\mathrm{End}_B(x_i)$ in K. Let c be the least common multiple of c_1 and c_2. Let $K_{c_i} := K(x_i)$. The field generated by K_{c_1} and K_{c_2} is contained in K_c and $[K_c : K_{c_1}K_{c_2}] \leq 3$. This last inequality comes from the unit groups of $O_{K,c}$, O_{K,c_1} and O_{K,c_2}. Hence the orbit of (x_1, x_2) under $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ has at least $\mathrm{Gal}(K_c/K)/3$ elements. Let $\sigma \in \mathrm{Gal}(K_c/K)$ be represented by a class $[I]$ in $\mathrm{Pic}(O_{K,c})$ of an invertible ideal I. There are isogenies from x_i to $\sigma(x_i)$ with kernels isomorphic, as $O_{K,c}$-modules, to $(O_{K,c}/I)^2$. Let p be a prime split in $O_{K,c}$, that is $O_{K,c} \otimes \mathbb{F}_p \cong \mathbb{F}_p \times \mathbb{F}_p$. Then taking the class of one of the 2 maximal ideals, say m, dividing p and denoting σ_m the corresponding Frobenius element, we see that there are isogenies from x_i to $\sigma_m(x_i)$ with kernel isomorphic to $\mathbb{F}_p^2$, hence $\sigma_m(x_1, x_2)$ is in $(T_p \times T_p)(x_1, x_2)$. Since $C_{\mathbb{Q}}$ and $(T_p \times T_p)(C_{\mathbb{Q}})$ are defined over $\mathbb{Q}$, $\sigma_m(x_1, x_2)$ is in $C_{\mathbb{Q}}(\overline{\mathbb{Q}}) \cap (T_p \times T_p)(C_{\mathbb{Q}})(\overline{\mathbb{Q}})$. All the conjugates of this point by $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ are contained in this intersection, hence $C_{\mathbb{Q}}(\overline{\mathbb{Q}}) \cap (T_p \times T_p)(C_{\mathbb{Q}})(\overline{\mathbb{Q}})$ contains at least $|\mathrm{Pic}(O_{K,c})|/3$ elements. Now we give a lower bound for the intersection number of $C_{\mathbb{Q}}(\mathbb{C})$ and $(T_p \times T_p)(C)(\mathbb{C})$. If B_i is isomorphic to $M_2(\mathbb{Q})$ we replace $S_i(\mathbb{C})$ by its compactification $\mathbb{P}_1(\mathbb{C})$ by adding one point, if B_i is not isomorphic to $M_2(\mathbb{Q})$ then $S_i(\mathbb{C})$ is already compact. We take closures of $C(\mathbb{C})$ and $(T_p \times T_p)(\mathbb{C})$ in $(S_1 \times S_2)(\mathbb{C})$ compactified. Let (d_1, d_2) be the bidegree of $C_{\mathbb{Q}}(\mathbb{C})$ then the bidegree of $(T_p \times T_p)(C)(\mathbb{C})$ is $((p+1)^2 d_1, (p+1)^2 d_2)$. Let f_i be a finite morphism $S_i(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ of degree D_i, then $f := (f_1, f_2)$ is a finite morphism from $(S_1 \times S_2)(\mathbb{C})$ to $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ of degree $D_1 D_2$. The Picard group of $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$, the isomorphism is given by the bidegree. The intersection form on $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ is given by $(a, b).(c, d) \mapsto ac + bd$. One easily sees that, the intersection number of $C(\mathbb{C})$ and $(T_p \times T_p)(C)(\mathbb{C})$ is bounded by $2(p+1)^2 (D_1 D_2)^2 d_1 d_2$. Let $k := 6(D_1 D_2)^2 d_1 d_2$. Bezout's theorem shows that if $|\mathrm{Pic}(O_{K,c})| > k(p+1)^2$ then $C_{\mathbb{Q}} \subset (T_p \times T_p)(C_{\mathbb{Q}})$. In [4, sect. 5], the following result is shown (it is a consequence of the effective form of Chebotarev density theorem which is true only under GRH, see [11]):

Lemma 3.3.2 *Assume the Generalized Riemann Hypothesis for imaginary quadratic fields. There exists $\epsilon > 0$ such that when K ranges through all the imaginary quadratic fields and c ranges through all positive numbers, then the number of primes $p < |\mathrm{discr}(O_{K,c})|^{1/4-\epsilon}$ that are split in $O_{K,c}$tends to infinity as $|\mathrm{discr}(O_{K,c})|$ tends to infinity.*

On the other hand, the Siegel's theorem tells that $\log |\mathrm{Pic}(O_{K,c})| = (1/2 + o(1)) \log |\mathrm{discr}(O_{K,c})|$ as $|\mathrm{discr}(O_{K,c})|$ tends to infinity . As C contains infinitely many special points, $|\mathrm{discr}(O_{K,c})|$

associated to them tends to infinity, then the two inequalities show that for infinitely many p, $C_{\mathbb{Q}} \subset (T_p \times T_p)(C_{\mathbb{Q}})$. We have shown the following proposition.

Proposition 3.3.3 *Assume GRH for all imaginary quadratic fields. Let $C_{\mathbb{Q}}$ be as before. Then there exist infinitely many primes p such that $C_{\mathbb{Q}}$ is contained in $(T_p \times T_p)(C_{\mathbb{Q}})$.*

3.4 Topological characterisation of Hodge type curves.

Theorem 3.4.1 *Let C in $(S_1 \times S_2)(\mathbb{C})$ be an irreducible closed complex algebraic curve. Let (d_1, d_2) be its bidegree i.e d_i is the degree of the projection $\mathrm{pr}_i : C \to S_i(\mathbb{C})$. Suppose that $C \subset (T_n \times T_n)(C)$ for some n square free and composed out of primes $p \geq \max\{13, d_1\}$ prime to the discriminants of B_1 and B_2. Then C is of Hodge type.*

If one of the projections of C is constant, then the result is trivial. Hence, we assume that none of the projections is constant. Using [9, prop. 5.2.11], one shows that there exist elements g_i $(i = 1, 2)$ in $B_{i\mathbb{Q}}$ of norm n such that the image of T_n in $S_i(\mathbb{C}) \times S_i(\mathbb{C})$ is the image of $S_{g_i}(\mathbb{C}) := \Gamma_i \cap g_i^{-1}\Gamma_i g_i \backslash \mathbb{H}$ in $\Gamma_i\backslash\mathbb{H} \times \Gamma_i\backslash\mathbb{H}$ given by $x \mapsto (x, g_i x)$. Let $T_{C,n}$ be the correspondence induced by $T_n \times T_n$ on C. This correspondence is defined as follows: take the inverse image of $C \times C$ in $S_{g_1}(\mathbb{C}) \times S_{g_2}(\mathbb{C})$ and delete its zero-dimensional part. This set with its two projections to C is $T_{C,n}$.

The image of $T_{C,n}$ under $(\mathrm{pr}_1, \mathrm{pr}_1)$ from $C \times C$ to $(S_1 \times S_1)(\mathbb{C})$ is the image of $T_n := \Gamma_1 \cap g_1\Gamma_1 g_1^{-1}\backslash\mathbb{H}$ in $\Gamma_1\backslash\mathbb{H} \times \Gamma_1\backslash\mathbb{H}$ under the map induced by $\tau \mapsto (\tau, g_1\tau)$ from $\mathbb{H}$ to $\mathbb{H} \times \mathbb{H}$. We have the following diagram.

$$\begin{array}{ccc} C & \xrightarrow{\mathrm{pr}_1} & S_1(\mathbb{C}) \\ \uparrow & & \uparrow \\ T_{C,n} & \longrightarrow & T_n \end{array}$$

Lemma 3.4.2 *The map from $T_{C,n}$ to $C \times_{S_1(\mathbb{C})} T_n$ induced by this diagram is surjective.*

Proof. The four morphisms in the diagram are finite morphisms of complex algebraic curves hence the morphism from $T_{C,n}$ to $C \times_{S_1(\mathbb{C})} T_n$ is finite. Hence, to show that the map from $T_{C,n}$ to $C \times_{S_1(\mathbb{C})} T_n$ is surjective, it suffices to show that $C \times_{S_1(\mathbb{C})} T_n$ is irreducible, or, equivalently, that the tensor product of the function fields of C and T_n over the function field of $S_1(\mathbb{C})$ is a field. Let $\Gamma_{1,n} := \{x \in \Gamma_1 | x \equiv 1 \mod n\}$, let $S_{1,n}(\mathbb{C})$ be the complex algebraic curve $S_{1,n}(\mathbb{C}) = \Gamma_n\backslash\mathbb{H}$. The curve $S_{1,n}(\mathbb{C})$ is a Galois cover of $S_1(\mathbb{C})$ and its function field is the Galois extension of that of $S_1(\mathbb{C})$ with Galois group isomorphic to $\Gamma_1/\Gamma_{1,n} \cong \mathrm{SL}_n(\mathbb{Z}/n\mathbb{Z}) \cong \prod_{p|n} \mathrm{SL}_2(\mathbb{F}_p)$. Since

$\mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ has no non proper subgroup of index at most d_1, the tensor product of the function fields of C and $S_{1,n}(\mathbb{C})$ over that of $S_1(\mathbb{C})$ is a field. This finishes the proof. $\square$

Lemma 3.4.3 *Let S be the Shimura curve attached to the datum $(B_{\mathbb{Q}}, B)$ and $n > 1$ a square free integer prime to the discriminant of B. The orbits of T_n acting on $S(\mathbb{C})$ are dense in $S(\mathbb{C})$.*

Proof. Let G be the algebraic group over $\mathbb{Z}$ of norm-one elements in B. This means that for any $\mathbb{Z}$-algebra R, one has $G(R) = \{x \in B \otimes R : n(x) = 1\}$. For any p that does not divide the discriminant of $B_{\mathbb{Q}}$, there is an isomorphism between $G(\mathbb{Q}_p)$ and $\mathrm{SL}_2(\mathbb{Q}_p)$ and between $G(\mathbb{Z}_p)$ and $\mathrm{SL}_2(\mathbb{Z}_p)$. One can write $S(\mathbb{C}) = G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty \times G(\widehat{\mathbb{Z}})$ where K_∞ is some maximal compact open subgroup of $G(\mathbb{R})$. Consider the element $s_p := \left(\begin{smallmatrix} p & 0 \\ 0 & p^{-1} \end{smallmatrix}\right)$ of $\mathrm{SL}_2(\mathbb{Q}_p)$. This element defines a correspondence on $S(\mathbb{C})$ by the right multiplication by s_p on $G(\mathbb{A})$ and clearly the orbits of this correspondence are contained in the orbits of T_p. The orbits of the correspondence defined by the right multiplication by the element g of $G(\mathbb{A}_{\mathrm{f}})$ that equals s_p at all places p dividing n and 1 elsewhere are contained in the orbits of T_n.

The subgroup of $\prod_{p|n} G(\mathbb{Q}_p)$ generated by g and $\prod_{p|n} G(\mathbb{Z}_p)$ is $\prod_{p|n} G(\mathbb{Q}_p)$. This is a consequence of the Cartan decomposition for SL_2 over a local field (see [12, p. 108, example]). In this case this is actually a consequence of the theory of elementary divisors.

Let x be a point of $S(\mathbb{C})$. Let t be an element in $G(\mathbb{A})$ in the preimage of x. The closure of the orbit of x by T_n contains the closure of the image of $G(\mathbb{Q})t\prod_{p|n} G(\mathbb{Q}_p)$ in $S(\mathbb{R})$. But, $G(\mathbb{Q})t\prod_{p|n} G(\mathbb{Q}_p) = G(\mathbb{Q})t\prod_{p|n} G(\mathbb{Q}_p)t^{-1}t = G(\mathbb{Q})\prod_{p|n} G(\mathbb{Q}_p)t$. The strong approximation theorem [13, thm. 4.3] shows that and $G(\mathbb{Q})\prod_{p|n} G(\mathbb{Q}_p)$ is dense in $G(\mathbb{A})$. The density of the orbits follows. $\square$

Remark 3.4.4 Using the strong approximation theorem for simply connected algebraic groups, one can prove the density of some Hecke orbits in any Shimura variety.

Let $\pi : \mathbb{H} \times \mathbb{H} \to \Gamma_1\backslash\mathbb{H} \times \Gamma_2\backslash\mathbb{H}$ be the quotient map. Let X be an irreducible component of the analytic variety $\pi^{-1}(C)$ and G_X the stabilizer of X in $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$. Let $\Gamma_X := G_X \cap (\Gamma_1 \times \Gamma_2)$. To prove the two following lemmas, one just has to follow the proofs of [4, lemma 6.6 and lemma 6.7]

Lemma 3.4.5 *The group G_X is an analytic subgroup of $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$.*

Lemma 3.4.6 *The kernels of the two projections from G_X to $\mathrm{SL}_2(\mathbb{R})$ are discrete.*

Lemma 3.4.7 *The topological group G_X is not discrete.*

Proof. Let us take representatives (t_i, t'_j) in $B^+_{1\mathbb{Q}} \times B^+_{2\mathbb{Q}}$ (where $+$ means "elements of positive norm") for the coset space $\Gamma_1 \times \Gamma_2(g_1, g_2)\Gamma_1 \times \Gamma_2/\Gamma_1 \times \Gamma_2$. The condition $C \subset (T_n \times T_n)(C)$ imply that there exist pairs (t_i, t'_j) such that $(t_i, t'_j)X$ is contained in $\pi^{-1}C$. Every such pair gives an element $g_{i,j}$ in G_X of the form

$$g_{i,j} = \gamma_{i,j,1}(n^{-1/2}g_1, n^{-1/2}g_2)\gamma_{i,j,2}$$

with $\gamma_{i,j,1}$ and $\gamma_{i,j,2}$ in $\Gamma_1 \times \Gamma_2$. Let H be the subgroup of G_X generated by Γ_X and all the $g_{i,j}$. Reproducing the proof of [4, lem. 6.9], one shows that H is not discrete hence G_X is not discrete.
$\square$

Now we can finish the proof of the theorem. As G_X is not discrete, the Lie algebra $\mathrm{Lie}(G_X)$ is non-zero. Since the kernels of the two projections of G_X are discrete, the images of the two projections of $\mathrm{Lie}(G_X)$ are non-zero. Since $\mathrm{pr}_i\mathrm{Lie}(G_X)$ is normalized by $\mathrm{pr}_i(\Gamma_X)$ which is Zariski dense in $\mathrm{SL}_2(\mathbb{R})$ (because it is of finite index in the arithmetic subgroup $\mathrm{SL}_2(\mathbb{Z})$ of $\mathrm{SL}_2(\mathbb{R})$ which is Zariski-dense by [10, thm 4.10]) and that $\mathrm{Lie}(\mathrm{SL}_2(\mathbb{R}))$ is simple, $\mathrm{pr}_i\mathrm{Lie}(G_X) = \mathrm{Lie}(\mathrm{SL}_2(\mathbb{R}))$. Since $\mathrm{SL}_2(\mathbb{R})$ is connected, G_X projects surjectively to the both factors. The kernel of $\mathrm{pr}_i : G_X \to \mathrm{SL}_2(\mathbb{R})$ is a normal subgroup of $\mathrm{SL}_2(\mathbb{R})$ (viewed as $\mathrm{SL}_2(\mathbb{R}) \times \{1\}$). This kernel is discrete and contains $\{1, -1\}$ so it is $\{1, -1\}$. Goursat's lemma then says that G_X is the inverse image in $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ of the graph of analytic automorphism σ of $\mathrm{SL}_2(\mathbb{R})/\{1, -1\}$. Now, since $\mathrm{pr}_i\Gamma_X$ is of finite index in $\mathrm{SL}_2(\mathbb{Z})$, σ comes from an isomorphism between the algebraic groups $B^\times_{1\mathbb{Q}}$ and $B^\times_{2\mathbb{Q}}$. If $B_{1\mathbb{Q}}$ is not isomorphic to $B_{2\mathbb{Q}}$ then such an isomorphism does not exist. Consider the case when $B_{1\mathbb{Q}}$ is isomorphic to $B_{2\mathbb{Q}}$. Then, as explained in the introduction, one takes an isomorphism which sends B_1 to B_2, hence one reduces to the case of $S \times S$ where S is the Shimura curve associated to the quaternion algebra $B_\mathbb{Q}$ with maximal order B. Let G be the algebraic group defined by $B^\times_\mathbb{Q}$ (that is $G(R) = (B \otimes R)^\times$ for any $\mathbb{Q}$-algebra R). The algebraic group $\mathrm{Aut}(B^\times_\mathbb{Q})$ is $G^{\mathrm{ad}} := G/\mathbb{G}_\mathrm{m}$ and the map $G(\mathbb{Q}) = B^\times_\mathbb{Q} \to G^{\mathrm{ad}}(\mathbb{Q})$ is surjective because of Hilbert 90, hence σ is given by conjugation by some element g of $B^\times_\mathbb{Q}$. We have shown that G_X is the set $\{(h, \pm ghg^{-1}) | h \in \mathrm{SL}_2(\mathbb{R})\}$.

In the previous version of this chapter there was a gap in the end of the proof, pointed out by Richard Pink. The problem was that we have assumed, without reason that g was actually in $\mathrm{SL}_2(\mathbb{R})$. In fact this gap was already present in the article of Edixhoven [19]. So Edixhoven has found a way out and has written an erratum to his article. I adopt it here to my situation.

Let $x = (x_1, x_2)$ be an element of X such that the two projections from X to $\mathbb{H}$ induce isomorphisms on tangent spaces $T_X(x) \to T_\mathbb{H}(x_1)$ and $T_X(x) \to T_\mathbb{H}(x_2)$. Let $G_{X,x}$ be the stabilizer of x in G_X. Let for $i = 1, 2$, the group $\mathrm{SL}_2(\mathbb{R})_{x_i}$ be the stabilizer of x_i in $\mathrm{SL}_2(\mathbb{R})$. These tangent spaces are naturally isomorphic to $\mathrm{Lie}(G_X)/\mathrm{Lie}(G_{X,x})$, $\mathrm{Lie}(\mathrm{SL}_2(\mathbb{R}))/\mathrm{Lie}(\mathrm{SL}_2(\mathbb{R})_{x_1})$

and $\mathrm{Lie}(\mathrm{SL}_2(\mathbb{R}))/\mathrm{Lie}(\mathrm{SL}_2(\mathbb{R})_{x_2})$. Since X is a complex analytic subvariety of $\mathbb{H} \times \mathbb{H}$, the isomorphisms between the tangent spaces are compatible with the complex structures. Write $x_2 = g'x_1$, with g' in $\mathrm{SL}_2(\mathbb{R})$. Then g' also induces an isomorphism $T_{\mathbb{H}}(x_1) \to T_{\mathbb{H}}(x_2)$ of one-dimensional complex vector spaces. It follows that conjugation by $g^{-1}g'$ on $\mathrm{SL}_2(\mathbb{R})$ induces an automorphism of $\mathrm{Lie}(\mathrm{SL}_2(\mathbb{R}))/\mathrm{Lie}(\mathrm{SL}_2(\mathbb{R})_{x_1}$ that preserves orientation. A simple computation shows that this implies that $g^{-1}g'$ is in the connected component of identity of the normalizer in $\mathrm{GL}_2(\mathbb{R})$ of $\mathrm{SL}_2(\mathbb{R})_{x_1}$, hence that g has positive determinant. (Note that $\mathrm{SL}_2(\mathbb{R})_{x_1}$ is a conjugate of $\mathbf{SO}_2(\mathbb{R})$, whose normalizer in $\mathrm{GL}_2(\mathbb{R})$ is $\mathbb{R}^*\mathrm{O}_2(\mathbb{R})$, a group with exactly two connected components.) Hence g has positive determinant, and we have $x_2 = gx_1$. This means that $X = \{(\tau, g\tau) \mid \tau \in \mathbb{H}\}$. The curve C is then $\Gamma_X \backslash X$ which is easily seen to be isomorphic to $g\Gamma g^{-1} \cap \Gamma \backslash \mathbb{H}$ hence it is of Hodge type.

