

Towards schemes over $\mathbb{F}_1$

A functorial characterization of open immersions

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The aim of this seminar is to present an alternative characterization of open immersions in the category of schemes. The results we present here are classical (1967), but are the starting point to generalize the concept of schemes.

Our ultimate goal is to present part of the proof of this theorem. All the terms used will be explained shortly.

Theorem 1 ([Gro67], IV.17.9.1). *Let $f : X \rightarrow Y$ be a map of schemes. The three following properties are equivalent:*

- i) f is an open immersion,*
- ii) f is an étale monomorphism,*
- iii) f is a flat monomorphism, locally of finite presentation.*

Due to lack of time and usage of non-trivial theorems in the proof, we won't be able to show the full proof of it.

The statement of the theorem may seem totally obscure at first sight. We want to stress out the fact that a purely “geometrical” notion (open immersion) is proved equivalent to other notions which can be described via the functor on rings represented by a scheme.

1 Some basic properties of morphisms of schemes

We now try to make all the notions involved in the theorem clear to the reader. We start by analysing the concept of open immersions. Let X be a scheme and let U be an open subset of X . We can put a scheme structure on U considering the structure sheaf of X and restricting it to U . Indeed, because U is open, any open subset of U is also an open subset of X . We can hence define without any problem $\mathcal{O}_U(V) := \mathcal{O}_X(V)$ for any open subspace V of U . We will denote this structure sheaf $\mathcal{O}_X|_U$.

Remark 2. $(U, \mathcal{O}_X|_U)$ is a scheme.

We will call U an open subscheme of X . We obviously want the inclusion $U \hookrightarrow X$ to be an open immersion. The definition of open immersions is then naturally obtained.

Definition 1. *A morphism $Y \rightarrow X$ of schemes is an open immersion if and only if it induces an isomorphism of Y with an open subscheme of X .*

From now on, we will often consider open immersions as being open inclusions. This is not restrictive, because in order to consider the general case it suffices to compose with an isomorphism, and isomorphisms have all properties we will define.

We have now to define all the categorical notions which appear in the main theorem.

Definition 2. *Let f be an arrow in $\text{Hom}(Y, Z)$ for any category $\mathcal{C}$. f is a monomorphism if $\text{Hom}(X, f)$ is injective for any X . Explicitly, f is a monomorphism if and only if for any object X and any couple of arrows h, k in $\text{Hom}(X, Y)$ we get*

$$f \circ h = f \circ k \iff h = k$$

and this is why we can briefly say that f is a monomorphism if and only if it can be “cancelled on the left”.

Note that in case of **Set**, an arrow is a monomorphism if and only if it is an injection. When we add further structure, being set-wise injective may not be sufficient in order to be a monomorphism. This is, for example, the case of **Sch**. Consider for example the diagram:

$$\mathrm{Spec} k' \otimes_k k' \rightrightarrows \mathrm{Spec} k' \longrightarrow \mathrm{Spec} k$$

where $k \hookrightarrow k'$ is a proper inclusion of fields, and the two parallel maps are induced by the two maps $k' \rightrightarrows k' \otimes_k k'$. We conclude that $\mathrm{Spec} k' \rightarrow \mathrm{Spec} k$ is not a monomorphism, despite the fact that it induces an identity on topological spaces.

Proposition 3. *An open immersion is a monomorphism.*

Proof. Clear. □

Definition 3. *A morphism $X \rightarrow Y$ of schemes is locally of finite presentation if for any $x \in X$ there exists an open affine neighbourhood $f(x) \in V = \mathrm{Spec} A$ and an open affine neighbourhood $x \in U = \mathrm{Spec} B \subset f^{-1}(V)$ such that the induced map $A \rightarrow B$ is of finite presentation.*

The fact that this property is closed under composition is not entirely trivial.

Proposition 4. *Let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be morphisms locally of finite presentation. Then gf is locally of finite presentation.*

Proof. Fix $x \in X$. By hypothesis, we can choose open subschemes $x \in \mathrm{Spec} C \subset X$, $f(x) \in \mathrm{Spec} B \subset Y$, $f(x) \in \mathrm{Spec} B' \subset Y$, $gf(x) \in \mathrm{Spec} A \subset Z$ such that $f(\mathrm{Spec} C) \subset \mathrm{Spec} B$ and $B \rightarrow C$ is of finite presentation, $g(\mathrm{Spec} B') \subset \mathrm{Spec} A$ and $A \rightarrow B'$ is of finite presentation. Let now $V = \mathrm{Spec} B'_g \subset \mathrm{Spec} B$, and $U = f^{-1}(V)$. We have $f^{-1}(V) = \mathrm{Spec} C \times_{\mathrm{Spec} B} \mathrm{Spec} B'_g = \mathrm{Spec}(C \otimes_B B'_g)$, and $C \otimes_B B'_g$ is a B'_g -algebra of finite presentation. Because $B'_g = B'[x]/(xg - 1)$ is a B' -algebra of finite presentation, and B' is a A -algebra of finite presentation, we conclude that $C \otimes_B B'_g$ is a A -algebra of finite presentation. Hence we can choose $x \in \mathrm{Spec}(C \otimes_B B'_g) = U$ and $gf(x) \in \mathrm{Spec} A = V$ as open affine neighbourhoods that satisfy the requirements of the definition. □

Proposition 5. *An open immersion is locally of finite presentation.*

Proof. In case of an open inclusion, the local morphism can be chosen to be identities of affine schemes, and the identity obviously defines a morphism of finite presentation. Since isomorphisms are of finite presentation, and because this property is closed under composition, we obtain the general case for open immersions. □

There are other characterizations of morphisms that are locally of finite presentation, which are equally important.

Proposition 6. *Let $f: X \rightarrow Y$ a morphism of schemes. The following are equivalent*

- (i) *f is locally of finite presentation.*
- (ii) *There exists a covering of affine open subschemes $\{\mathrm{Spec} A_i \rightarrow Y\}$ of Y and affine open coverings $\{\mathrm{Spec} B_{ij} \rightarrow f^{-1}(\mathrm{Spec} A_i)\}$ of each inverse image of $\mathrm{Spec} A_i$ such that the induced morphisms of rings $A_i \rightarrow B_{ij}$ are of finite presentation.*
- (iii) *For any open affine subscheme $W = \mathrm{Spec} C$ of Y , there exists an affine open covering $\{\mathrm{Spec} B_i\}$ of $f^{-1}(W)$ such that each induced map $C \rightarrow B_i$ is of finite presentation.*

Proof. The implications (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (ii) are clear. Now suppose (i), and consider an open affine subscheme $W = \mathrm{Spec} C$ of Y . Let x be in $f^{-1}(W)$, and let U, V be as in the definition. Consider now another affine neighbourhood $\bar{V} = \mathrm{Spec} C_g$ of $f(x)$ such that $\bar{V} \subset W \cap V$, and the open affine neighbourhood of x defined as $\bar{U} = f^{-1}(\bar{V}) = \mathrm{Spec}(B \otimes_A C_g)$. Consider now the map of rings $C \rightarrow C_g \rightarrow B \otimes_A C_g$. Being the composite of two maps of finite presentation, it is of finite presentation. If we now consider the collection of opens $\bar{U}$ as x runs in the set $f^{-1}(W)$, we obtain an open covering $\{\mathrm{Spec} B_i\}$ of it, and by construction all maps $C \rightarrow B_i$ are of finite presentation. This proves (iii), hence the claim. □

Definition 4. *A morphism $X \rightarrow Y$ of schemes is flat if for any $x \in X$ the induced map of rings $\mathcal{O}_{Y, f(x)} \rightarrow \mathcal{O}_{X, x}$ is flat.*