3.5 End of the proof.

Using the results of the two preceding sections, one can easily finish the proof of the main result. Let C be as in the introduction. There are infinitely many primes p that are prime to the discriminant of B such that $C_{\mathbb{Q}}$ is contained in $(T_p \times T_p)(C_{\mathbb{Q}})$. Let $\mathcal{C}$ be the set of irreducible components of $C_{\mathbb{Q}}$, then every p as above induces a correspondence $T_{\mathcal{C},p}$ on $\mathcal{C}$ commuting with the transitive action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\mathcal{C}$. Let $x_0 \in \mathcal{C}$ be the element corresponding to the irreducible component C. One then easily shows that there exists a finite product $n := p_1 \ldots p_r$ such that $x_0 \in T_{\mathcal{C},n}x_0$, which means that $C \subset (T_n \times T_n)C$. The result of the preceding section finishes the proof.

3.6 Some remarks.

Remark 3.6.1 Our result is a special case of the André-Oort conjecture saying that an irreducible component of the Zariski closure of a set of special points in a Shimura variety is a sub-variety of Hodge type. Very recently Edixhoven ([6]) generalized his method to prove under GRH the André-Oort conjecture for the product of an arbitrary number of modular curves. It should be easy to extend these methods to the arbitrary products of Shimura curves.

If one makes an additional assumption that the curve contains an infinite subset of special points such that the corresponding abelian varieties lie in one isogeny class, then the method we used in this article show that the curve is of Hodge type without assuming the GRH. Indeed, in this case one does not need the effective form of the Chebotarev density theorem but only the classical form.

Remark 3.6.2 The methods of [5] can be used to give another proof of the topological result, in a somewhat weaker form. But these methods does not seem to be generalizable to the higher dimensional case whereas one can expect a generalization of the methods of the topological part of this article to the case of sub-varieties of arbitrary dimension in general Shimura varieties.

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Chapter 4

Main results

In this chapter we state the main results of this thesis and make some preliminary reduction steps. We also give the strategy of the proof of the main results.

So we consider a Shimura datum (G, X) and a compact open subgroup K of $G(\mathbb{A}_{\mathrm{f}})$. We consider an irreducible closed subvariety Z of $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$ such that Z contains a Zariski dense subset of special points. Recall that the general André-Oort conjecture asserts that in this situation, Z is a subvariety of Hodge type. Since for Z of dimension 0, the André-Oort conjecture is true by definition of special points, we assume that the dimension of Z is at least 1. In the rest of this thesis we only consider the situation when Z contains a Zariski-dense subset of special points that lie in one Hecke orbit. This means that there exists a special element x_0 in X such that Z contains a Zariski dense subset of points that are in the image of $\{x_0\} \times G(\mathbb{A}_{\mathrm{f}})$ in $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$. We take such a Z an we try to prove that it is of Hodge type. As we can not do it in full generality for a subvariety of arbitrary dimension, we impose some conditions on Z.

4.1 Main results and some reduction steps

The main result is the following.

Theorem 4.2 *Let (G, X) be a Shimura datum and let K be a compact open subgroup of $G(\mathbb{A}_{\mathrm{f}})$. Let Z be an irreducible algebraic curve contained in $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$ such that $Z(\mathbb{C})$ contains an infinite set of special points that lie in one Hecke orbit. Then Z is of Hodge type.*

Let us now make some reduction steps. In order to prove a result about irreducibility of the transforms of irreducible subvarieties by some Hecke correspondences we will need to apply results about the monodromy such as a theorem of André (2.3.3) so we want Z to be Hodge generic in the sense of the definition that follows. This condition "Hodge generic" plays a crucial role in the proof of 5.2.1.

Definition 4.2.1 Let (G, X) be a Shimura datum and K a compact open subgroup of $G(\mathbb{A}_{\mathrm{f}})$. An irreducible closed subvariety Z of $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$ is said to be Hodge generic if Z is not contained in any proper subvariety of Hodge type of $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$.

Remark 4.2.2 The property of subvariety to be Hodge generic does not depend on the compact open subgroup K.

Remark 4.2.3 The property of being Hodge generic can be characterized in terms of variations of Hodge structure attached to various representations of G. Indeed, "Hodge generic" means that for such a variation of Hodge structure, the generic Mumford-Tate group on this variety is G. To be more precise , let us consider a representation ρ of G over $\mathbb{Q}$ which factors through a faithful representation of G^{ad}. We also assume that K is neat. Hence we get a variation of $\mathbb{Q}$-Hodge structure $\mathcal{V}(\rho)$ over $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$. Suppose that a point $[x, g]$ of X is *not* Hodge generic in the sense of the preceding definition. This means that $[x, g]$ is contained in a proper subvariety of Hodge type. Hence the Mumford-Tate group $\mathrm{MT}(x)$ is strictly smaller than G. Using Chevalley's theorem, we see that there exists a line l in V such that $\rho(\mathrm{MT}(x))$ is the stabilizer of l. Let v be an element of V generating l. Let Y be the locus of points of X where v is a Hodge class. Then $[x, g]$ is contained in the image of Y in $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$ hence it is in the Hodge exceptionnal locus for $\mathcal{V}(\rho)$. Conversely, suppose that $[x, g]$ is in the Hodge exceptionnal locus for $\mathcal{V}(\rho)$. Let v be an element of V which is an "extra" Hodge class for the Hodge structure on V determined by x (this means that v is not fixed by $\rho(G)$). Let M be the stabilizer in $\mathrm{GL}(V)$ of the line generated by v. Then the neutral component G' of the identity of $\rho^{-1}(M)$ is a proper subgroup of G and x factors through G'. Let X' be the $G'(\mathbb{R})$-conjugacy class of x. The Shimura datum (G', X') and the element g determine a proper subvariety of Hodge type that contains $[x, g]$.

Example 4.2.4 Consider $\mathbb{C} \times \mathbb{C}$ as a Shimura variety defined by the Shimura datum $(\mathrm{GL}_2 \times \mathrm{GL}_2, \mathbb{H}^{\pm} \times \mathbb{H}^{\pm})$ where $\mathbb{H}^{\pm}$ is the union of the upper and lower half plane. Special points of $\mathbb{C}$ are the $j(\tau)$ where j is the classical modular function and $\tau \in \mathbb{H}$ is imaginary quadratic. Let x be a *non-special* point (then $\{x\}$ is Hodge generic in $\mathbb{C}$) and consider the fiber $\{x\} \times \mathbb{C} \subset \mathbb{C} \times \mathbb{C}$ of the first projection over $\{x\}$. This variety is Hodge generic. Obviously it is not of Hodge type (if it were it would be equal to $\mathbb{C} \times \mathbb{C}$). However this variety has an interesting property to be totally geodesic in the sense of [24].

There exists a smallest subvariety of Hodge type S' of S that contains Z in the sense that Z is not contained in a proper subvariety of Hodge type of S'. This follows form the fact that the irreducible components of the intersection of two subvarieties of Hodge type is again of Hodge type. One can prove this last assertion using the characterization of subvarieties of Hodge type as irreducible components of the loci where certain classes are Hodge classes. The Shimura datum (G', X') describing S' is given by the following proposition.

Proposition 4.2.5 *Let (G, X) be a Shimura datum, K a compact open subgroup of $G(\mathbb{A}_f)$ and let Z be a closed irreducible subvariety of $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$. Let s in Z be Hodge generic. Let (x, g) in $X \times G(\mathbb{A}_f)$ lie over s and let G' be the Mumford-Tate group of x. Then we have a morphism of Shimura data from (G', X') to (G, X) with X' the $G'(\mathbb{R})$-conjugacy class of x. Let K' be the intersection of $G'(\mathbb{A}_f)$ and gKg^{-1}. Then the inclusion of G' in G, followed by right multiplication by g induces a morphism $f \colon \mathrm{Sh}_{K'}(G', X')_{\mathbb{C}} \to \mathrm{Sh}_K(G, X)_{\mathbb{C}}$. This morphism is finite and its image contains Z. Let Z' be an irreducible component of $f^{-1}Z$. Then Z is of Hodge type if and only if Z' is.*

Proof. The proof is essentially the same as in the Remark 4.2.3. For details we refer to the Proposition [24, Prop 2.9] and [24, ß2.9]. $\square$

Remark 4.2.6 This proposition gives a "candidate" for Z. Indeed, the variety Z is of Hodge type if and only if Z equals the irreducible component of the image of f containing it. To prove this equality it will be sufficient to prove that Z contains an orbit of a suitable Hecke correspondence. Indeed, in the next Chapter we will prove that such an orbit is dense in the smallest subvariety of Hodge type containing Z. The proof of the fact that Z contains such a Hecke orbit is, however, the hardest part of the proof and I still do not know how to do it in full generality.

Example 4.2.7 As an example, let us consider the Shimura variety $\mathbb{C} \times \mathbb{C} \times CC$ which is a moduli space for triples of elliptic curves. Let Z be $\mathbb{C} \times \{x\} \times \{y\}$ where x is a non-special point of $\mathbb{C}$ and y is special. The group G' given by the preceding proposition is $\mathrm{GL}_2 \times \mathrm{GL}_2 \times M \subset \mathrm{GL}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_2$ where M is the Mumford-Tate group of y. The smallest subvariety of Hodge type containing Z is $\mathbb{C} \times \mathbb{C} \times \{y\}$.

With the notations of this proposition, S' is an irreducible component containing Z of the image of $\mathrm{Sh}_{K'}(G', X')_{\mathbb{C}}$ by f. This proposition shows that the general André-Oort conjecture is true if and only if it is true for the sets of special points whose Zariski closure is irreducible and Hodge generic. On the other hand, the proposition does not reduce the proof of Theorem 4.2 to the case where Z is Hodge generic, because the inverse image in $\mathrm{Sh}_{K'}(G', X')_{\mathbb{C}}$ of a Hecke orbit in $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$ is a disjoint union of a possibly infinite number of Hecke orbits. This means that we cannot simply replace G by G' in our proof. The point is that an infinite set of tori in G' that lie in one $G(\mathbb{Q})$-conjugacy class does not, in general, intersect only a finite set of $G'(\mathbb{Q})$-conjugacy classes of tori in G'. However, we will overcome the difficulty by considering the situation only over $\mathbb{Q}_p$ for some fixed p. In this case using some Theorems of Grothendieck on the liftings properties of morphisms from tori to group schemes.

It will be convinient for us to work with Shimura varieties that are defined by semisimple groups of adjoint type. The following proposition shows that it is actually possible to reduce ourselves to this case.

Proposition 4.2.8 *Let (G, X) be a Shimura datum, let G^{ad} be the quotient of G by its center, and let X^{ad} be the $G^{\mathrm{ad}}(\mathbb{R})$ conjugacy class of morphisms from $\mathbb{S}$ to $G^{\mathrm{ad}}_{\mathbb{R}}$ that contains the image of X. Let K^{ad} be a compact open subgroup of $G^{\mathrm{ad}}(\mathbb{A}_{\mathrm{f}})$, and let K be a compact open subgroup of $G(\mathbb{A}_{\mathrm{f}})$ whose image in $G^{\mathrm{ad}}(\mathbb{A}_{\mathrm{f}})$ is contained in K^{ad}. Then the induced morphism from $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$ to $\mathrm{Sh}_{K^{\mathrm{ad}}}(G^{\mathrm{ad}}, X^{\mathrm{ad}})_{\mathbb{C}}$ is finite. Let Z be a closed irreducible subvariety of $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$, and let Z^{ad} be its image in $\mathrm{Sh}_{K^{\mathrm{ad}}}(G^{\mathrm{ad}}, X^{\mathrm{ad}})_{\mathbb{C}}$. Then Z is of Hodge type if and only if Z^{ad} is.*

Proof. By [24, §2.1], X is just a set of connected components of X^{ad}. Let S and S^{ad} be the connected components of $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$ and of $\mathrm{Sh}_{K^{\mathrm{ad}}}(G^{\mathrm{ad}}, X^{\mathrm{ad}})_{\mathbb{C}}$ that contain Z and Z^{ad}, respectively. Let X^+ be a connected component of X and let g in $G(\mathbb{A}_{\mathrm{f}})$ be such that S is the image in $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$ of $X^+ \times \{g\}$. Then the inverse images of Z and Z^{ad} in $X^+ \times \{g\}$ are equal, hence the property of being of Hodge type for them is equivalent. □

We also see that Z contains a Zariski dense subset of special points lying in one Hecke orbit if and only if Z^{ad} does. Now we state an auxiliary result that we obtain in this thesis.

Theorem 4.3 *Let (G, X) be a Shimura datum where G is a semisimple group of adjoint type and let $K \subset G(\mathbb{A}_{\mathrm{f}})$ be a compact open subgroup. Let Z be a closed irreducible subvariety of $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$ which is Hodge generic. There exists a prime p_0 such that the following holds. Suppose that Z contains a Zariski dense set of special points $[x_0, g]$ (with x_0 in X a special element) such that there exists a prime $p \geq p_0$ that splits $\mathrm{MT}(x_0)$ and for all points in this set $g_p \in H(\mathbb{Q}_p)K_p$ (H is the centralizer of $\mathrm{MT}(x_0)$ in G). Then Z is of Hodge type.*

4.4 Reduction

In this section we use the results form the previous section in order to reduce the proof of the general theorem 4.2 to a more particular situation. So we start with an arbitrary Shimura datum (G, X) and K a compact open subgroup of $G(\mathbb{A}_{\mathrm{f}})$. We are also given an irreducible closed subvariety Z containing a Zariski-dense subset of special points lying in one Hecke orbit. Recall that our aim is to prove that Z is of Hodge type. Using 4.2.8 we readily see that we can assume that G is semisimple of adjoint type without loosing any of our assumptions. So from now on we suppose that

> The group G is semisimple of adjoint type.

Let (G', X') be as given in the Proposition 4.2.5 (after the choice of a Hodge generic point in Z). Let K', g and the morphism

$$f : \mathrm{Sh}_{K'}(G', X')_{\mathbb{C}} \to \mathrm{Sh}_K(G, X)_{\mathbb{C}}$$

be as in the proposition 4.2.5. We let Z' be an irreducible component of $f^{-1}(Z)$. From the Proposition 4.2.5 we know that Z is of Hodge type if and only if Z' is. Let X^+ be a connected component of X and let X'^+ be a connected component of X'. Let S and S' be the connected components of $\mathrm{Sh}_{K'}(G', X')_{\mathbb{C}}$ and $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$ that are images of $X'^+ \times 1$ and $X^+ \times 1$ in $\mathrm{Sh}_{K'}(G', X')_{\mathbb{C}}$ and $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$ respectively.

> After possibly changing Z by its transform by some Hecke correspondence, we can and do assume that Z lies in the image of S' by f (hence in S).

Let $(G'^{\mathrm{ad}}, X'^{\mathrm{ad}})$ be obtained from (G', X') as in 4.2.8. Let K'^{ad} be an open compact subgroup of $G'^{\mathrm{ad}}(\mathbb{A}_{\mathrm{f}})$. After possibly replacing K by a smaller compact open subgroup of $G(\mathbb{A}_{\mathrm{f}})$ the image of K' in $G'^{\mathrm{ad}}(\mathbb{A}_{\mathrm{f}})$ is contained in K'^{ad}. Let Z'^{ad} be the image of Z' in $\mathrm{Sh}_{K'^{\mathrm{ad}}}(G'^{\mathrm{ad}}, X'^{\mathrm{ad}})_{\mathbb{C}}$ and let S'^{ad} be the connected component of $\mathrm{Sh}_{K'^{\mathrm{ad}}}(G'^{\mathrm{ad}}, X'^{\mathrm{ad}})_{\mathbb{C}}$ containing Z'^{ad}. I wish to add one obvious remark: since Z'^{ad} is a Hodge generic variety of dimension ≥ 1, the group G'^{ad} is non-trivial. Various objects that we have introduced can be represented by the following diagram:

$$\begin{tikzcd}
Z' \arrow[r, "f"] \arrow[dd] & Z & \\
 & S' \arrow[r, "f"] \arrow[d] & S \\
Z'^{\mathrm{ad}} & S'^{\mathrm{ad}} &
\end{tikzcd}$$

All the arrows in this diagram are finite morphisms. To prove that Z is of Hodge type it suffices to prove that Z'^{ad} is of Hodge type. Recall that the property of Z'^{ad} to be of Hodge type is independent of the choice of the compact open subgroup K'^{ad} in $G'^{\mathrm{ad}}(\mathbb{A}_{\mathrm{f}})$. Hence we can safely assume that K'^{ad} is a product $\prod_p K'^{\mathrm{ad}}_p$ of compact open subgroups K'^{ad}_p of $G'^{\mathrm{ad}}(\mathbb{Q}_p)$. The result of our reduction steps is hence the following.

1. The group G is semisimple of adjoint type
2. The variety Z'^{ad} is contained in the image S'^{ad} of $X'^{+} \times 1$ in $\mathrm{Sh}_{K'^{\mathrm{ad}}}(G', X')_{\mathbb{C}}$ where K'^{ad} is a product $\prod_p K'^{\mathrm{ad}}_p$ of compact open subgroups K'^{ad}_p of $G'^{\mathrm{ad}}(\mathbb{Q}_p)$.
3. To prove that Z is of Hodge type, it suffices to prove that the variety Z'^{ad} is. The last statement is equivalent to proving that Z'^{ad} equals S'^{ad}.

We recall once again that with our assumptions we can not say that Z'^{ad} contains an infinite set of special points that lie in one Hecke orbit. Hence we can not simply forget the existence of Z.

4.5 Strategy of the proofs

At this point we can describe the strategy of the proof of the main results. So let the notation now be as in the introduction to this chapter. Recall that Z denotes an irreducible closed subvariety of $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$ containing a Zariski dense subset of special points that lie in one Hecke orbit. To avoid exposition of details here, let us assume that Z is Hodge generic and that G is semisimple of adjoint type.

In the next chapter we will prove that Z is of Hodge type if and only if there exists an element g of $G(\mathbb{A}_{\mathrm{f}})$ that satisfies some conditions that we will make explicit in the next chapter such that

$$Z \subset T_g(Z)$$

The ingredients for a proof of this characterization is a result about irreducibility of images of Z by some irreducible components of Hecke correspondences and a result about the density of the orbits of some Hecke correspondences.

The hardest part of the proof is the proof of such an inclusion that is carried out in our cases in chapter 7 and that uses a hard Theorem 6.1 by Edixhoven giving lower bounds for Galois orbits of special points. The idea is basically the same as in the preceding chapter where we have cosidered the case of a Shimura variety which is a product of two Shimura curves attached to indefinite quaternion algebras over $\mathbb{Q}$. We use the action on special points in Z of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$, F being some number field of definition of Z. The explicit description of this action given by the reciprocity morphism allows one to prove that the intersection of Z with its images by some Hecke correspondences is not proper. The strategical novelty of this part of the proof is the idea to consider two cases depending on the behavior of the given set of special points on Z. In the first case we can conclude directly that the intersection of Z with its image by some Hecke correspondence is infinite and in the second we really need the lower bounds for Galois orbits given by the theorem of Edixhoven 6.1.

Chapter 5

Irreducibility of transforms by Hecke correspondences and density of Hecke orbits

This chapter contains

1. A result about irreducibility of the transforms of an irreducible *Hodge generic* subvariety of a Shimura variety defined by a semisimple group of adjoint type by irreducible components of Hecke correspondences.

2. A result about density of orbits of irreducible components of some Hecke correspondences.

The two results will actually provide a criterion that we will use to prove that some subvariety is of Hodge type. We will prove in the last section of the chapter that subvarieties of Hodge type are actually characterized by the property that there exists a Hecke correspondence T_g verifying certain conditions such that

$$Z \subset T_g(Z)$$

We consider the following data.

- A Shimura datum (G, X) with G a semisimple algebraic group of adjoint type (except the first section where we consider general Shimura datum).

- A compact open subgroup K of $G(\mathbb{A}_{\mathrm{f}})$ which is a product $\prod_p K_p$ where K_p are compact open subgroups of $G(\mathbb{Q}_p)$ and we moreover assume K to be neat (see [28] for definition). Recall that this implies that rational faithful representations of G induce polarizable variations of Hodge structure over $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$.

- A closed irreducible Hodge generic subvariety Z of $\mathrm{Sh}_K(G,X)_{\mathbb{C}}$.

Let X^+ be some connected component of X. We suppose that the connected component S of $\mathrm{Sh}_K(G,X)_{\mathbb{C}}$ containing Z is the image of $X^+\times 1$ in $\mathrm{Sh}_K(G,X)_{\mathbb{C}}$. Let $G(\mathbb{Q})^+$ be the intersection of $G(\mathbb{R})^+$ with $G(\mathbb{Q})$. Since G is adjoint $G(\mathbb{Q})^+ = G(\mathbb{Q})_+$ (with the notations of chapter 2). Let us define $\Gamma := G(\mathbb{Q})^+ \cap K$. Then Γ is an arithmetic group acting on X^+ and

$$S = \Gamma\backslash X^+$$

5.1 Hecke correspondences

We consider a Shimura datum (G,X) with G a compact open subgroup K of $G(\mathbb{A}_{\mathrm{f}})$. In this section we do not assume G to be semisimple of adjoint type. Let X^+ be some connected component of X. We recall from Chapter 2 that $G(\mathbb{Q})_+$ denotes the intersection of $G(\mathbb{R})^+$ with the inverse image of the intersection $G^{\mathrm{ad}}(\mathbb{Q})$ with $G^{\mathrm{ad}}(\mathbb{R})^+$. Let X^+ be a connected component of X. We let S be the image of $X^+\times 1$ in $\mathrm{Sh}_K(G,X)_{\mathbb{C}}$. Let us define $\Gamma := G(\mathbb{Q})_+ \cap K$. Then Γ is an arithmetic group acting on X^+ and

$$S = \Gamma\backslash X^+$$

In this section we recall the definition of Hecke correspondences on $\mathrm{Sh}_K(G,X)_{\mathbb{C}}$ defined by the elements of $G(\mathbb{A}_{\mathrm{f}})$. Then we explain how the correspondences on S induced by Hecke correspondences on $\mathrm{Sh}_K(G,X)_{\mathbb{C}}$ decompose into a union of irreducible components and we describe these components in terms of correspondences defined by the elements of $G(\mathbb{Q})$. This will provide us with the definition of correspondences T_q^0 defined by the elements of $G(\mathbb{Q})_+$ whose properties we shall study in the next sections.

5.1.1 Definition of Hecke correspondences defined by elements of $G(\mathbb{A}_{\mathrm{f}})$

The first kind of correspondences on $\mathrm{Sh}_K(G,X)_{\mathbb{C}}$ are the Hecke correspondences that we have already defined in the second chapter. Let us recall their definition. Let s be an element of $G(\mathbb{A}_{\mathrm{f}})$. Then s defines a correspondence T_s on $\mathrm{Sh}_K(G,X)_{\mathbb{C}}$ whose graph is the image of $\mathrm{Sh}_{sKs^{-1}\cap K}(G,X)_{\mathbb{C}}$ in $\mathrm{Sh}_K(G,X)_{\mathbb{C}}\times \mathrm{Sh}_K(G,X)_{\mathbb{C}}$ by the map $[x,g]\mapsto ([x,g],[x,g\cdot s])$. In other words, the Hecke correspondence T_s on $\mathrm{Sh}_K(G,X)_{\mathbb{C}}$ is defined by the diagram

$$\mathrm{Sh}_K(G,X)_{\mathbb{C}} \leftarrow \mathrm{Sh}_{sKs^{-1}\cap K}(G,X)_{\mathbb{C}} \xrightarrow{\cdot s} \mathrm{Sh}_K(G,X)_{\mathbb{C}}$$

where the left arrow is the obvous map induced by the inclusion $sKs^{-1}\cap K \subset K$ and the right map - by right multiplication by s. We will identify the correspondence T_s

with its graph in $\mathrm{Sh}_K(G,X)_{\mathbb{C}} \times \mathrm{Sh}_K(G,X)_{\mathbb{C}}$ and sometimes speak of T_s as a subset of $\mathrm{Sh}_K(G,X)_{\mathbb{C}} \times \mathrm{Sh}_K(G,X)_{\mathbb{C}}$.

Remark 5.1.2 It is obvious that a Hecke correspondence defined by $s \in K$ is the identity.

Example 5.1.3 Consider the moduli space of elliptic curves over $\mathbb{C}$ which is identified with $\mathbb{C}$ by the classical modular j-function. It can be viewed as a Shimura variety defined by the Shimura datum $(\mathrm{GL}_2, \mathbb{H}^{\pm})$ and the open compact subgroup $\mathrm{GL}_2(\widehat{\mathbb{Z}})$ of $\mathrm{GL}_2(\mathbb{A}_{\mathrm{f}})$. We can consider Hecke correspondences T_p for any p, defined by the matrix $g := \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Given a point x in $\mathbb{C}$, the image $T_p(x)$ consists of $p+1$ points representing elliptic curves isogeneous to the curve represented by x by a cyclic isogeny of degree p.

More generally, suppose that the Shimura variety $\mathrm{Sh}_K(G,X)_{\mathbb{C}}$ admits a modular interpretation in terms of abelian varieties and Hodge classes. For x in $\mathrm{Sh}_K(G,X)(\mathbb{C})$, let us denote by A_x the corresponding abelian variety. Let $s \in G(\mathbb{A}_{\mathrm{f}})$. Then T_s sends a point x to the set of points $\{y\}$ such that the abelian varieties A_x and A_y are isogeneous by some isogeny determined by s.

5.1.4 Decomposition of the correspondenes induced on the connected component

In this section we introduce the correspondence T_s^1 induced by T_s on S. We explain how this correspondence decomposes into a set of irreducible components and we describe these irreducible components in terms of elements of $G(\mathbb{Q})^+$.

The Hecke correspondence T_s defined by s induces a correspondence T_s^1 on S whose graph is the intersection of T_s with $S \times S \subset \mathrm{Sh}_K(G,X)_{\mathbb{C}} \times \mathrm{Sh}_K(G,X)_{\mathbb{C}}$. Of course this correspondence might be empty. We will also identify T_s^1 with its graph in $S \times S$.

Now we define another type of correspondence on S which will appear to be connected components of the previous ones. Let $q \in G(\mathbb{Q})_+$. Recall that S is the quotient of X^+ by the left action of Γ. For x in X^+ we denote by $[x]$ the image of x in S. We define the correspondence T_q^0 to be the one whose graph is the image of $q^{-1}\Gamma q \cap \Gamma \backslash X^+$ in $S \times S$ by the map $[x] \mapsto ([x], [q \cdot x])$. As before this correspondence can be represented by the following diagram:

$$\Gamma \backslash X^+ \leftarrow q^{-1}\Gamma q \cap \Gamma \backslash X^+ \xrightarrow{q \cdot} \Gamma \backslash X^+$$

This correspondence is obviously irreducible in the sense that $q^{-1}\Gamma q \cap \Gamma \backslash X^+$ is irreducible. As before, we consider T_q^0 as a subvariety of $S \times S$. The next proposition shows that if T_s^1 is non-empty, then its irreducible components are preciesly of the form T_q^0 with q in $G(\mathbb{Q})_+$ defined in terms of s.

Proposition 5.1.5 *Let s be an element of $G(\mathbb{A}_{\mathrm{f}})$. Let $G(\mathbb{Q})_+ \cap KsK = \bigcup_{i\in I}\Gamma q_i \Gamma$ be a decomposition of $G(\mathbb{Q})_+ \cap KsK$ into a union of double classes modulo left and right action of Γ on $G(\mathbb{Q})^+$. Then*

$$T_s^1 = \coprod_{i\in I} T^0_{q_i^{-1}}$$

is a decomposition of T_s^1 into a set of irreducible components.

Proof. We consider the intersection $T_s \cap (S\times S)$. Let $([x,1],[x',1])$ be a point in the intersection $T_s \cap (S\times S)$. Then, by definition of T_s, there exists some k_1 in K such that $[x',1]=[x,k_1 s]$. This means that there exists some q in $G(\mathbb{Q})$ such that $q\cdot x' = x$ and $q = k_1\cdot s\cdot k_2$ for some k_2 in K. The condition $q\cdot x' = x$ implies that $q\in G(\mathbb{R})_+ \cap G(\mathbb{Q}) = G(\mathbb{Q})_+$ and, with the condition $q\in KsK$, we see that $q\in KsK\cap G(\mathbb{Q})_+$. The fact that $qx'=x$ shows that $[x']\in T^0_{q^{-1}}([x])$. Hence any element in the intersection $T_s\cap(S\times S)$ is in $T^0_{q^{-1}}$ where q is an element in $KsK\cap G(\mathbb{Q})_+$. And conversely, for any q in $KsK\cap G(\mathbb{Q})_+$, the correspondence $T^0_{q^{-1}}$ is contained in T_s^1. Hence T_s^1 is the union of the $T^0_{q^{-1}}$ over all q in the intersection $KsK\cap G(\mathbb{Q})_+$. Any correspondence $T^0_{q^{-1}}$ is determined by the double class $\Gamma q\Gamma$. The intersection $KsK\cap G(\mathbb{Q})_+$ is a union of double classes $(\Gamma q_i \Gamma)_{i\in I}$ with $q_i\in G(\mathbb{Q})_+$ indexed by some set I. We then have $T_s^1 = \coprod_{i\in I} T^0_{q_i^{-1}}$. $\square$

5.2 Irreducibility of the images under Hecke correspondences

In the previous section we have introduced irreducible correspondences on S defined by the elements of $G(\mathbb{Q})$. Now we are interested in the properties of irreducibility of the images of irreducible closed subvarieties of S by these correspondences. In this section we suppose that the Shimura datum that we consider is defined by a group G which is semisimple of adjoint type. We suppose that the compact open subgroup K of $G(\mathbb{A}_{\mathrm{f}})$ is a product $\prod_p K_p$ where K_p are compact open subgroups of $G(\mathbb{Q}_p)$ and we moreover assume K to be neat (see [28] for definition). This condition implies that faithful representations of G induce variations of Hodge structure over $\mathrm{Sh}_K(G,X)_{\mathbb{C}}$. We consider a closed irreducible subvariety Z of $\mathrm{Sh}_K(G,X)_{\mathbb{C}}$. Let X^+ be some connected component of X. We suppose that the connected component S of $\mathrm{Sh}_K(G,X)_{\mathbb{C}}$ containing Z is the image of $X^+\times 1$ in $\mathrm{Sh}_K(G,X)_{\mathbb{C}}$. Since G is semisimple of adjoint type, we have $G(\mathbb{Q})_+ = G(\mathbb{R})^+\cap G(\mathbb{Q})$. The result is the following.

Proposition 5.2.1 *Keep the above assumptions (G, X, K, Z etc.) For almost all primes p the following holds. Let $s\in G(\mathbb{Q}_p)$ and let $T_s^1 = \coprod_{i\in I} T^0_{q_i^{-1}}$ be the decomposition of T_s^1 into a union of irreducible correspondences $T^0_{q_i^{-1}}$ as in 5.1.5. Then each $T^1_{q_i^{-1}}(Z)$ is irreducible.*

Proof. The idea of the proof is to exploit the relation between the "generic Mumford-Tate group on Z" and the algebraic monodromy group which is the Zariski closure of the image in it of the fundamental group $\pi_1(Z)$ acting on the locally constant sheaf underlying the variation of Hodge structure over the smooth locus of Z. In our case (the group G is semisimple and Z is Hodge generic) André's theorem allows us to say that these two groups are the same. The theorem of Nori stated after the proof of this proposition allows us to conclude that $\pi_1(Z)$ acts transitively on the fibers of the covering

$$q_i^{-1}\Gamma q_i \cap \Gamma\backslash X^+ \to \Gamma\backslash X^+$$

which will prove the irreducibility.

Let p be a prime and s an element of $G(\mathbb{Q}_p)$. Let $T_s^1 = \coprod_{i\in I} T^0_{q_i^{-1}}$ be the decomposition of T_s^1 into a union of irreducible correspondences $T^0_{q_i^{-1}}$, each being given by some q_i in $G(\mathbb{Q})^+$ as in 5.1.5. Let us take some i in I and $q := q_i^{-1}$ the corresponding element of $G(\mathbb{Q})^+$ defining $T^0_{s,i}$ (the element q is defined up to right and left multiplication by the elements of Γ). We will prove that $T_q^0(Z)$ is irreducible if p is larger then some prime p_0.

Let $\pi : \Gamma\cap q^{-1}\Gamma q\backslash X^+ \to \Gamma\backslash X^+$ be the natural map induced by the inclusion $\Gamma\cap q^{-1}\Gamma q \subset \Gamma$. Let $Z_q := \pi^{-1}(Z)$. To prove that $T_q^0(Z)$ is irreducible, it suffices to prove that Z_q is irreducible. Let Z^{sm} denote the smooth locus of Z. Let $x \in Z^{\mathrm{sm}}$ be a nonsingular point which is Hodge generic (it does not lie on a proper subvariety of Hodge type of $\Gamma\backslash X^+$). The (topological) fundamental group $\pi_1(Z^{\mathrm{sm}}, x)$ acts on $\pi^{-1}(x)$. The irreducibility of Z_q is equivalent to the property that this action is transitive. We will hence prove the transitivity. Let $\mathcal{V}(\rho)$ be the polarized variation of Hodge structure over S that we have constructed in 2.6 from a faithful representation ρ of G on a $\mathbb{Q}$-vector space V. A choice of a Γ-invariant lattice $V_{\mathbb{Z}}$ in V gives a variation of $\mathbb{Z}$-Hodge structure that we denote $\mathcal{V}(\rho)_{\mathbb{Z}}$. We define $G_{\mathbb{Z}}$ to be the Zariski closure of G in $\mathrm{GL}(V_{\mathbb{Z}})$. We choose an isomorphism between the fibre $\mathcal{V}(\rho)_{\mathbb{Z},x}$ of $\mathcal{V}(\rho)_{\mathbb{Z}}$ above x and $V_{\mathbb{Z}}$. Let $\xi : \pi_1(Z^{\mathrm{sm}}, x) \to \mathrm{GL}(V_{\mathbb{Z}})$ be the monodromy representation. A theorem of André (2.3.3) implies that the Zariski closure in $\mathrm{GL}(V_{\mathbb{Z}})$ of $\xi(\pi_1(Z^{\mathrm{sm}}(\mathbb{C}), x))$ is $G_{\mathbb{Z}}$. Since K is neat, Γ acts without fixed points on X^+, hence $\pi_1(S, x) = \Gamma$ and the image of Γ by the monodromy representation is a finitely generated Zariski-dense subgroup of $G(\mathbb{Z})$.