Proposition 7. *An open immersion is flat.*

Proof. In case of an inclusion, the morphisms at stalks are identities, and the identity obviously defines a flat morphism. Because isomorphisms are flat, and the composition of flat morphisms is flat, we conclude the general statement. $\square$

Definition 5. *A morphism $X \rightarrow Y$ of schemes is formally étale if for any affine scheme $\text{Spec } A$ and any closed subscheme $\text{Spec } A/I$ such that $I^k = 0$ for some k , the map induced by the closed immersion $i : \text{Spec } A/I \rightarrow \text{Spec } A$:*

$$\text{Hom}_Y(\text{Spec } A, X) \rightarrow \text{Hom}_Y(\text{Spec } A/I, X)$$

is bijective. Equivalently, f is formally étale if for any commutative square

$$\begin{array}{ccc} \text{Spec } A/I & \xrightarrow{g} & X \\ i \downarrow & \exists! \nearrow & \downarrow f \\ \text{Spec } A & \xrightarrow{h} & Y \end{array}$$

where I is a nilpotent ideal, there exists a unique lift $\text{Spec } A \rightarrow X$ (the dotted map in the diagram).

Proposition 8. *An open immersion is formally étale.*

Proof. Let's focus on the diagram, where now $X \rightarrow Y$ is an open inclusion. Because $I^k = 0$, we have $I \subset \sqrt{0}$, hence the map i is an identity on the topological spaces underneath. As topological spaces then, by the commutativity of the diagram, we have $g = h$ and the lifting map is g itself. There is one only map that we can define $\mathcal{O}|_U \rightarrow h_*(\mathcal{O}_{\text{Spec } A})$ so that it commutes with the diagram. This is the map induced by the equalities and maps

$$\mathcal{O}|_U(V \cap U) = \mathcal{O}_X(V \cap U) \rightarrow (h_*\mathcal{O}_{\text{Spec } A})(V \cap U) = (g_*\mathcal{O}_{\text{Spec } A})(V \cap U)$$

for any open subset V of X .

Since composition of formally étale morphism is formally étale, and since isomorphisms are obviously formally étale, we conclude the proof for arbitrary open immersions. $\square$

Note that in order to prove this proposition, we have only used the fact that $I \subset \sqrt{0}$, and not the stronger hypothesis that I is nilpotent.

Note also that in the definition we can replace the condition $I^k = 0$ with the weaker $I^2 = 0$. Indeed, in that case the general statement can be proved using as test affine schemes $\text{Spec } A/I^k \hookrightarrow \text{Spec } A/I^{k+1}$ and composing.

The fundamental property of formally étale morphisms is to preserve the Kähler differential bundle. We can't describe this property with all the details. We reduce ourselves to observe something related to the tangent bundle over a scheme.

Let's see what is implied by the condition in the definition in case we consider $A = k[\varepsilon] := k[t]/(t^2)$, $I = (\varepsilon) = (t)$ and two schemes X and Y over $\text{Spec } k$. We obtain $\text{Spec } A/I = \text{Spec } k$. To give a map g from $\text{Spec } k$ to X is equivalent to choose a point in X and an inclusion map $k(x) \hookrightarrow k$. Indeed, the topological space associated to $\text{Spec } k$ is constituted by just one point $\mathbf{0}$. Let x be its image in X . The map at stalks $g^\sharp : \mathcal{O}_{X,x} \rightarrow k$ is a local morphism, hence induces a morphism of fields $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x \rightarrow k$. If we add the further condition to be a morphism over k , then from the identity splitting $k \rightarrow k(x) \rightarrow k$, we conclude that the point x is k -rational, i.e. is such that $k(x) = k$.