The covering $Z_q^{\mathrm{sm}} \to Z^{\mathrm{sm}}$ is defined by the $\pi_1(Z^{\mathrm{sm}}, x)$-set $\Gamma/\Gamma\cap q^{-1}\Gamma q$. From the fact that $q \in KsK \cap G(\mathbb{Q})$ and that $s \in G(\mathbb{Q}_p)$, it follows that there exists an integer $m \geq 1$ such that the group $\Gamma \cap q^{-1}\Gamma$ contains the kernel of the natural map $\Gamma \to G(\mathbb{Z}/p^m\mathbb{Z})$. Now we apply a theorem of Nori ([26, thm 5.3]) stated below and the remark that follows its statement. It implies that the closure in $\mathrm{GL}(V_{\mathbb{Z}} \otimes \widehat{\mathbb{Z}})$ of the image of $\pi_1(Z^{\mathrm{sm}}(\mathbb{C}), x)$ by the monodromy representation is open in the closure of $G(\mathbb{Z})$ in $\mathrm{GL}(V_{\mathbb{Z}} \otimes \widehat{\mathbb{Z}})$. Hence there exists a prime p_0 such that if p is larger than p_0, then for any $m \geq 1$, the images in $G(\mathbb{Z}/p^m\mathbb{Z})$ of Γ and of $\pi_1(Z^{\mathrm{sm}}(\mathbb{C}), z)$ are the same. Hence $\pi_1(Z^{\mathrm{sm}}, z)$ acts transitively on $\pi^{-1}(x)$. The irreducibility of Z_q follows. $\square$

Theorem 5.2.2 (Nori) *Let π be a finitely generated subgroup of $\mathrm{GL}_n(\mathbb{Z})$. Let A be its Zariski closure in $\mathrm{GL}_{n,\mathbb{Z}}$. Suppose that $A(\mathbb{C})$ has a finite fundamental group, then the closure of π in $\mathrm{GL}_n(\widehat{\mathbb{Z}})$ is open in the closure of $A(\mathbb{Z})$ in $\mathrm{GL}_n(\widehat{\mathbb{Z}})$.*

Remark 5.2.3 We apply this theorem to the case of semisimple group. The set of complex valued points of such a group is a semisimple complex Lie group. The fundamental group of such a group is finite by [32, thm. 2 (c)]

Remark 5.2.4 The condition that Z contains a special point in the above statement can not be omitted. Indeed, consider the Shimura variety defined by the Shimura datum $(\mathrm{GL}_2, \mathbb{H}^{\pm})$ where $\mathbb{H}^{\pm}$ is the union of the upper and lower half complex plane and let $K := \mathrm{GL}_2(\widehat{\mathbb{Z}})$. Let S be the connected component of $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$ isomorphic to $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$. Let x be a Hodge generic point in S. Actually any nonspecial point in $\mathbb{C}$ is Hodge generic. For any prime p, the transform $T_p(x)$ of x by the Hecke operator T_p defined by the matrix $g := \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$ consists of $p+1$ points hence is not irreducible.

5.3 Density of Hecke orbits

In this section we prove a result about the density of Hecke orbits in Shimura varieties. We are actually interested in the orbits of the irreducible correspondences T_q^0 described in the first section. We actually prove that for suitably chosen q in $G(\mathbb{Q})$, for any point x of S, the set $\bigcup_{m,n} T_q^{0m} T_{q^{-1}}^{0n}(x)$ is dense in S for the Archimedean topology.

Theorem 5.3.1 *Let (G, X) be a Shimura datum with G semisimple algebraic group of adjoint type. Let K be a compact open subgroup of $G(\mathbb{A}_{\mathrm{f}})$ that is a product of compact open subgroups K_p in $G(\mathbb{Q}_p)$. Let X^+ be a connected component of X and let S be the image of $X^+ \times 1$ in $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$. Then all but finitely many primes p have the following property. Let q be an element of $G(\mathbb{Q})^+$. Suppose that for any prime $l \neq p$ the image of q in $G(\mathbb{Q}_l)$ is in K_l and that no image of q under a projection to a simple factor of $G_{\mathbb{Q}_p}$ is contained in a compact subgroup. Then, the for any point x in S, the set $\bigcup_{m,n} T_q^{0m} T_{q^{-1}}^{0n}(x)$ is dense in S for the Archimedean topology.*

Proof. The idea of the proof is to reduce ouselves to the case where the group G is simply connected (to work in the simply connected cover of G). The reason for doing this is that we will apply results about topology of the set of points of such groups over local fields and adele-valued points like the strong approximation theorem.

Let q be an element of $G(\mathbb{Q})^+$ satisfying the conditions of the theorem and let s be a point of S. Let $\rho : \widetilde{G} \to G$ be the universal covering of G. Let us choose some $\mathbb{Z}$-structure on G and $\widetilde{G}$. Write $X^+ = G(\mathbb{R})^+/K_\infty$ where K_∞ is a maximal compact subgroup of $G(\mathbb{R})^+$. The group $\widetilde{G}(\mathbb{R})$ is connected by the Proposition [30, Prop. 7.6] and its image by ρ in $G(\mathbb{R})$ is $G(\mathbb{R})^+$. Then $\widetilde{K}_\infty := \rho^{-1}(K_\infty)$ is a maximal compact subgroup of $\widetilde{G}(\mathbb{R})$ and ρ induces an isomorphism between $\widetilde{G}(\mathbb{R})/\widetilde{K}_\infty$ and X^+. Let $\widetilde{K} := \rho^{-1}(K)$, then $\widetilde{K}$ is a compact open subgroup of $\widetilde{G}(\mathbb{A}_{\mathrm{f}})$ which is a product $\widetilde{K} = \prod_p \widetilde{K}_p$ of compact open subgroups of $\widetilde{G}(\mathbb{Q}_p)$. Since $\widetilde{K}$ is a compact open subgroup of $\widetilde{G}(\mathbb{A}_{\mathrm{f}})$, for p large enough $\widetilde{K}_p$ is $\widetilde{G}(\mathbb{Z}_p)$. Then, from [38, 3.9 and 3.8.2], it follows that for p large enough $\widetilde{K}_p$ is a maximal compact subgroup of $\widetilde{G}(\mathbb{Q}_p)$. We assume that p is so large that $\widetilde{K}_p$ is a maximal compact subgroup of $\widetilde{G}(\mathbb{Q}_p)$. Then $\widetilde{S} := \widetilde{G}(\mathbb{Q})\backslash X^+ \times \widetilde{G}(\mathbb{A}_{\mathrm{f}})/\widetilde{K}$ is a finite covering of S. Let $\widetilde{s}$ be a point in the inverse image of s. Let F be the kernel of ρ (viewed as a morphism of algebraic groups), then F is a finite group scheme of order n, hence $\mathrm{H}^1(\mathbb{Q}, F)$ is killed by n. Let t be an element of $\widetilde{G}(\mathbb{Q})$ in $\rho^{-1}(q^n)$. The correspondence on $\widetilde{S}$ given by left multiplication by t is the same as the correspondence given by right multiplication by t^{-1} on $G(\mathbb{A}_{\mathrm{f}})$. Let $\widetilde{T}_t$ be this correspondence. Obviously, the image in S of $\bigcup_{m,n} \widetilde{T}_t^m \widetilde{T}_{t^{-1}}^n(\widetilde{s})$ is contained in $\bigcup_{m,n} T_q^{0m} T_{q^{-1}}^{0n}(s)$

The density of $\bigcup_{m,n} \widetilde{T}_t^m \widetilde{T}_{t^{-1}}^n(\widetilde{s})$ implies the density of $\bigcup_{m,n} T_q^{0m} T_{q^{-1}}^{0n}(s)$ by lemma 5.4.5 applied to the covering $\widetilde{S} \to S$. Let $x \times 1$ be a representative in $X^+ \times \widetilde{G}(\mathbb{A}_{\mathrm{f}})$ for the point $\widetilde{s} \in \widetilde{S}$. Since for any $l \neq p$, the image of t in $\widetilde{G}(\mathbb{Q}_l)$ is in $\widetilde{K}_l$, the set $\bigcup_{m,n} \widetilde{T}_t^m \widetilde{T}_{t^{-1}}^n(\widetilde{s})$ is the image in $\widetilde{S}$ of $x \times H$ where H is the subgroup of $\widetilde{G}(\mathbb{Q}_p)$ generated by $\widetilde{K}_p$ and the image t_p of t in $\widetilde{G}(\mathbb{Q}_p)$.

Lemma 5.3.2 *The group H is $\widetilde{G}(\mathbb{Q}_p)$.*

Proof. Let $\widetilde{G} = \widetilde{G}_1 \times \cdots \times \widetilde{G}_m$ be the decomposition of $\widetilde{G}$ into a product of simple factors over $\mathbb{Q}_p$. Let $t_{p,i}$ be the image of t_p in $\widetilde{G}_i(\mathbb{Q}_p)$. Since $\widetilde{K}_p$ is a maximal compact subgroup of $\widetilde{G}_i(\mathbb{Q}_p)$, $\widetilde{K}_p = \widetilde{K}_{p,i} \times \cdots \times \widetilde{K}_{p,m}$ where $\widetilde{K}_{p,i}$ are maximal compact subgroups of $\widetilde{G}_i(\mathbb{Q}_p)$. By 5.4.3, for any i, the group generated by $t_{p,i}$ and $\widetilde{K}_{p,i}$ is $\widetilde{G}_i(\mathbb{Q}_p)$ (indeed this group is open because it contains an open subgroup $\widetilde{K}_{p,i}$ which is open and it is not compact because $t_{p,i}$ is not contained in $\widetilde{K}_{p,i}$) . This proves that all the projections of H to the $\widetilde{G}_i(\mathbb{Q}_p)$ are surjective. One easily checks that the intersections $H \cap \widetilde{G}_i(\mathbb{Q}_p)$ are normal (5.4.1) subgroups of $\widetilde{G}_i(\mathbb{Q}_p)$. Since in a semisimple group the normalizer of an open compact subgroup is compact (5.4.2) , any intersection $H \cap \widetilde{G}_i(\mathbb{Q}_p)$ is equal to $\widetilde{G}_i(\mathbb{Q}_p)$. It follows that $H = \widetilde{G}(\mathbb{Q}_p)$ $\square$

To prove the density of the orbit, it now suffices to prove that the image $\overline{x \times \widetilde{G}(\mathbb{Q}_p)}$ of $x \times \widetilde{G}(\mathbb{Q}_p)$ in $\widetilde{S}$ is dense in $\widetilde{S}$. Let x^+ be an element of $\widetilde{G}(\mathbb{A})$ lying above $x \times 1$. Then $\overline{x \times \widetilde{G}(\mathbb{Q}_p)}$ is the same as the image in $\widetilde{S}$ of the subset $\widetilde{G}(\mathbb{Q})x^+\widetilde{G}(\mathbb{Q}_p)$ of $\widetilde{G}(\mathbb{A})$. Since $\widetilde{G}(\mathbb{Q}_p)$ is a normal subgroup of $\widetilde{G}(\mathbb{A})$, we have $\widetilde{G}(\mathbb{Q})x^+\widetilde{G}(\mathbb{Q}_p) = \widetilde{G}(\mathbb{Q})\widetilde{G}(\mathbb{Q}_p)x^+$. Since any simple factor of

$\widetilde{G}(\mathbb{Q}_p)$ contains an element which is not contained in any compact subgroup, any simple factors of $\widetilde{G}(\mathbb{Q}_p)$ is not compact. The strong approximation theorem (see 5.4.6) then proves the density of $\widetilde{G}(\mathbb{Q})\widetilde{G}(\mathbb{Q}_p)$ in $\widetilde{G}(\mathbb{A})$. The result follows. $\square$

Remark 5.3.3 Richard Pink suggested to me that this theorem is, although it is correct and sufficient for our purposes, is not optimal. One can actually prove a stronger result (without assuming that Hecke correspondences are concentrated at a single prime p). He suggests the following theorem, from which 5.3.1 follows (indeed, is easy to see that a q verifying the conditions of 5.3.1 verifies the conditions of the following theorem).

Theorem 5.3.4 (Pink) *Let G be a semisimple group over $\mathbb{Q}$. Consider the decomposition $G^{\mathrm{ad}} = \prod_{i=1}^{r} G_i^{\mathrm{ad}}$ into $\mathbb{Q}$-simple factors, denoting the projection $G \to G_i^{\mathrm{ad}}$ by π_i. Consider an arithmetic subgroup Γ and an element g of $G(\mathbb{Q})$. Let Γ_g be the subgroup generated by Γ and g. Assume that $\pi_i(\Gamma)$ has infinite index in $\pi_i(\Gamma_g)$ for every i. (In other words, $\pi_i(\Gamma_g)$ is not an arithmetic subgroup of $G_i^{\mathrm{ad}}(\mathbb{Q})$.) Then the closure of Γ_g in $G(\mathbb{R})$ contains the identity component $G(\mathbb{R})^+$.*

Example 5.3.5 The theorem above is a generalization of the lemma 3.4.3 of Chapter 3 where we considered the special case of the Shimura variety which is a product of two Shimura curves $S_1 \times S_2$ associated to indefinite quaternion algebras B_1 and B_2 over $\mathbb{Q}$. We have considered Hecke correspondences $T_p \times T_p$ defined by the element $q_1 \times q_2$ of $B_1 \times B_2$, whose image in $B_{1\mathbb{Q}_p} \times B_{2\mathbb{Q}_p} = \mathrm{GL}_2 \times \mathrm{GL}_2$ is $s_p \times s_p$ with $s_p = \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)$. We note that the method of the proof is exactly the same: we lift the square of the image of $q_1 \times q_2$ in the adjoint group to the simply connected covering (for PGL_2, it is SL_2) and then prove the density using the strong approximation theorem.

5.4 Some lemmas and theorems

In this section we state with proofs or references some lemmas and theorems used in the preceding section.

Lemma 5.4.1 *Let $H_1, \ldots, H_m$ be abstract groups. Let H be a subgroup of a product $H_1 \times \cdots \times H_m$. Assume that all the projections of H to H_i are surjective. Then, for any $i = 1, \cdots, m$, the intersection $H \cap H_i$ is a normal subgroup of H_i.*

Proof. Let i be an integer between 1 and m and let us prove that $H \cap H_i$ is a normal subgroup of H_i. Take $i = 1$ for simplicity. Let us prove that $H \cap H_1$ is a normal subgroup of H_1.

Let $(x, 1, \cdots, 1)$ be an element of $H \cap H_1$ and let y be an element of H_1. Let us prove that $(yxy^{-1}, 1, \cdots, 1)$ is in $H \cap H_1$. Since the projection from H to H_1 is surjective, there exists an element in H of the form $Y := (y, y_1, \ldots, y_m)$, then $(yxy^{-1}, 1, \cdots, 1) = Y(x, 1, \cdots, 1)Y^{-1}$ hence $(yxy^{-1}, 1, \cdots, 1)$ is in H_1. $\square$

Lemma 5.4.2 *Let G be a semisimple group over a local filed F. The normalizer of any compact open subgroup of $G(F)$ is compact.*

Proof. See [30, prop. 3.16]. $\square$

Theorem 5.4.3 *Let G be an almost simple simply connected algebraic group over a non-archimedian local field F. Then $G(F)$ doesn't contain a proper open non-compact subgroup of $G(F)$.*

Proof. If $G(F)$ is compact, then the result is obviously true. Assume now that $G(F)$ is not compact. The theorem [6, thm. 8.2] asserts that if G is anisotropic (contains no split torus) then $G(F)$ is compact. Hence G is isotropic. Let H be an open non-compact subgroup of $G(F)$. Let $G^+(F)$ denote the subgroup of $G(F)$ generated by the F-valued points of the unipotent radicals of the parabolic subgroups of G. By [30, prop. 7.5], $G^+(F) = G(F)$. We conclude that $H = G(F)$ using the following theorem [29, thm. (T)]

Theorem 5.4.4 (Tits) *Let G be an almost simple group over a non-archimedian local field F. Then any proper open subgroup of $G^+(F)$ is contained in a compact subgroup.*

This finishes the proof of the theorem. $\square$

Lemma 5.4.5 *Let $f : X \to Y$ be a continuous map of topological spaces. Let F be a subset of X, then $f(\overline{F}) \subset \overline{f(F)}$*

Proof. Since f is continuous, $f^{-1}(\overline{f(F)})$ is closed and it contains F, hence it contains $\overline{F}$. $\square$

Theorem 5.4.6 (Strong approximation theorem) *Let G be a semisimple simply connected algebraic group over $\mathbb{Q}$. If G is almost simple and p is a prime such that $G(\mathbb{Q}_p)$ is not compact, then $G(\mathbb{Q})G(\mathbb{Q}_p) \subset G(\mathbb{A})$ is dense in $G(\mathbb{A})$.*

Proof. The proof of this theorem can be found for example in [30, Thm 7.2]. $\square$

5.5 Characterization of subvarieties of Hodge type

In this section we use the preceding results in order to prove a characterization of subvarieties of Hodge type that will be used in chapter 7 in the proof of the main results. As usual we start with a Shimura datum (G, X), a compact open subgroup K of $G(\mathbb{A}_\mathrm{f})$ and an irreducible closed subvariety of $\mathrm{Sh}_K(G, X)_\mathbb{C}$ containing at least one special point. It is important that

We do not assume Z to be Hodge generic.

The notations G', Z', Z'^{ad}, S'^{ad} are those of the preceding section. Recall from the Section 4.4 from the previous Chapter that to prove that Z is of Hodge type we can assume that

1. The group G is semisimple of adjoint type
2. The variety Z'^{ad} is contained in the image S'^{ad} of $X'^+ \times 1$ in $\mathrm{Sh}_{K'^{\mathrm{ad}}}(G', X')_\mathbb{C}$ where K'^{ad} is a product $\prod_p K'^{\mathrm{ad}}_p$ of compact open subgroups K'^{ad}_p of $G'^{\mathrm{ad}}(\mathbb{Q}_p)$.
3. To prove that Z is of Hodge type, it suffices to prove that the variety Z'^{ad} is. The last statement is equivalent to proving that Z'^{ad} equals S'^{ad}.

The following proposition is the actual caracterizarition of subvarieties of Hodge type that we will use to conclude that with our assumptions, Z is of Hodge type.

Proposition 5.5.1 *Keep the above notations and assumptions. Suppose that there exists an s in $G'(\mathbb{Q}_p)$ for p so large that 5.2.1 and 5.3.1 hold, that none of the projections of s to the simple factors of $G'_{\mathbb{Q}_p}$ is contained in a compact subgroups and that s is such that*

$$Z' \subset T_s(Z')$$

(T_s is the Hecke correspondence defined by s on $\mathrm{Sh}_{K'}(G', X')_\mathbb{C}$). Then Z is of Hodge type.