On the other hand, to give a map h over $\text{Spec } k$ from $\text{Spec } k[\varepsilon]$ to Y is equivalent to give a k -rational point $x \in X$, and an element of the dual of the k -space $\mathfrak{m}_x/\mathfrak{m}_x^2$. Indeed, the topological space associated to $\text{Spec } k[\varepsilon]$ is again one point (ε) . Call y the image of this point in Y . Again by the maps $k \rightarrow k(y) \rightarrow k(\varepsilon) = k$, we conclude that y is k -rational. The map at stalks $\mathcal{O}_{Y,y} \rightarrow k[\varepsilon]$ is local, hence if we restrict it to $\mathfrak{m}_y$ we get a map $\mathfrak{m}_y \rightarrow (\varepsilon)$ such that $\mathfrak{m}_y^2 \mapsto 0$. We end up with a morphism of k -algebras $\mathfrak{m}_y/\mathfrak{m}_y^2 \rightarrow (\varepsilon)$, which is equivalent to a map of k -vector spaces $\mathfrak{m}_y/\mathfrak{m}_y^2 \rightarrow k \cong (\varepsilon)$, hence to an element t of the dual of $\mathfrak{m}_y/\mathfrak{m}_y^2$. It is easy to show that, viceversa, such a couple y, t determines uniquely a morphism over k from $\text{Spec } k[\varepsilon]$ to Y (cfr. [Har77], Ex.II.2.8).

The dual space of $\mathfrak{m}_x/\mathfrak{m}_x^2$ is called the Zariski tangent space and denoted T_x , because it is isomorphic to the k -vector space of the derivations at x .

Now let $f : X \rightarrow Y$ a formally étale morphism of schemes over k , where we consider in the definition only morphisms over k . The commutative square

$$\begin{array}{ccc} \mathrm{Spec} A/I & \xrightarrow{g} & X \\ i \downarrow & & \downarrow f \\ \mathrm{Spec} A & \xrightarrow{h} & Y \end{array}$$

is then equivalent to a choice of a k -rational point $x \in X$ and a vector t in the tangent space of $f(x)$. The existence of the dotted arrow means that t can be lifted to a tangent vector at x , and the uniqueness means that t can be lifted uniquely to a tangent vector at x , i.e. that the map induced by f between the two tangent spaces $T_x \rightarrow T_{f(x)}$ is indeed bijective.

Definition 6. A morphism $X \rightarrow Y$ of schemes is étale if it is locally of finite presentation and formally étale.

Proposition 9. An open immersion is étale.

Proof. Clear by Propositions 5 and 8. □

2 Direct limits

We have promised to give a totally categorical characterisation of our definitions. Indeed, this can be done for morphism which are locally of finite presentation, as we will show in the next section. We start by introducing the concept of a direct limit and its basic properties.

Let $\mathcal{I}$ and $\mathcal{C}$ be two categories. We denote by $\mathcal{C}^{\mathcal{I}}$ the category of functors from $\mathcal{I}$ to $\mathcal{C}$. Given an object X of $\mathcal{C}$, we can define a functor $\Delta X : \mathcal{I} \rightarrow \mathcal{C}$ such that $\Delta X(i) = X$ and $\Delta X(f) = id_X$ for all objects i and all arrows f in $\mathcal{I}$. We can define a functor $\Delta : \mathcal{C} \rightarrow \mathcal{C}^{\mathcal{I}}$ that associates to each object c of $\mathcal{C}$ the constant functor Δc and to each arrow the obvious natural transformation. Suppose there exists a functor $\mathrm{colim} : \mathcal{C}^{\mathcal{I}} \rightarrow \mathcal{C}$ which is a left adjoint of Δ . We will say that $\mathcal{C}$ has all colimits indexed by $\mathcal{I}$, and for any $F \in \mathcal{C}^{\mathcal{I}}$, we will call the object $\mathrm{colim} F \in \mathcal{C}$ the colimit of F . Observe that the colimit of a functor has the adjointness property:

$$\mathrm{Hom}_{\mathcal{C}^{\mathcal{I}}}(F, \Delta X) \cong \mathrm{Hom}_{\mathcal{C}}(\mathrm{colim} F, X) \quad (1)$$

for any $X \in \mathcal{C}$, and the isomorphisms are functorial. In particular, the functor colim is right exact, hence it commutes with quotients, cokernels, disjoint unions etc. (indeed, with any other colimit [ML98], Theorem V.5.1).