Proof. Let s be an in the proposition, hence in particular $Z' \subset T_s(Z')$. From this inclusion it follows that $Z'^{\mathrm{ad}} \subset (T_s(Z'))^{\mathrm{ad}}$ where $T_s(Z')$ is the image of $(T_s(Z'))^{\mathrm{ad}}$ in $\mathrm{Sh}_{K'^{\mathrm{ad}}}(G'^{\mathrm{ad}}, X'^{\mathrm{ad}})$. Let s^{ad} be the image of s in $G'^{\mathrm{ad}}(\mathbb{Q}_p)$. It is clear that $(T_s(Z'))^{\mathrm{ad}}$ is contained in $T_{s^{\mathrm{ad}}}(Z'^{\mathrm{ad}})$. Hence

$$Z'^{\mathrm{ad}} \subset T_{s^{\mathrm{ad}}}(Z'^{\mathrm{ad}})$$

It follows from 5.1.5 that there exists some q in $G'^{\mathrm{ad}}(\mathbb{Q})$ which satisfies the conditions of the Theorems 5.2.1 and 5.3.1 such that $Z'^{\mathrm{ad}} \subset T^0_q(Z'^{\mathrm{ad}})$. Then 5.2.1 shows that

$$Z'^{\mathrm{ad}} = T^0_q(Z'^{\mathrm{ad}})$$

If we prove that

$$Z'^{\mathrm{ad}} = T^0_{q^{-1}}(Z'^{\mathrm{ad}})$$

then 5.3.1 will prove that $Z'^{\mathrm{ad}} = S'^{\mathrm{ad}}$ and hence Z is of Hodge type. It then suffices to prove that $Z'^{\mathrm{ad}} = T^0_{q^{-1}}(Z'^{\mathrm{ad}})$. But this is a formal consequence of the equality $Z'^{\mathrm{ad}} = T^0_q(Z'^{\mathrm{ad}})$. Indeed, let Γ^{ad} be the intersection of $G'^{\mathrm{ad}}(\mathbb{Q})^+$ with K'^{ad} where $G'^{\mathrm{ad}}(\mathbb{Q})^+ = G'^{\mathrm{ad}}(\mathbb{R})^+ \cap G'^{\mathrm{ad}}(\mathbb{Q})$. Then $S'^{\mathrm{ad}} = \Gamma^{\mathrm{ad}} \backslash X'^+$. We write $[x]$ for the image of the point x in X'^+ in S'^{ad}. Given $[x]$ in S'^{ad}, we have $T^0_s([x]) = \{[q\gamma x] : \gamma \in \Gamma^{\mathrm{ad}}\}$. Then

$$Z'^{\mathrm{ad}} = T^0_q(Z'^{\mathrm{ad}}) = \{[q\gamma z] : \gamma \in \Gamma^{\mathrm{ad}}, [z] \in Z'^{\mathrm{ad}}\}$$

It follows that

$$T^0_{q^{-1}}(Z'^{\mathrm{ad}}) = \{[q^{-1}\gamma'^{-1}\gamma' q\gamma'' z] : \gamma', \gamma'' \in \Gamma^{\mathrm{ad}}, [z] \in Z'^{\mathrm{ad}}\} = Z'^{\mathrm{ad}}$$

This finishes the proof that $T^0_q(Z'^{\mathrm{ad}}) = Z'^{\mathrm{ad}} = T^0_{q^{-1}}(Z'^{\mathrm{ad}})$. Now 5.3.1 shows that $Z'^{\mathrm{ad}} = S'^{\mathrm{ad}}$ and hence Z is of Hodge type. $\square$

Chapter 6

Lower bounds for Galois orbits (written by Bas Edixhoven)

The aim of this chapter is to give certain lower bounds for the sizes of Galois orbits of special points in a fixed Hecke orbit on a Shimura variety. To be precise, we will prove the following theorem.

Theorem 6.1 *Let (G, X) be a Shimura datum, with G semi-simple of adjoint type. Let K be a compact open subgroup of $G(\mathbb{A}_{\mathrm{f}})$. Via a suitable faithful representation we view G as a closed algebraic subgroup of $\mathrm{GL}_{n,\mathbb{Q}}$. Let x_0 in X be a special point, let M be its Mumford-Tate group, and let H' be the centralizer of M in $\mathrm{GL}_{n,\mathbb{Q}}$. Let F be a number field over which the Shimura variety $\mathrm{Sh}_K(G,X)_{\mathbb{C}}$ has a canonical model $\mathrm{Sh}_K(G,X)_F$ (for example, the splitting field of the torus M). For g in $G(\mathbb{A}_{\mathrm{f}})$, let $\overline{(x_0, g)}$ denote the $\overline{\mathbb{Q}}$-valued point of $\mathrm{Sh}_K(G,X)_F$ given by (x_0, g). Then there exist real numbers $c_1 > 0$ and $c_2 > 0$ such that for all g in $G(\mathbb{A}_{\mathrm{f}})$ we have:*

$$|\mathrm{Gal}(\overline{\mathbb{Q}}/F)\cdot\overline{(x_0,g)}| > c_1 \prod_{\{p \text{ prime} \mid g_p \notin H'(\mathbb{Q}_p)\mathrm{GL}_n(\mathbb{Z}_p)\}} c_2 p.$$

Let us note that varying F and K does not affect the statement of the theorem: if F' and K' satisfy the same hypotheses as F and K, then the sizes of the Galois orbits differ by a bounded factor, and $K' \cap K$ has finite index in both K and K'. In the course of the proof of Theorem 6.1 we will assume that M splits over F.

6.2 The "Shimura varieties" associated to H and M.

Let the notation be as in Theorem 6.1, and let H be the centralizer in G of M, i.e., $H = H' \cap G$. In order to prove the theorem, we first study the maps induced by the g in $G(\mathbb{A}_{\mathrm{f}})$ from the

Shimura variety $\mathrm{Sh}(M)$ associated to $(M, \{x_0\})$ to $\mathrm{Sh}_K(G, X)$. All these maps have their image contained in the Hecke orbit of $\overline{(x_0, 1)}$. This Hecke orbit is in bijection with $H(\mathbb{Q})\backslash G(\mathbb{A}_{\mathrm{f}})/K$, via the map $\overline{g} \mapsto \overline{(x_0, g)}$. To simplify notation, we will also write, for U a compact open subgroup of $H(\mathbb{A}_{\mathrm{f}})$, $\mathrm{Sh}_U(H)$ for $H(\mathbb{Q})\backslash H(\mathbb{A}_{\mathrm{f}})/U$. (Note that $(H, \{x_0\})$ is not a Shimura datum if H^{ad} is non-trivial; this is simply the third condition in the definition of what is a Shimura datum.) We note that $H(\mathbb{R})$ is compact: the kernel of the action of $G(\mathbb{R})$ on X consists precisely of the product of the compact factors, i.e., if $G_{\mathbb{R}}$ is the product of simple G_i's, then the kernel is the product of the $G_i(\mathbb{R})$ with $G_i(\mathbb{R})$ compact, and $H(\mathbb{R})$ is the stabilizer in $G(\mathbb{R})$ of the point x_0 in the hermitian manifold X.

We have, for g in $G(\mathbb{A}_{\mathrm{f}})$, the following maps:

(6.2.1) $$\mathrm{Sh}_{M(\mathbb{A}_{\mathrm{f}})\cap gKg^{-1}}(M) \longrightarrow \mathrm{Sh}_{H(\mathbb{A}_{\mathrm{f}})\cap gKg^{-1}}(H) \xrightarrow{\cdot g} H(\mathbb{Q})\backslash G(\mathbb{A}_{\mathrm{f}})/K,$$

with the first one induced by the inclusion of M into H and the second one by the inclusion of H into G, followed by right multiplication by g. The composed map is $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$-equivariant, hence useful for studying Galois orbits in the Hecke orbit.

Proposition 6.2.2 *Let g be in $G(\mathbb{A}_{\mathrm{f}})$, and consider the two maps in (6.2.1).*

1. *The second map is injective, and the fibers of the first map are finite, and with cardinality bounded uniformly in g.*

2. *If K is neat (see [5, §17] for a definition), then the first map is injective.*

Proof. Let us first prove that the second map is injective. This map can be split into two maps, first from $\mathrm{Sh}_{H(\mathbb{A}_{\mathrm{f}})\cap gKg^{-1}}(H)$ to $H(\mathbb{Q})\backslash G(\mathbb{A}_{\mathrm{f}})/gKg^{-1}$, simply induced by the injection of H in G, and then from $H(\mathbb{Q})\backslash G(\mathbb{A}_{\mathrm{f}})/gKg^{-1}$ to $H(\mathbb{Q})\backslash G(\mathbb{A}_{\mathrm{f}})/K$ via the isomorphism induced by right multiplication by g. Hence it suffices to show that the first map in this splitting is injective. Let us write this map out:

$$H(\mathbb{Q})\backslash H(\mathbb{A}_{\mathrm{f}})/(H(\mathbb{A}_{\mathrm{f}}) \cap U) \longrightarrow H(\mathbb{Q})\backslash G(\mathbb{A}_{\mathrm{f}})/U,$$

with $U = gKg^{-1}$. Let h_1 and h_2 be in $H(\mathbb{A}_{\mathrm{f}})$. Then h_1 and h_2 have the same image in $H(\mathbb{Q})\backslash G(\mathbb{A}_{\mathrm{f}})/U$ if and only if h_2 is in $H(\mathbb{Q})h_1U$ (in $G(\mathbb{A}_{\mathrm{f}})$). But that is equivalent to the existence of h in $H(\mathbb{Q})$ and u in U such that $h_2 = hh_1u$. But now note that such a u is in $H(\mathbb{A}_{\mathrm{f}})$, since it is $h_1^{-1}h^{-1}h_2$. Hence the condition is equivalent to the existence of h in $H(\mathbb{Q})$ and u in $H(\mathbb{A}_{\mathrm{f}}) \cap U$ such that $h_2 = hh_1u$. This finishes this part of the proof.

Let us now consider the first map. We remark that the second part of the Proposition we are proving implies the first part, as an arbitrary K contains a neat one with finite index. So let us assume that K is neat, and prove the injectiveness. Let us write U for $H(\mathbb{A}_{\mathrm{f}}) \cap gKg^{-1}$. The map we consider is the map of $M(\mathbb{A}_{\mathrm{f}})$-sets:

$$M(\mathbb{A}_{\mathrm{f}})/M(\mathbb{Q})(M(\mathbb{A}_{\mathrm{f}}) \cap U) \longrightarrow H(\mathbb{Q})\backslash H(\mathbb{A}_{\mathrm{f}})/U.$$

The image of this map is the $M(\mathbb{A}_{\mathrm{f}})$-set $M(\mathbb{A}_{\mathrm{f}})/((H(\mathbb{Q})U) \cap M(\mathbb{A}_{\mathrm{f}}))$. So what we want to show in fact is that the inclusion of $M(\mathbb{Q})(M(\mathbb{A}_{\mathrm{f}}) \cap U)$ into $(H(\mathbb{Q})U) \cap M(\mathbb{A}_{\mathrm{f}})$ is an equality.

To do so, let us consider the quotient (of algebraic groups) $H \to \overline{H} = H/M$ (recall that M is central in H, hence normal). The image $\overline{U}$ of U in $\overline{H}(\mathbb{A}_{\mathrm{f}})$ is neat. Since $H(\mathbb{R})$ is compact, $\overline{H}(\mathbb{R})$ is compact as well, as this depends only on the connected component (archimedean topology) of the identity element, and since the map from $H(\mathbb{R})$ to $\overline{H}(\mathbb{R})$ is surjective on identity components. Hence $\overline{H}(\mathbb{Q})$ is discrete in $\overline{H}(\mathbb{A}_{\mathrm{f}})$. Since $\overline{U}$ is compact, $\overline{H}(\mathbb{Q}) \cap \overline{U}$ is finite, hence torsion. But then the neatness of $\overline{U}$ says that $\overline{H}(\mathbb{Q}) \cap \overline{U}$ is trivial.

Now suppose that h is in $H(\mathbb{Q})$ and u in U such that hu is in $M(\mathbb{A}_{\mathrm{f}})$. Then, in $\overline{H}(\mathbb{A}_{\mathrm{f}})$, we have $\overline{h} \cdot \overline{u} = 1$, hence $\overline{h} = \overline{u} = 1$ in $\overline{H}(\mathbb{A}_{\mathrm{f}})$. But that means that h is in $M(\mathbb{Q})$ and u is in $M(\mathbb{A}_{\mathrm{f}}) \cap U$. $\square$

Proposition 6.2.2 enables us to replace the Galois orbit $\mathrm{Gal}(\overline{\mathbb{Q}}/F) \cdot \overline{(x_0, g)}$ in Theorem 6.1 by the Galois orbit $\mathrm{Gal}(\overline{\mathbb{Q}}/F) \cdot \overline{x_0}$ in $\mathrm{Sh}_{M(\mathbb{A}_{\mathrm{f}}) \cap gKg^{-1}}(M)$.

6.3 Reduction to a problem about representations of tori.

The aim of this section is to reduce the proof of Theorem 6.1 to a statement about representations of tori. We keep the notation of the previous section. Let E be the splitting field of the torus M, and let O_E be its ring of integers. Then the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ acts on $\mathrm{Sh}(M) = M(\mathbb{Q})\backslash M(\mathbb{A}_{\mathrm{f}})$ via its maximal abelian quotient, which we view, via class field theory, as a quotient of $(\mathbb{A} \otimes E)^*/(\mathbb{R} \otimes E)^{*,+}$. The action of $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ is then given via a surjective morphism of tori:

$$r \colon \mathrm{Res}_{E/\mathbb{Q}} \mathbb{G}_{\mathrm{m}E} \longrightarrow M.$$

We note that $(\mathbb{A} \otimes E)^*/(\mathbb{R} \otimes E)^{*,+}$ is the product of $(\mathbb{A}_{\mathrm{f}} \otimes E)^*$ and the finite group of connected components of $(\mathbb{R} \otimes E)^*$. Hence the sizes of the $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$-orbits in the $\mathrm{Sh}_{M(\mathbb{A}_{\mathrm{f}}) \cap gKg^{-1}}(M)$ are, up to a bounded factor, the sizes of the $(\mathbb{A}_{\mathrm{f}} \otimes E)^*$-orbits. Since $E^{*,+}$ in $(\mathbb{A}_{\mathrm{f}} \otimes E)^*$ acts trivially, and since the class group $E^*\backslash(\mathbb{A}_{\mathrm{f}} \otimes E)^*/(\hat{\mathbb{Z}} \otimes O_E)^*$ is finite, proving the lower bound we want for the $(\mathbb{A}_{\mathrm{f}} \otimes E)^*$-orbits is equivalent to proving it for the $(\hat{\mathbb{Z}} \otimes O_E)^*$-orbits.

The image of $(\hat{\mathbb{Z}} \otimes O_E)^*$ in $M(\mathbb{A}_{\mathrm{f}})$ is a compact subgroup. In order to reduce our problem on $(\hat{\mathbb{Z}} \otimes O_E)^*$-orbits in $M(\mathbb{Q})\backslash M(\mathbb{A}_{\mathrm{f}})/(M(\mathbb{A}_{\mathrm{f}}) \cap gKg^{-1})$ to a problem on $(\hat{\mathbb{Z}} \otimes O_E)^*$-orbits in $M(\mathbb{A}_{\mathrm{f}})/(M(\mathbb{A}_{\mathrm{f}}) \cap gKg^{-1})$ we use the following lemma.

Lemma 6.3.1 *Let U be a compact subgroup of $M(\mathbb{A}_\mathrm{f})$. Then the index of $U \cap gKg^{-1}$ in $U \cap (M(\mathbb{Q})(M(\mathbb{A}_\mathrm{f}) \cap gKg^{-1})$ is bounded uniformly in g (g ranging over $G(\mathbb{A}_\mathrm{f})$). In particular, there is a real number $c > 0$ such that, under the map of $M(\mathbb{A}_\mathrm{f})$-sets from $M(\mathbb{A}_\mathrm{f})/(M(\mathbb{A}_\mathrm{f})\cap gKg^{-1})$ to $M(\mathbb{Q})\backslash M(\mathbb{A}_\mathrm{f})/(M(\mathbb{A}_\mathrm{f})\cap gKg^{-1})$, the sizes of the U-orbits change by a factor at least c.*

Proof. We may replace U and K by open subgroups of finite index. Hence we may assume that the images of U and K in $M(\mathbb{Q}_3)$ and $G(\mathbb{Q}_3)$ are pro-3-groups. Suppose then that m_1 and m_2 are in $M(\mathbb{Q})$ and $M(\mathbb{A}_\mathrm{f}) \cap gKg^{-1}$, respectively, such that m_1m_2 is in U. Then m_1 is in $U(M(\mathbb{A}_\mathrm{f}) \cap gKg^{-1})$. But this last group has finite intersection with $M(\mathbb{Q})$, since it is compact and since $M(\mathbb{Q})$ is discrete in $M(\mathbb{A}_\mathrm{f})$ (as $M(\mathbb{R})$ is compact). By our assumptions, $U(M(\mathbb{A}_\mathrm{f}) \cap gKg^{-1})$ is neat (because it is pro-3 at 3). Hence $M(\mathbb{Q})$ has even trivial intersection with it. So $m_1 = 1$, which proves what we want. □

Lemma 6.3.2 *Let U be a compact open subgroup of $M(\mathbb{A}_\mathrm{f})$ which is the product of compact open subgroups U_p in $M(\mathbb{Q}_p)$. Then there exists an integer c such that, for all p, the intersection of $r((\mathbb{Z}_p\otimes O_E)^*)$ and U_p in $M(\mathbb{Q}_p)$ has index at most c in $r((\mathbb{Z}_p\otimes O_E)^*)$ and in U_p. In particular, the sizes of the $(\mathbb{Z}_p\otimes O_E)^*$-orbits and of the U_p-orbits in the $M(\mathbb{A}_\mathrm{f})/M(\mathbb{A}_\mathrm{f})\cap gKg^{-1}$ are bounded uniformly in terms of each other.*

Proof. For all p, $r((\mathbb{Z}_p\otimes O_E)^*)$ is a compact open subgroup of $M(\mathbb{Q}_p)$. We choose models over $\mathbb{Z}$ for M and for $\mathrm{Res}_{E/\mathbb{Q}}\mathbb{G}_{\mathrm{m}E}$ (for the last one we take $\mathrm{Res}_{O_E/\mathbb{Z}}\mathbb{G}_{\mathrm{m}O_E}$). For almost all p, these models give group schemes over $\mathbb{Z}_p$ and r induces a morphism of group schemes over $\mathbb{Z}_p$, with a smooth kernel. For almost all p, $U_p = M(\mathbb{Z}_p)$ and $r((\mathbb{Z}_p \otimes O_E)^*)$ is contained in $M(\mathbb{Z}_p)$ with an index bounded independently of p. To see that these indices are uniformly bounded, consider the exact sequence:

$$r((\mathbb{Z}_p \otimes O_E)^*) \hookrightarrow M(\mathbb{Z}_p) \longrightarrow \mathrm{H}^1(\mathbb{Z}_p, \ker(r)),$$

and use Lang's Theorem ([21]) and that the group of connected components of $\ker(r)_{\overline{\mathbb{F}}_p}$ has uniformly bounded order. □

With notation as in Lemma 6.3.2, the U-orbits in the $M(\mathbb{A}_\mathrm{f})/M(\mathbb{A}_\mathrm{f}) \cap gKg^{-1}$ are products over all primes, as both U and K are such products. Hence the proof of Theorem 6.1 is now reduced to the following proposition, that will be proved in the next section.

Proposition 6.3.3 *Let M be a torus in some $\mathrm{GL}_{n,\mathbb{Q}}$, and let H be its centralizer. Let U be a compact open subgroup of $M(\mathbb{A}_\mathrm{f})$. Assume that U is a product of compact open subgroups*

U_p of $M(\mathbb{Q}_p)$. Then there exists $c > 0$ such that for p prime and g in $\mathrm{GL}_n(\mathbb{Q}_p)$ but not in $H(\mathbb{Q}_p)\mathrm{GL}_n(\mathbb{Z}_p)$, we have:

$$|U_p/U_p \cap g\mathrm{GL}_n(\mathbb{Z}_p)g^{-1}| > cp.$$

Equivalently: the U_p-orbits in $\mathrm{GL}_n(\mathbb{Q}_p)/\mathrm{GL}_n(\mathbb{Z}_p)$ that are not in the $H(\mathbb{Q}_p)$-orbit of the identity element have cardinality at least cp.

6.4 Lattices in representations of tori.

The aim of this section is to prove Proposition 6.3.3. We start by rephrasing the proposition in terms of lattices.

Proposition 6.4.1 *Let M be a torus over $\mathbb{Q}$, let V be a finite dimensional $\mathbb{Q}$-vector space with a faithful M-action. Let H be the centralizer in $G := \mathrm{GL}(V)$. Let $V_\mathbb{Z}$ be a lattice in V. This induces $\mathbb{Z}$-structures on G, M and H by taking Zariski closures in $\mathrm{GL}(V_\mathbb{Z})$. For p prime, the set $\mathcal{L}_p := G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$ is the set of $\mathbb{Z}_p$-lattices in $V_{\mathbb{Q}_p}$. There exists $c > 0$ such that for p prime and L in $\mathcal{L}_p$ not in the $H(\mathbb{Q}_p)$-orbit of $V_{\mathbb{Z}_p}$, we have $|M(\mathbb{Z}_p){\cdot}L| > cp$.*

Remark 6.4.2 In the proof we will see that, for all p such that $M_{\mathbb{Z}_p}$ is a torus, the $H(\mathbb{Q}_p)$-orbit in question is exactly the set of fixed points in $\mathcal{L}_p$ of the group scheme $M_{\mathbb{Z}_p}$, in the following sense. An element L of $\mathcal{L}_p$ is called fixed by $M_{\mathbb{Z}_p}$ if for all finite extensions K of $\mathbb{Q}_p$ the lattice L_{O_K} is fixed by all elements of $M_{\mathbb{Z}_p}(O_K)$. This is an important point, since it underlies the principle of the proof of the proposition. Namely, for a lattice L that is not fixed by $M_{\mathbb{Z}_p}$, we consider a suitable non-trivial $M_{\mathbb{F}_p}$-orbit, and only then study what happens with $\mathbb{F}_p$-valued points.

Proof. We start with noting that, by taking c small enough, we can assume p to be large enough, so that a finite set of "bad" primes can be avoided. Let $X := \mathrm{Hom}_{\overline{\mathbb{Q}}}(M_{\overline{\mathbb{Q}}}, \mathbb{G}_{\mathrm{m}\,\overline{\mathbb{Q}}})$ be the character group of M, which we view as a $\mathbb{Z}$-module (free of finite rank) with a continuous $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action. Let E be the splitting field in $\overline{\mathbb{Q}}$ of M, i.e., the finite Galois extension of $\mathbb{Q}$ in $\overline{\mathbb{Q}}$ such that $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on X with image $\Gamma := \mathrm{Gal}(E/\mathbb{Q})$. The action of $M_\mathbb{Q}$ on V corresponds to an X-grading of V_E such that, for all γ in Γ and all χ in X we have: $\gamma V_{E,\chi} = V_{E,\gamma\chi}$. (To see this, use the representation theory of split tori as in [16, Exp. I, §4.7], and finite etale descent of quasi-coherent modules as in [7, Ch. 6].)

Let p be a prime such that $M_{\mathbb{Z}_p}$ is a torus (in particular, this implies that p is not ramified in E). Note that this condition on p excludes only finitely many primes. We choose a place of E above p, i.e., a factor E_p of $\mathbb{Q}_p \otimes E$. Let Γ_p be the corresponding decomposition subgroup of Γ, i.e., $\Gamma_p = \mathrm{Gal}(E_p/\mathbb{Q}_p)$. The fact that $\mathbb{Q} \to E$ is unramified at p implies that $\mathbb{Q}_p \to E_p$ is unramified, and that Γ_p is a cyclic subgroup of Γ, with a canonical generator: the Frobenius at p. We let $O_{E,p}$ be the ring of integers of E_p. For L a $\mathbb{Z}_p$-lattice in $V_{\mathbb{Q}_p}$ and for χ in X, we define $L_\chi := L_{O_{E,p}} \cap V_{E,\chi}$.

Lemma 6.4.3 *With the above notations and assumptions, the morphism of $O_{E,p}$-modules $\oplus_\chi L_\chi \to L_{O_{E,p}}$ is injective, with finite cokernel. It is surjective if and only if L is in the $H(\mathbb{Q}_p)$-orbit of $V_{\mathbb{Z}_p}$.*

Proof. We begin by noting that to give an $M_{\mathbb{Z}_p}$-action on a $\mathbb{Z}_p$-module L is to give an X-grading $L_{O_{E,p}} = \oplus_\chi L_\chi$ such that for all γ in Γ_p we have $\gamma L_\chi = L_{\gamma\chi}$ (as before, use finite etale descent, but now from O_{E_p} to $\mathbb{Z}_p$). In this description, $M_{O_{E,p}}$ acts on L_χ via the character χ. For example, the $M_{\mathbb{Z}_p}$-action on $V_{\mathbb{Z}_p}$ gives such an X-grading. It follows that for each L in the $H(\mathbb{Q}_p)$-orbit of $V_{\mathbb{Z}_p}$ the morphism $\oplus_\chi L_\chi \to L_{O_{E,p}}$ is an isomorphism.

Now suppose that L is a $\mathbb{Z}_p$-lattice in $V_{\mathbb{Q}_p}$ such that the morphism $\oplus_\chi L_\chi \to L_{O_{E,p}}$ is an isomorphism. Then we have an $M_{\mathbb{Z}_p}$-action on L, compatible with the $M_{\mathbb{Q}_p}$-action on $L_{\mathbb{Q}_p} = V_{\mathbb{Q}_p}$. All we have to show now is that L and $V_{\mathbb{Z}_p}$ are isomorphic as representations of $M_{\mathbb{Z}_p}$. The easiest way to do this is to use the classification of representations of $M_{\mathbb{Z}_p}$ on $\mathbb{Z}_p$-modules that are free of finite rank. For each such a representation W, one gets a direct sum decomposition $W = \oplus_x W_x$, indexed by the elements x of $\Gamma_p\backslash X$, such that for each x one has $O_{E_p} \otimes_{\mathbb{Z}_p} W_x = \oplus_{\chi\in x} W_\chi$ (finite etale descent from $O_{E,p}$ to $\mathbb{Z}_p$). The classification says that two representations W and W' are isomorphic if and only if for each x the ranks of W_x and W'_x are equal. It would be convenient to have a reference for this, but we could not find one.

A down to earth proof in our case can be given as follows. Let W and W' be given, such that for each x the ranks for W_x and W'_x are equal. Of course, it suffices to show that, for each x, W_x and W'_x are isomorphic as representations of $M_{\mathbb{Z}_p}$. Let χ be in X, let x be its Γ-orbit, and let W and W' of equal rank be given with $W = W_x$, $W' = W'_x$. We may replace E_p by its minimal subextension over which χ is defined. Then all elements of x are defined over E_p, since Γ_p is commutative. Now one just chooses any isomorphism f of $O_{E,p}$-modules from W_χ to W'_χ. For γ in Γ_p, one then gets an isomorphism γg from $W_{\gamma\chi}$ to $W'_{\gamma\chi}$. The direct sum of these isomorphisms is an isomorphism $F\colon W_{O_{E,p}} \to W'_{O_{E,p}}$ that is compatible with the descent datum to $\mathbb{Z}_p$. Hence it is an isomorphism from W to W'.

Another way to prove the classification is to use the infinitesimal lifting properties for morphisms of a torus into a smooth group scheme as for example in [16, Exp. IX]. These lifting properties, applied to the GL_n, imply that each finite dimensional representation of $M_{\mathbb{F}_p}$ has a unique lift to $\mathbb{Z}_p$. Hence this reduces the proof of the classification to $\mathbb{F}_p$, in which case we consider it standard. $\square$

We continue the proof of Proposition 6.4.1. Let now L be given such that it is not in the $H(\mathbb{Q}_p)$-orbit of $V_{\mathbb{Z}_p}$. Define $L' := (\oplus_\chi L_\chi) \cap L_{\mathbb{Q}_p}$ (intersection in L_{E_p}). Then L' is a sublattice of L, and it has the property that $L'_{O_{E,p}}$ is the direct sum of the L'_χ, and moreover, $L_\chi = L'_\chi$ for each χ (again, use finite etale descent). By the previous lemma, L' is in the $H(\mathbb{Q}_p)$-orbit of $V_{\mathbb{Z}_p}$. In fact, it is the biggest sub-$\mathbb{Z}_p$-module of L with that property. We have to prove that the $M(\mathbb{Z}_p)$-orbit of L in $\mathcal{L}_p$ has at least cp elements, for some $c > 0$ independent of p. Since $H(\mathbb{Q}_p)$ commutes with $M(\mathbb{Z}_p)$, we may as well replace L by hL, with h in $H(\mathbb{Q}_p)$ such that $hL' = V_{\mathbb{Z}_p}$. Then L strictly contains $V_{\mathbb{Z}_p}$, and for each χ we have $L_\chi = (V_{\mathbb{Z}_p})_\chi$.

Let W be the subspace $(pL \cap V_{\mathbb{Z}_p})/pV_{\mathbb{Z}_p}$ of the $\mathbb{F}_p$-vector space $V_{\mathbb{F}_p}$. By construction, W is non-zero, and has the property that, for each χ in X, $W_{\overline{\mathbb{F}}_p} \cap V_{\overline{\mathbb{F}}_p,\chi} = 0$. This implies that W is not stabilized by the action of $M_{\mathbb{F}_p}$. Let $M'_{\mathbb{F}_p}$ be the stabilizer in $M_{\mathbb{F}_p}$ of W. Lemma 6.4.4 below says that the order of the group of connected components of $M'_{\overline{\mathbb{F}}_p}$ is bounded independently of p and$L$, and that $M'_{\overline{\mathbb{F}}_p}$ is smooth for almost all p. Put $M''_{\mathbb{F}_p} := M_{\mathbb{F}_p}/M'_{\overline{\mathbb{F}}_p}$. Then $M''_{\mathbb{F}_p}$ is a non-trivial torus over $\mathbb{F}_p$, and by Lemma 6.4.5, the morphism $M(\mathbb{F}_p) \to M''_{\mathbb{F}_p}(\mathbb{F}_p)$ has its cokernel of order bounded independently of p and L. The proof of Proposition 6.4.1 is now finished by noting that $M''_{\mathbb{F}_p}(\mathbb{F}_p)$ has at least $p-1$ elements (see [26, Lemma 3.5]). □

Lemma 6.4.4 *Let k be an algebraically closed field, and T a k-torus. Let V be a finite dimensional k-vector space with an action by T. Then the set of stabilizers T_W, for W running through the set of subspaces of V, is finite. The set of groups of connected components of these stabilizers, up to isomorphism, is finite, and bounded in terms of the dimension of V and the set of characters of T that do occur in V.*

Proof. Let us consider the set S of subspaces W of a fixed dimension, call it d. Then we have a natural injection from S into $\mathbb{P}(\Lambda^d(V))$, compatible with the $T(k)$-action. The image of a W under this map is the line generated by $w_1 \wedge \cdots \wedge w_d$, where w is any k-basis of W. Hence the set of stabilizers of the elements of S is contained in the set of stabilizers of elements of $\mathbb{P}(\Lambda^d(V))$. This reduces the proof of the lemma to the case of one-dimensional subspaces (we replace V by $\mathbb{P}(\Lambda^d(V))$).

Let X be the character group of T, and let $V = \oplus_\chi V_\chi$ be the X-grading of V given by the T-action. Of course, almost all V_χ are zero. For v in V, we have $v = \sum_\chi v_\chi$, and we let $\mathrm{Supp}(v)$ be the set of χ with $v_\chi \neq 0$. For v non-zero, the stabilizer in T of the element kv in $\mathbb{P}(V)$ is the intersection of the kernels of the $\chi - \chi'$ with χ and χ' in $\mathrm{Supp}(v)$. Since the set of such differences is finite, the claim follows. □

Lemma 6.4.5 *Let $0 \to T' \to T \to T'' \to 0$ be a short exact sequence of tori over k, with k a finite field. Let Φ be the group of connected components of $T''_{\overline{k}}$. Then we have the exact* *sequence:*

$$0 \longrightarrow T'(k) \longrightarrow T(k) \longrightarrow T''(k) \longrightarrow \mathrm{H}^1(\mathrm{Gal}(\overline{k}/k), \Phi),$$

and the upper bound:

$$|\mathrm{H}^1(\mathrm{Gal}(\overline{k}/k), \Phi)| \leq |\Phi|.$$

Proof. One has, of course, the long exact sequence coming from etale cohomology:

$$0 \longrightarrow T'(k) \longrightarrow T(k) \longrightarrow T''(k) \longrightarrow \mathrm{H}^1(\mathrm{Gal}(\overline{k}/k), T'(\overline{k})).$$

We combine this with the sequence

$$\mathrm{H}^1(\mathrm{Gal}(\overline{k}/k), (T')^0(\overline{k})) \longrightarrow \mathrm{H}^1(\mathrm{Gal}(\overline{k}/k), T'(\overline{k})) \longrightarrow \mathrm{H}^1(\mathrm{Gal}(\overline{k}/k), \Phi)$$

coming from the short exact sequence

$$0 \longrightarrow (T')^0 \longrightarrow T' \longrightarrow T'/(T')^0 \longrightarrow 0.$$

Lang's Theorem ([21]) implies that $\mathrm{H}^1(\mathrm{Gal}(\overline{k}/k), (T')^0(\overline{k})) = 0$. The upper bound for $|\mathrm{H}^1(\mathrm{Gal}(\overline{k}/k), \Phi)|$ follows from the fact that this cohomology group is just the group of coinvariants for the action of $\mathrm{Gal}(\overline{k}/k)$ on Φ. $\square$

Chapter 7

Proof of the main results

In this chapter we eventually prove the main results 4.2 and 4.3 of the thesis using the results of the previous chapters.

7.1 Notations and assumptions

In this section we recall some notations and assumptions. We also make some first reduction steps. We start with the following data.

1. A Shimura datum (G, X).

2. A compact open subgroup K of $G(\mathbb{A}_{\mathrm{f}})$.

3. An irreducible closed subvariety Z of $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$ of dimension at least 1 which contains a Zariski-dense subset Σ of special points lying in one Hecke orbit. Recall that this last condition means that there exists an element x_0 in X such that Σ consists of points of the form $[x_0, g]$ with g varying in $G(\mathbb{A}_{\mathrm{f}})$. Actually another choice of x_0 changes x_0 by a $G(\mathbb{Q})$-conjugate.

We try to prove that Z is of Hodge type using the characterization 5.5.1. We will suceed when Z is a curve and when Z contains a Zariski dense subset of special points that verify the condition of 4.3. As we have seen in the Chapter 4 we can and do assume

1. That G is semisimple of adjoint type

2. That K is neat and is a product $K = \prod_p K_p$ of compact open subgroups of $G(\mathbb{Q}_p)$.

Futhermore we fix once and for all a faithful representation $G \hookrightarrow \mathrm{GL}_n$, this gives a $\mathbb{Z}$-structure to G. After possibly replacing K by a smaller subgroup, we can and do assume that for p large enough $K_p = G(\mathbb{Z}_p) \subset \mathrm{GL}_n(\mathbb{Z}_p)$.

We now recall the reduction that is done in Chapter 4, Section 4.4. Let (G', X') and K' be as given in the Proposition 4.2.5 (after the choice of a Hodge generic point in Z). We have a morphism

$$f : \mathrm{Sh}_{K'}(G', X')_{\mathbb{C}} \to \mathrm{Sh}_K(G, X)_{\mathbb{C}}$$

of Shimura varieties and we let Z' be an irreducible component of $f^{-1}(Z')$. Let X^+ be a connected component of X and let X'^+ be a connected component of X'. Let S and S' be the connected components of $\mathrm{Sh}_{K'}(G', X')_{\mathbb{C}}$ and $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$ that are images of $X'^+ \times 1$ and $X^+ \times 1$ in $\mathrm{Sh}_{K'}(G', X')_{\mathbb{C}}$ and $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$ respectively. To prove that Z is of Hodge type it suffices to do so under assumption that Z is contained in S, that Z' is contained in S'. We let Σ' be the inverse image of the set Σ in S'. Our criterion 5.5.1, which is the main result of Chapter 5, says that to prove that Z is of Hodge type it suffices to prove that there exists a suitable element g of $G(\mathbb{A}_{\mathrm{f}})$ such that $Z' \subset T_g(Z')$. As explained in the Chapter 4, to prove such an inclusion, we exploit Galois action and use (in the case where Z is a curve) the result of Edixhoven from chapter 6. Let us now introduce some more notation.

1. We fix an element x_0 of X' such that Σ consists of points $[x_0, g]$ with g varying in $G(\mathbb{A}_{\mathrm{f}})$.

2. We let M be the Mumford-Tate group of x_0, M is then a subtorus of G'.

3. We let F be a number field such that S and S' admit canonical models over F and Z' is defined and is geometrically irreducible over F. Then F splits M.

4. We let C be the centralizer of M in GL_n.

5. We let Σ' be the inverse image of Σ in S'.

6. For any point $[x', 1]$ in Σ' we let M' be the Mumford-Tate group of x'.

7. We let r denote the reciprocity morphism $\mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_{\mathrm{m}F} \to M$ attached to the element x_0 of X.

8. For any point $[x', 1]$ in Σ' we let r' be the reciprocity morphism $\mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_{\mathrm{m}F} \to M'$ attached to the element x' of X'.

Let us make some comments about these definitions.

1. The field F of the item 3 does actually exist. Indeed, S' admits a canonical model over some number field. One of the properties of a canonical model is that special points of S' are in $S'(\overline{\mathbb{Q}})$. Since Z' contains a Zariski dense subset of special points, Z' is defined over $\overline{\mathbb{Q}}$. To be more precise, there is a unique closed subscheme $Z'_{\overline{\mathbb{Q}}}$ of $S'_{\overline{\mathbb{Q}}}$ that gives Z' after base change to $\mathbb{C}$. Hence we actually can take a number field F such that S' admits a canonical model over F such that Z' is defined over F. Possibly replacing F by a finite extension, we can assume Z' is geometrically irreducible over F.

2. As was already remarked in Chapter 4, the points of Σ' do not in general lie in a finite number of Hecke orbits in S'. This will cause some technical complications.

3. The element x' can be written as $x' = qx$ for some q in $G(\mathbb{Q})$. Then the torus M' is actually qMq^{-1}. The morphism r' is the conjugate qrq^{-1}.