In general, the colimit of a functor $F : \mathcal{I} \rightarrow \mathcal{C}$ is an object X of $\mathcal{C}$ and a natural transformation $\eta : F \rightarrow \Delta X$ such that the couple (X, η) is the initial object in the comma category $(F \downarrow \Delta)$.

These definitions are definitely too obscure to be understandable without any further comments. Hence, we want now to give a better idea of the notion of a colimit, restricting ourselves to the case of direct limits (which are special colimits).

Definition 7. Let I be a preordered set such that every couple of elements has an upper bound. Let also $\{A_i, f_{ij}\}$ be a collection of objects A_i in a category $\mathcal{C}$, indexed by elements of I together with arrows $f_{ij} : A_i \rightarrow A_j$ indexed by couples $(i, j) \in I^2$ with $i \leq j$ such that:

1. $f_{ii} = id_{A_i}$,
2. $f_{ik} = f_{jk} \circ f_{ij}$ for all $i \leq j \leq k$.

We will call this data a direct system. The direct limit of a direct system is an object $\varinjlim_{i \in I} A_i$ of $\mathcal{C}$, equipped with a collection of morphisms $\varphi_i : A_i \rightarrow \varinjlim_{i \in I} A_i$ which is universal with respect to the property $\varphi_i = \varphi_j \circ f_{ij}$ for all $i \leq j$. This means that for any other object B and any collection of morphisms $\psi_i : A_i \rightarrow B$ such that $\psi_i = \psi_j \circ f_{ij}$ for any $i \leq j$, then all the morphisms ψ_j factor through the φ_j via a

unique map $\varinjlim_{i \in I} A_i \rightarrow B$ independent on j , as pictured below

$$\begin{array}{ccc} A_i & \xrightarrow{\varphi_i} & \varinjlim_{i \in I} A_i \\ f_{ij} \downarrow & \nearrow \varphi_j & \downarrow \exists! \\ A_j & \xrightarrow{\psi_j} & B \end{array}$$

This definition is consistent with the previous one. It suffices to consider the data $F := \{A_i, f_{ij}\}$ as a functor from the poset category I to $\mathcal{C}$. The collection $\{\varphi_i\}$ is nothing but the natural transformation of functors $F \rightarrow \Delta \varinjlim_{i \in I} A_i$ that corresponds to the identity via the isomorphism (1). Note that, because it is defined through a universal property, the direct limit is uniquely determined, up to a unique isomorphism.

Proposition 10. *The categories of sets, of A -modules, of rings and the category of A -algebras have direct limits. The direct limit functor on A -modules is exact.*

Proof. The category of rings is the category of $\mathbb{Z}$ -algebras, so we can reduce the proof for rings to the proof of A -algebras. The construction we will make in this case is also valid for the category of sets or A -modules, forgetting the additional structure. Let $\{A_i\}$ a direct system of A -algebras. Consider the set $L := \coprod_i A_i / \sim$ where $\sim$ is the equivalence relation defined as: $A_i \ni x_i \sim x_j \in A_j$ if and only if there exists a $k \geq i, j$ such that $f_{ij}(x_i) = f_{jk}(x_j)$. It is easy to show that $\sim$ is well defined and is an equivalence relation. We can define also a ring structure via $[x_i] + [y_j] = [x_k y_k]$, $[x_i][y_j] = [x_k y_k]$ where $k \geq i, j$, $x_k = f_{ik}(x_i)$ and $y_k = f_{jk}(y_j)$. There exists a ring morphism $A \rightarrow L$ that can be defined as the composite $A \rightarrow A_i \rightarrow L$, independent by the choice of i . So L has the structure of a A -algebra. We also have A -morphisms $A_i \rightarrow L$, $x_i \mapsto [x_i]$. We are left to prove the universal property. Let B be an A -algebra and let $\psi_i : A_i \rightarrow B$ compatible with the f_{ij} 's. In order to define a morphism $L \rightarrow B$ that commutes with the base of the direct set, the element $[x_i] \in L$ must be mapped into $\psi_i(x_i)$. We then have to prove that this definition is well posed and is a morphism of A -algebras, but this is straightforward.