7.1.1 Some preliminary lemmas

In this section we prove some general lemmas that will be used in the proof.

The following lemma explains in some sense the significance of the condition $g_p \in C(\mathbb{Q}_p)\mathrm{GL}_n(\mathbb{Z}_p)$.

Lemma 7.1.2 *For almost all primes p the following holds. Let $[x_0, g]$ be a point in Σ such that g_p is in $C(\mathbb{Q}_p)\mathrm{GL}_n(\mathbb{Z}_p)$. Let $[x', 1]$ be a point in S' lying over $[x_0, g]$ and let M' be the Mumford-Tate group of x'. Then there exists an element k in $\mathrm{GL}_n(\mathbb{Z}_p)$ such that*

$$M'_{\mathbb{Q}_p} = kM_{\mathbb{Q}_p}k^{-1}$$

In particular $M_{\mathbb{Z}_p}$ is a torus if and only if $M'_{\mathbb{Z}_p}$ is.

Proof. In S we have an equality $[x_0, g] = [x', 1]$ (here $[x', 1]$ actually stands for the image of the point $[x', 1]$ of S' in S). Then there exists q in $G(\mathbb{Q})$ and k' in K such that $qx_0 = x'$ and $qgk' = 1$. Hence $q = {k'}^{-1}g^{-1}$. The group K is a product of compact open subgroups K_p of $G(\mathbb{Q}_p)$. The group K_p is $G(\mathbb{Z}_p)$ for almost all p. From $qx_0 = x'$, it follows that $M' = qMq^{-1}$. Then $M'_{\mathbb{Q}_p} = q_pM_{\mathbb{Q}_p}q_p^{-1}$, where$q_p$ denotes for the image of q in $G(\mathbb{Q}_p)$. Since g_p is in $C(\mathbb{Q}_p)\mathrm{GL}_n(\mathbb{Z}_p)$, we can write $g_p = c \cdot k''$ with c in $C(\mathbb{Q}_p)$ and k'' in $\mathrm{GL}_n(\mathbb{Z}_p)$. It follows that

$$q_p = {k'_p}^{-1}g_p^{-1} = {k'_p}^{-1}{k''}^{-1}c^{-1} = kc^{-1}$$

with k the element ${k'_p}^{-1}{k''}^{-1}$ of $G(\mathbb{Z}_p)$ It follows that there exists an element k in $\mathrm{GL}_n(\mathbb{Z}_p)$ such that $M'_{\mathbb{Q}_p} = kM_{\mathbb{Q}_p}k^{-1}$. □

Next proposition is a generality on the conjugacy classes of algebraic groups over $\mathbb{Z}_p$. This lemma will allow us to overcome the problem that we can not directly reduce ourselves to the case where Z is Hodge generic.

Lemma 7.1.3 *We consider a reductive group G' over $\mathbb{Q}_p$ with a finite dimensional faithful representation $G' \hookrightarrow \mathrm{GL}_n$ over $\mathbb{Q}_p$. This gives a $\mathbb{Z}_p$-structure to G'. We suppose that $G'_{\mathbb{Z}_p}$ is a reductive group scheme (this condition is satisfied for almost all p). Let M be a subtorus of G' such that $M_{\mathbb{Z}_p}$ is a torus. The set W of tori in G' that are conjugated to M by an element of $\mathrm{GL}_n(\mathbb{Z}_p)$ splits into a finite number of $G'(\mathbb{Z}_p)$-conjugacy classes.*

Proof. Consider the set W_p consisting of reductions modulo p of tori in W. Since the group $\mathrm{GL}_n(\mathbb{F}_p)$ is finite, the set W_p is finite hence it splits into a finite number of $G'(\mathbb{F}_p)$-conjugacy classes. Then, using lemma 7.1.4 below, we see that every $G'(\mathbb{F}_p)$-conjugacy class lifts to a $G'(\mathbb{Z}_p)$-conjugacy class. Hence W splits into a finite number of $G'(\mathbb{Z}_p)$-conjugacy classes classes. $\square$

Lemma 7.1.4 *Keep the notations and assumtions of lemma 7.1.3. Let M' be a torus in W such that $M'_{\mathbb{F}_p}$ is conjugated to $M_{\mathbb{F}_p}$ by an element of $G'(\mathbb{F}_p)$. Then $M'_{\mathbb{Z}_p}$ is conjugated to $M_{\mathbb{Z}_p}$ by an element of $G'(\mathbb{Z}_p)$.*

Proof. Let k be an element of $\mathrm{GL}_n(\mathbb{Z}_p)$ such that M and M' are conjugated by k. We apply the following theorem of Grothendieck (see [16, Corollaire 5.2, Exposé 11]).

Theorem 7.1.5 (Grothendieck) *Let S be a scheme. Let H and G be two S-group schemes with G smooth and affine over S and H of multiplicative type and of finite type. Let $u_1, u_2 : H \to G$ be two S-group homomorphisms. Let us consider the functor $\mathrm{Transp}(u_1, u_2)$ that to any S-scheme T associates the set of g in $G(T)$ such that $u_2 = gu_1g^{-1}$. The functor $\mathrm{Transp}(u_1, u_2)$ is representable by a smooth closed subscheme of G.*

We apply this theorem in the case $H = M$, $G = G'$, $S = \mathrm{Spec}(\mathbb{Z}_p)$ and with u_1 the inclusion $M \hookrightarrow G'$ and $u_2 = ku_1k^{-1}$. By our hypothesis $\mathrm{Transp}(u_1, u_2)(\mathbb{F}_p)$ is not empty hence this scheme is actually non-empty. Next we apply Hensel's lemma (see for example [30, Thm. 3.11]). It follows from it that we actually can apply this lemma. This lemma implies that we can lift to $\mathrm{Transp}(u_1, u_2)(\mathbb{Z}_p)$ the element of $G'(\mathbb{F}_p)$ that conjugates $u_{1\mathbb{F}_p}$ to $u_{2\mathbb{F}_p}$. This finishes the proof of the lemma. $\square$

The remaining Lemmas of this section answer the following question. For any point $[x', 1]$ of Σ', for any prime p, we are looking for an element x in $M'(\mathbb{Q}_p)$ such that

1. The projections of x to the simple factors of G' over $\mathbb{Q}_p$ are not contained in compact subgroups.

2. The index $[K_p : xK_px^{-1}]$ is bounded by p^k where k does not depend on p and on the point $[x', 1]$.

The existence of such an x is proved in Proposition 7.1.8.

Lemma 7.1.6 *All the projections of M to the simple factors of G'^{ad} over $\mathbb{Q}$ are non-trivial.*

Proof. The torus M is the Mumford-Tate group of some special element $x_0 : \mathbb{S} \to G'_{\mathbb{R}}$ in X'. Let G'' be a simple factor of G'^{ad} over $\mathbb{Q}$. Then the projection of M to G'' is non-trivial if and only if the composition of x_0 with $G'_{\mathbb{R}} \to G'^{\mathrm{ad}}_{\mathbb{R}} \to G''_{\mathbb{R}}$ is non-trivial. But X' is a $G'(\mathbb{R})$-conjugacy class of x_0, hence, by the axiom 3, the composition of x_0 with $G'_{\mathbb{R}} \to G'^{\mathrm{ad}}_{\mathbb{R}} \to G''_{\mathbb{R}}$ is non-trivial. $\square$

Lemma 7.1.7 *Let G be a simple algebraic group over $\mathbb{Q}$. Let M be a subtorus of G. Let $G = G_1 \times \cdots \times G_k$ be a decomposition of G over $\overline{\mathbb{Q}}$. Then the projections of M (over $\overline{\mathbb{Q}}$) to the simple factors G_i of G over $\overline{\mathbb{Q}}$ are non-trivial.*

Proof. Let N be the subgroup of $G(\mathbb{Q})$ generated by the conjugates $gM(\mathbb{Q})g^{-1}$ with $g \in G(\mathbb{Q})$. The group N is invariant by conjugation by the elements of $G(\mathbb{Q})$. Since G is connected, $G(\mathbb{Q})$ is Zariski-dense in G hence the Zariski closure $\overline{N}$ of N in G is a normal algebraic subgroup of G. As G is simple, $\overline{N} = G$. Assume that the projection of $M_{\overline{\mathbb{Q}}}$ to some simple factor G' of G over $\overline{\mathbb{Q}}$ is trivial. Let W be the group generated by the conjugates of $M(\overline{\mathbb{Q}})$ by the elements of $G(\overline{\mathbb{Q}})$. Then the projection of W to $G'(\overline{\mathbb{Q}})$ is trivial. Then the Zariski closure of this projection is trivial. But obviously, the Zariski closure of this projection contains the projection of $N_{\overline{\mathbb{Q}}} = G_{\overline{\mathbb{Q}}}$. This gives a contradiction. $\square$

Proposition 7.1.8 *There exists an integer k such that the following holds. For any $[x', 1]$ in Σ', and for almost all primes p splitting E and such that $M'_{\mathbb{Z}_p}$ is a torus, there exists an element s_p in $M'(\mathbb{Q}_p)$ that satisfies the following conditions.*

1. *Let $G'^{\mathrm{ad}} = G_1 \times \cdots \times G_m$ be a decomposition of G'^{ad} into a product of simple factors over $\mathbb{Q}_p$, then the image of s_p in any $G_i(\mathbb{Q}_p)$ is not contained in a compact subgroup.*

2. *The index $[K : s_pKs_p^{-1} \cap K]$ is at most p^k.*

Proof. For any $x \in G'(\mathbb{Q}_p)$, we have $[K : xKx^{-1} \cap K] = [K_p : xK_px^{-1} \cap K_p]$ and the index $[K_p : xK_px^{-1} \cap K_p]$ is the size of the K_p-orbit of the lattice $x\mathbb{Z}_p^n$ (recall that we have chosen a representation). Of course this orbit is smaller than the orbit of $x\mathbb{Z}_p^n$ for the action of $\mathrm{GL}_n(\mathbb{Z}_p)$.

Hence it suffices to find an integer k and for almost any p an element x in $M'(\mathbb{Q}_p)$ that satisfies the condition that its projections to simple factors of G'^{ad} are not contained in compact subgroups and such that the size of the orbit $\mathrm{GL}_n(\mathbb{Z}_p) \cdot x\mathbb{Z}_p^n$ is bounded by p^k.

The point is that given an x in $M'(\mathbb{Q}_p)$, one obtains bounds for the size of the $\mathrm{GL}_n(\mathbb{Z}_p)$-orbit of the lattice $x \cdot \mathbb{Z}_p^n$ from the bounds of the valuations of the eigenvalues of the characters of the torus M' that are actually involved in the representation evaluated at x. More preciesly we have the following lemma.

Lemma 7.1.9 *Let M be a torus over $\mathbb{Q}$ with a finite dimensional faithful representation $\rho : M \hookrightarrow \mathrm{GL}_n$. Let p be a prime that splits M and such that $M_{\mathbb{Z}_p}$ is a torus. Then the following holds.*

1. *The representation $\rho \times \mathbb{Q}_p$ of $M_{\mathbb{Q}_p}$ on $\mathbb{Q}_p^n$ splits as a direct sum*

$$\mathbb{Q}_p^n = \oplus_{\chi \in X_\rho} V_\chi$$

Here X_χ is some finite set of characters and V_χ is the subspace of $\mathbb{Q}_p$ on which $M_{\mathbb{Q}_p}$ acts via χ (x in $M_{\mathbb{Q}_p}$ sends v in V_χ to $\chi(x) \cdot v$).

2. *The lattice $\mathbb{Z}_p^n$ splits as*

$$\mathbb{Z}_p^n = \oplus_{\chi \in X_\rho} V_{\mathbb{Z},\chi}$$

where $V_{\mathbb{Z},\rho}$ is some lattice in V_χ on which $M_{\mathbb{Z}_p}$ acts via χ.

3. *Let x be an element of $M(\mathbb{Q}_p)$. Let λ be the maximum of absolute values of p-adic valuations of the $\chi(x)$ with $\chi \in X_\rho$. Then*

$$|\mathrm{GL}_n(\mathbb{Z}_p) \cdot x\mathbb{Z}_p^n| \leq p^{2\lambda n^2}$$

Proof. The first two statements are generalities about representation of split tori. They are explained in detail in Chapter 6. Let us prove the third statement. Let x in $M(\mathbb{Q}_p)$ and and let k be as in the statement. Consider the direct sum decomposition as in 1

$$\mathbb{Q}_p^n = \oplus_{\chi \in X_\rho} V_\chi$$

and let $V_{\mathbb{Z},\chi}$ be a lattice in V_χ as in 2. Let E_χ be a basis of each $V_{\mathbb{Z},\chi}$, then

$$E = \oplus_{\chi \in X_\rho} E_\chi$$

is a basis of $\mathbb{Z}_p^n$ such that the action $\rho_{\mathbb{Q}_p}(x)$ with respect to this basis is given by the diagonal matrix

$$\begin{pmatrix} \chi_1(x) & & & & & & & \\ & \ddots & & & & & & \\ & & \chi_1(x) & & & & & \\ & & & \chi_2(x) & & & & \\ & & & & \ddots & & & \\ & & & & & \chi_2(x) & & \\ & & & & & & \ddots & \\ & & & & & & & \chi_m(x) \end{pmatrix}$$

where $\chi_1, \ldots, \chi_m$ are the characters that actually occur in X_ρ. The passage from the standard basis to E is given by a matrix in $\mathrm{GL}_n(\mathbb{Z}_p)$. Hence the elemnts of $\mathrm{GL}_n(\mathbb{Z}_p)$ in the basis E the are given by matrices with entries in $\mathbb{Z}_p$. We have supposed that the absolute value of p-adic valuation of every $\chi_i(x)$ is bounded by λ. By a direct computation one sees that the group $x\mathrm{GL}_n(\mathbb{Z}_p)x^{-1} \cap \mathrm{GL}_n(\mathbb{Z}_p)$ contains the principal congruence subgroup

$$\Gamma(2\lambda) := \{y \in \mathrm{GL}_n(\mathbb{Z}_p) : y \equiv I_n \mod p^{2\lambda}\}$$

The index $[G(\mathbb{Z}_p) : \Gamma(2\lambda)]$ is at most $p^{2\lambda n^2}$. The statement 3 follows from it. $\square$

From this lemma it follows that to prove the Proposition 7.1.8, it suffices to find an integer k' and, for almost any p splitting M, an element x in $M'(\mathbb{Q}_p)$ such that the valuations of the eigenvalues of the characters of $M'_{\mathbb{Q}_p}$ that actually occur in the representation of $M'_{\mathbb{Q}_p}$ induced from ρ by base change from $\mathbb{Q}$ to $\mathbb{Q}_p$ are bounded by k' in absolute value.

Let X' the character group of M'. From the Lemma 7.1.7, it follows that the projections of $M'_{\overline{\mathbb{Q}}}$ to the simple factors of $G'_{\overline{\mathbb{Q}}}$ are non-trivial. Hence, for any p, the projections of $M_{\mathbb{Q}_p}$ and $M'_{\mathbb{Q}_p}$ onto the simple factors of $G'_{\mathbb{Q}_p}$ are non-trivial. Let X'_i be the character groups of the images of the projections of M' to the simple factors of G over $\overline{\mathbb{Q}}$.

Lemma 7.1.10 *For any prime p that splits M', there exists a surjective morphism*

$$\phi\colon M'(\mathbb{Q}_p) \to \mathrm{Hom}(X', \mathbb{Z})$$

whose kernel is exactly $M'(\mathbb{Z}_p)$. Futhermore, the kernel of the map $M'(\mathbb{Q}_p) \to \mathrm{Hom}(X', \mathbb{Z}) \to \mathrm{Hom}(X'_i, \mathbb{Z})$ is exactly the set of elements of $M'(\mathbb{Q}_p)$ whose projection to the simple factor $G'_{i\mathbb{Q}_p}$ is contained in a compact subgroup.

Proof. Let p be a prime that splits M'. Then all the characters of $M'_{\mathbb{Q}_p}$ are defined over $\mathbb{Q}_p$ and we can identify $M'(\mathbb{Q}_p)$ with $\mathrm{Hom}(X', \mathbb{Q}_p^\times)$ by the evaluation map. One has a surjective morphism ϕ from $\mathrm{Hom}(X', \mathbb{Q}_p^\times)$ to $\mathrm{Hom}(X', \mathbb{Z})$ which is induced from the map $\mathbb{Q}_p^\times \to \mathbb{Z}$ which takes the p-adic valuation. The kernel of ϕ is $\mathrm{Hom}(X', \mathbb{Z}_p^\times)$ which is identified with $M'(\mathbb{Z}_p)$ via evaluation. So ϕ gives us a surjection from $M'(\mathbb{Q}_p)$ to $\mathrm{Hom}(X', \mathbb{Q}_p^\times)$ whose kernel is $M'(\mathbb{Z}_p)$. The second statement can be seen in the same way. □

Recall that X' is a free $\mathbb{Z}$-module of some rank, say r. Let us choose a basis of X'. Then we can define a euclidiean norm on X' by

$$||(\lambda_1, \cdots, \lambda_r)|| = \max_{1 \leq i \leq r}(|\lambda_i|)$$

Note that all the X' are isomorphic, the isomorphism is given by conjugation with elements of $G(\mathbb{Q})$. Combining the two previous lemmas we obtain the following lemma.

Lemma 7.1.11 *We keep the above assumptions. Suppose that there exists a integer k such that the following holds. Suppose that for any subtorus M' of G' conjugated to M by an element of $\mathrm{GL}_n(\mathbb{Q})$, there exists an element λ in $\mathrm{Hom}(X', \mathbb{Z})$ such that*

1. *For any i, the images of λ in $\mathrm{Hom}(X'_i, \mathbb{Z})$ are non-zero.*
2. *When we view $\mathrm{Hom}(X', \mathbb{Z})$ as a $\mathbb{Z}$-free module, the euclidian norm $||\lambda||$ is at most k.*

Then the conclusion of 7.1.8 holds.

Our last aim is to prove the existence of such a λ. Using the fact that all the tori M' that we consider are conjugated by elements of $G(\mathbb{Q})$, we see that this follows from the following lemma.

Lemma 7.1.12 *Let $r > 0$ and $l \geq 0$ be two integers. There exists an integer k such that the following holds. Let N be a free $\mathbb{Z}$-module of finite rank r. Let l be an integer and $N_i, 1 \geq i \geq l$ be l proper free submodules of N such that $\mathrm{rank}(N_i) = r_i < r$. Then there exists in $N - \bigcup_i N_i$ an element λ whose norm is strictly smaller than k.*

Proof. This can be seen by reduction modulo some prime q. Indeed, let q be a prime. The number of points in $N \otimes \mathbb{F}_q$ is q^m, the number of points in the union $\bigcup_i N_i \otimes \mathbb{F}_q$ is at most lq^{m-1}. Hence if we take q larger than l, there exists an element $\overline{x}$ in $N \otimes \mathbb{F}_q$ which is not contained in $\bigcup_i N_i \otimes \mathbb{F}_q$. We can lift $\overline{x}$ to an element x of $N - \bigcup_i N_i$ such that $||x|| \leq q$. Then one takes $k := q + 1$ and $\lambda := x$. □

□

We have now in our posession all the ingredients to prove the main results of our thesis, the Theorems 4.2 and 4.3.

7.2 Case of a curve

In this section we treat the case where Z is a closed irreducible algebraic curve in S containing an infinite set Σ of points lying in one Hecke orbit. We actually prove the Theorem 4.2. In the proof we will distinguish two cases depending on the form of the elements g that correspond to the points $[x_0, g]$ in Σ.

Definition 7.2.1 For x in Σ, let $i(x)$ be the number of primes p such that g_p is not in $C(\mathbb{Q}_p)\mathrm{GL}_n(\mathbb{Z}_p)$.

Note that this function $i : \Sigma \to \mathbb{Z}$ depends on the choice of x_0 but another choice changes it by a bounded function. Also, replacing Z by an irreducible component of its image by a Hecke correspondence, changes i by a bounded function. Hence it makes sense to distinguish two cases, whether i is bounded or not.

7.2.2 The case where i is bounded

In this section we assume that i is bounded. We will use this property to prove that there exists an irreducible Hecke correspondence on S' satisfying the conditions of 5.2.1 and 5.3.1 such that for infinitely many points in Σ, Galois orbits of these points intersect their images by this correspondence. This will allow us to conclude that Z' is contained in its image by some Hecke correspondence on S' satisfying the conditions of 5.5.1. Then 5.5.1 will finish the proof in this case.

Recall that Σ is the set of points of S of the form $[x_0, g]$ with g varying in $G(\mathbb{A}_\mathrm{f})$, that M is the Mumford-Tate group of x_0 and that the F that we have taken for the field of definition of Z' splits M. Let p be a prime such that

1. The Theorems 5.2.1 and 5.3.1 of the preceding chapter can be applied to ${S'}^{\mathrm{ad}}$;

2. $M_{\mathbb{Z}_p}$ is a torus;

3. there exists an infinite subset Σ_1 of Σ such that for all $[x_0, g]$ in Σ_1 we have $g_p \in C(\mathbb{Q}_p)\mathrm{GL}_n(\mathbb{Z}_p)$;

4. p splits F (hence p splits M)

Such primes do exist. Indeed the first two conditions only exclude a finite number of primes. The last condition is verified by infinitely many primes (by Chebotarev density theorem). In fact, Chabotarev density theorem asserts that the number of primes split in F and which are smaller than some striclty positive real x is roughly speaking $x/[F : \mathbb{Q}]\log(x)$. The third condition is also satisfied by almost all primes.

Let Σ_1' be the inverse image of Σ_1 in Z'. For any point $[x', 1]$ of Σ_1', we let as before M' be the Mumford-Tate group of x'. Recall that M' is conjugate to M by an element of $G(\mathbb{Q})$, say $M' = qMq^{-1}$ with q in $G(\mathbb{Q})$. We also recall that we have defined

$$r' : \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_{\mathrm{m}F} \to M'$$

as the reciprocity morphism attached to x'. Let s_p be an element in the image of the morphism

$$r'(\mathbb{Q}_p) : (F \otimes \mathbb{Q}_p)^\times \to M'(\mathbb{Q}_p)$$

induced from r' by taking $\mathbb{Q}_p$-valued points. Then $[x', s_p]$ is a $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$-conjugate of $[x', 1]$. The image of $(F \otimes \mathbb{Q}_p)^\times$ by the above morphism is of finite index in $M'(\mathbb{Q}_p)$, bounded independently of p. Let n be a bound for this index which is independent of p. Since for different points in σ_1', the associated morphisms r' are conjugated by elements of $G(\mathbb{Q})$, this boun is actually independent of the point of Σ_1' that we take.

We wish to extract from Σ_1' an infinite subset of points for which the tori $M_{\mathbb{Q}_p}$ and $qM_{\mathbb{Q}_p}q^{-1}$ over $\mathbb{Q}_p$ are actually conjugated by an element of $G'(\mathbb{Z}_p)$. This is important because we wish to prove that the intersection of Z' with its image by one well-chosen Hecke operator is infinite. The theorem 7.1.3 implies that any Mumford-Tate group M' of a point in Σ_1' is of the form $k'M_ik'^{-1}$ with k' in $G'(\mathbb{Z}_p)$ and M_i is a subtorus of G' belonging to a finite set $M_1, \ldots, M_j$ of tori in $G'_{\mathbb{Q}_p}$. The tori M_i are all conjugated by elements of $\mathrm{GL}_n(\mathbb{Z}_p)$ by 7.1.2 hence any $M_{i\mathbb{Z}_p}$ is a torus. It follows that for an infinite set, say Σ_1'', of Σ_1', the corresponding torus qMq^{-1} is conjugated in $G'(\mathbb{Z}_p)$ to the same of the M_i. Say $qMq^{-1} = kM_1k^{-1}$ with k in $G'(\mathbb{Z}_p)$ for any $[qx_0, 1]$ in Σ_1''.

Let us take one point $[x', 1]$ in Σ_1''. Let s_p be the element in $M'(\mathbb{Q}_p)$ given by 7.1.8. Then there exists an element k in $G'(\mathbb{Z}_p)$ and $t_p \in M_1(\mathbb{Q}_p)$ such that $s_p = kt_pk^{-1}$. The element $kt_p^nk^{-1}$ is in the image of $r'(\mathbb{Q}_p)$. It follows that for any point in Σ_1'' some Galois conjugate is contained in $T_{t_p^n}(Z')$ and hence

$$\Sigma_1'' \subset Z' \cap T_{t_p^n}(Z')$$

It follows that $Z' \subset T_{t_p^n}(Z')$ and the Proposition 5.5.1 allows us to conclude that Z' is of Hodge type.

7.2.3 The case where i is not bounded

In this section we treat the case where i is not bounded. As in the preceding section, our aim is to prove that Z' is contained in its image by some Hecke correspondence on $\mathrm{Sh}_{K'}(G', X')_{\mathbb{C}}$ but the strategy is somewhat different in this case. The point is that in the previous section, the fact that i was bounded, allowed us to prove directly that the intersection of Z' with its image by some Hecke correspondence contained infinitely many points. In the case that we consider here (i is not bounded), we choose one Hecke operator for one point and we exploit the fact that Galois orbits are large. The crucial ingredients of the proof exposed in this section are results of Edixhoven on the growth of Galois orbits and on the number of points in the intersection $Z' \cap T_g(Z')$ for g un $G(\mathbb{A}_{\mathrm{f}})$.

Number of points in the intersection of Z' with $T_g(Z')$

Since we consider intersections of Z with its images by Hecke correspondences, we will need a bound on the number of points in the intersection $|Z(\mathbb{C}) \cap T_g(Z)(\mathbb{C})|$ for g in $G(\mathbb{A}_{\mathrm{f}})$ in the case when this intersection is finite. Let us recall the following result proved by Edixhoven in [18, thm. 7.2].

Theorem 7.2.4 (Edixhoven) *Let (G, X) be a Shimura datum and let K_1 and K_2 be compact open subgroups of $G(\mathbb{A}_{\mathrm{f}})$ and let Z_1 and Z_2 be closed subvarieties of the Shimura varieties $S_1 := \mathrm{Sh}_{K_1}(G, X)_{\mathbb{C}}$ and $S_2 := \mathrm{Sh}_{K_2}(G, X)_{\mathbb{C}}$ respectively. Suppose that Z_1 or Z_2 of dimension at most one. Then there exists an integer c such that for all g in $G(\mathbb{A}_{\mathrm{f}})$ for which $T_g(Z_1) \cap Z_2$ is finite, one has:*

$$|T_g(Z_1) \cap Z_2| \leq c \cdot \deg(\pi_1)$$

where π_1 is the natural map

$$\mathrm{Sh}_{K_1 \cap g K_2 g^{-1}}(G, X) \to \mathrm{Sh}_{K_1}(G, X)$$

induced by the inclusion $K_1 \cap g K_2 g^{-1} \subset K_1$.

In our case G is adjoint, $K_1 = K_2 = K$ and Z is a curve. Next lemma gives a bound on the degree of the map $\pi_1 : \mathrm{Sh}_{K \cap g K g^{-1}}(G, X)_{\mathbb{C}} \to \mathrm{Sh}_K(G, X)_{\mathbb{C}}$.

Lemma 7.2.5 *The number $\deg(\pi_1)$ is $[K : K \cap g K g^{-1}]$.*

Proof. This follows from the fact that, since K is neat, the action of K on $\mathrm{Sh}(G, X)_{\mathbb{C}} := G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_{\mathrm{f}})$ is free. $\square$

Proof of the equality between Z'^{ad} and its image by some Hecke correspondence

By Theorem 5.5.1, to prove that Z' is of Hodge type, it suffices to prove that Z' is contained in its image by some well-chosen Hecke correspondence. The idea to prove such an inclusion is to apply some element of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ to some point of Z' and to consider the intersection of Z' and its image by the Hecke correspondence coming from the element of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ that we have applied. Such an intersection contains the $\mathrm{Gal}(\overline{F}/F)$-orbit of the point that we have considered because Hecke correspondences are defined over F. Our aim is to prove that the intersection is not proper, hence the lower bounds for Galois orbits given in the preceding chapter will play crucial role. First we prove the following proposition.

Proposition 7.2.6 *Suppose that there exists a point $[x_0, g]$ in Σ and a prime p such that*

1. *The prime p is so large that the Theorems 5.2.1 and 5.3.1 hold*
2. *The prime p splits F*
3. *The prime p is such that $g_p \in C(\mathbb{Q}_p)\mathrm{GL}_n(\mathbb{Z}_p)$*
4. *The prime p is that the size of the $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$-orbit of $[x_0, g]$ exceeds $c \cdot p^k$ (where k is given by 7.1.8 and c is a constant from 7.2.4).*

Then $Z' \subset T_{s_p}(Z')$.

Proof. Let $[x_0, g]$ be a point in Σ and let p be a prime as in the proposition. Since $[x_0, g]$ is in the image of Σ', there exists $[x', 1]$ in Σ' with $x' = qx_0$ for some $q \in G(\mathbb{Q})$ whose image in S is $[x_0, g]$. The Mumford-Tate group $M' = qMq^{-1}$ of x' is such that $M'_{\mathbb{Z}_p}$ is a torus because by 7.1.2 $M'_{\mathbb{Q}_p}$ is conjugated to $M_{\mathbb{Q}_p}$ by an element of $\mathrm{GL}_n(\mathbb{Z}_p)$. Take an element s_p in $M'(\mathbb{Q}_p)$ as in 7.1.8. Possibly replacing s_p by a suitable power (independent of p0, we can and do assume that s_p is in the image of $r'(\mathbb{Q}_p)$. Let σ be the corresponding element of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$. Then $\sigma([x', 1]) = [x', s_p]$, hence $\sigma([x', 1])$ is in the intersection of Z' and $T_{s_p}(Z')$ and hence all the $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$-orbit of $[x', 1]$ is contained in the intersection. Suppose that the intersection is finite, then the number of points in the intersection is bounded by $c \cdot p^k$. Since the size of the $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$-orbit of $[x', 1]$ exceeds $c \cdot p^k$, the intersection can not be finite. It follows that $Z' \subset T_{s_p}(Z')$. $\square$

Our next and last aim is to prove the existence of a prime p as in the proposition. Let us recall the Theorem 6.1 of Edixhoven about Galois orbits of points in Σ. For any $x = [x_0, g]$ in Σ we let $p_1, \cdots, p_{i(x)}$ be the $i(x)$ primes such that $g_p \notin C(\mathbb{Q}_p)\mathrm{GL}_n(\mathbb{Z}_p)$. Then there exists a real $c_1 > 0$ and $c_2 \geq 0$ such that for any $[x_0, g]$ in Σ, the size of the $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$-orbit of $x := [x_0, g]$ is at least

$c_1 \cdot {c_2}^{i(x)}$ times the product $p_1 \times \cdots \times p_{i(x)}$. We have to find a point x in Σ and a prime p verifying the conditions of 7.2.6 such that

$$(\frac{c_1}{c}){c_2}^{i(x)} p_1 \times \cdots \times p_{i(x)} > p^k$$

In other words

$$p < (\frac{c_1}{c})^{1/k} c'^{i(x)/k} (p_1 \cdots p_{i(x)})^{1/k}$$

For any real $y \geq 0$, define

$$\pi_F(y) = |\{p \text{ prime splitting } F \text{ and } p \leq y\}|$$

Recall that the Chebotarev density theorem says that

$$\pi_F(y) \backsim_{y \to \infty} \frac{y}{[F:\mathbb{Q}] \log(y)}$$

Let $y := (\frac{c_1}{c})^{1/k} c'^{i(x)/k} (p_1 \cdots p_{i(x)})^{1/k}$. It follows that for any x in Σ with $i(x)$ large enough, the number of primes p with $p < y$ is larger than $\frac{y}{2[F:\mathbb{Q}] \log(y)}$. Since y tends to infinity when $i(x)$ tends to infinity, for any x with $i(x)$ large enough, $\frac{y}{2[F:\mathbb{Q}] \log(y)}$ is larger than $y^{1/2}$.

Now we use that the ith prime is larger than i and we replace k by $2k$. It follows that

$$y^{1/2} > (\frac{c_1}{c})^{1/k} (c_2)^{i(x)/k} (i(x)!)^{1/k}$$

Stirling's formula asserts that

$$n! \backsim_{n \to \infty} (\frac{n}{e})^n \sqrt{2\pi n}$$

Hence for any x with $i(x)$ large enough,

$$i(x)! > (1/2)(\frac{i(x)}{e})^{i(x)} \sqrt{2\pi i(x)}$$

Hence

$$y^{1/2} > (c_1 \sqrt{2\pi}/2c)^{1/k} i(x)^{i(x)/2} i(x)^{1/2k} (\frac{c_2}{e^{k/2}})^{i(x)/k}$$

It is now clear that the right expression is bigger than $i(x)$ is x is such that $i(x)$ is large enough. It follows that for x in Σ with $i(x)$ large enough

$$\frac{y}{2[E:\mathbb{Q}] \log(y)} > i(x)$$

Hence the primes verifying the condition of 7.2.6 do exist. This finishes the proof in this case.

7.3 Case of an arbitrary subvariety

In this section we prove 4.3. Recall the assumptions. We suppose that Z is an arbitrary subvariety of S which *is Hodge generic* and that the generic Mumford-Tate group on Z is semisimple of adjoint type. Suppose that Z conatins a Zariski dense subset Σ of special points lying in one Hecke orbit. Let x_0 be an element of X such that Σ consists of elements of the form $[x_0, g]$. We let M be the Mumford-Tate group of G and H its centralizer in G. As in the preceding section, we choose a number field F which splits M and such that Z is defined and is geometrically irreducible over F.

We suppose that there exists a prime p auch that the Theorems 5.2.1 and 5.3.1 hold, such that p splits F and such that the following condition is verified: for any $[x_0, g]$ in Σ the p-component g_p of g is in $H(\mathbb{Q}_p)K_p$ where H is the centralizer of M in G. Let

$$r\colon \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_{\mathrm{m}F} \to M$$

be the reciprocity law corresponding to x_0. We take an element s_p in $M(\mathbb{Q}_p)$ which is in the image of the map

$$r(\mathbb{Q}_p)\colon (F \otimes \mathbb{Q}_p)^\times \to M(\mathbb{Q}_p)$$

induced from r by taking $\mathbb{Q}_p$-valued points. Moreover, we take s_p such that the projections of s_p to the simple factors of G over $\mathbb{Q}_p$ are non-trivial. The existence of such an element is guaranteered by the Lemma 7.1.8. Since s_p is in the image of $r(\mathbb{Q}_p)$, there exists σ in $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ such that for any $[x_0, g]$ in Σ

$$\sigma([x_0, g]) = [x_0, s_p \cdot g]$$

It follows from the fact that $g_p \in H(\mathbb{Q}_p)K_p$ that for any $[x_0, g]$ in Σ, $\sigma([x_0, g])$ is in $T_{s_p}(Z)$, hence Σ is in the intersection of Z and $T_{s_p}(Z)$. Since Σ is Zariski-dense in Z, $Z \subset T_{s_p}(Z)$. We conclude that Z is of Hodge type using 5.5.1.

Remark 7.3.1 The result above should be easily improuved. One can try to remove the conditions G is adjoint and Z is Hodge generic. One can also consider the question to know to which extent we really need the condition that points lie in one Hecke orbit. It should be also interesting to compare this result to the result of Moonen [25]. The lack of time prevented the author from doing this.

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Resumé

Dans cette thèse on traite certains cas de la conjecture d'André-Oort qui affirme que les composantes irréducibles de l'adherence de Zariski d'un ensemble de points spéciaux dans une variété de Shimura est une sousvariété de type Hodge. Le premier resultat traite le cas des courbes dans un produit $S_1 \times S_2$ où S_1 et S_2 sont des courbes de Shimura associées aux algèbres de quaternions indéfines sur $\mathbb{Q}$. On démontre la conjecture d'André-Oort pour un tel produit en supposant l'Hypothèse de Riemann Généralisée.

Le deuxième résultat affirme qu'une courbe irréductible fermée dans une variété de Shimura quelconque S contenant un ensemble infini de points qui sont dans une orbite de Hecke est de type Hodge. Ce resultat implique, via le travail de Cohen, Wolfart et Wüstholz, une conjecture sur la transcendence de valeurs de fonctions hypergéométriques.

Mots clés

Variété de Shimura, point spécial, sous-variété de type Hodge, groupe de Mumford-Tate, monodromie

Abstract

In this thesis we deal with some cases of the André-Oort conjecture which asserts that the Zariski closure of a set of special points in a Shimura variety is a subvariety of Hodge type. The first result deals with curves contained in a product $S_1 \times S_2$ where S_1 and S_2 are Shimura curves attached to indefinite quaternion algebras over $\mathbb{Q}$. We prove the André-Oort conjecture for such a product assuming the Generalized Riemann Hypothesis.

The second result asserts that an irreducible closed algebraic curve in any Shimura variety S containing an infinite subset of special points lying in one Hecke orbit is of Hodge type. This result implies, via the work of Cohen, Wolfart and Wüstholz, a conjecture about the transcendence of values of hypergeometric functions.

Key words

Shimura variety, special point, subvariety of Hodge type, Mumford-Tate group, monodromy