Let's now turn to the exactness property in the category of A -modules. Because of the adjointness property, we are left to prove that the direct limit is left exact. Being in an abelian category, it suffices to prove it preserves injections. Let $\{\alpha_i : A_i \rightarrow B_i\}$ be a collection injective maps between two direct systems $\{A_i, f_{ij}\}$, $\{B_i, g_{ij}\}$ indexed by I , which are compatible with the maps f_{ij} , g_{ij} . By universal property, there is an induced map $\alpha : \varinjlim_{i \in I} A_i \rightarrow \varinjlim_{i \in I} B_i$, $[x_i] \mapsto [\alpha_i(x_i)]$. Suppose $\alpha[x_i] = 0$. This means that there exists a $k \geq i$ such that $(g_{ik} \circ \alpha_i)(x_i) = 0$. Because of the compatibility conditions, we have $(\alpha_k \circ f_{ik})(x_i) = (g_{ik} \circ \alpha_i)(x_i) = 0$. By the injectivity of the α_i 's we deduce $f_{ik}(x_i) = 0$, hence $[x_i] = 0$.

We remark that the preservation of surjections can be proved using a similar, explicit argument. $\square$

Before proceeding, we will make some examples of direct limits.

Example 1. *Any A -algebra B is the direct limit of its finitely generated sub- A -algebras B_i . In particular, $\mathbb{Q} = \varinjlim \mathbb{Z}[\frac{1}{n}]$.*

Proof. We obviously imply that the transition maps are inclusion, and the fact that they form a direct system is straightforward by the inclusion $A[b_1, \dots, b_n] \cup A[b_{n+1}, \dots, b_{n+m}] \subset A[b_1, \dots, b_{n+m}]$. The set of inclusion maps in B induces a map $\varinjlim B_i \rightarrow B$. This map is injective, because of its definition $[b] \mapsto b$, then $b \in A[b]$. The map is also surjective. Let $b \in B$, it is the image of $[b]$ which is an element of the direct limit, being $b \in A[b]$.

As for the last statement, recall that any subring $\mathbb{Z}[\frac{1}{d_1}, \dots, \frac{1}{d_n}]$ of $\mathbb{Q}$ which is finitely generated over $\mathbb{Z}$ is equal to $\mathbb{Z}[\frac{1}{\text{lcm}(d_i)}]$. $\square$

In general, an A -algebra B is not the direct limit of its sub- A -algebras of finite type, because in general $A[b]$ is not of finite type. Through a similar proof, it can be shown that the limit in this case is the algebraic closure of A in B .

Example 2. *Let A be a ring and $\mathfrak{p}$ a prime ideal. $A_{\mathfrak{p}}$ is the direct limit of the rings A_f with $f \notin \mathfrak{p}$, with respect to the natural maps $A_f \rightarrow A_g$ induced whenever $g \in \sqrt{fA}$. In particular, $\mathbb{Q} = \varinjlim \mathbb{Z}[\frac{1}{n}]$.*

Example 3. *As a set, $\text{Spec}(A)$ is the colimit of the system formed by $\{\text{Hom}(A, K)\}$ as K runs through fields and where transition maps are induced by inclusion of fields. This is not a direct system, but the colimit still exists.*

3 Characterisation of morphisms locally of finite presentation

We are finally ready to prove an alternative characterisation of A -algebras of finite presentation, using the categorical notions such as commutation with colimits.

Proposition 11. *Let B a A -algebra. It is of finite presentation if and only if the functor $\mathrm{Hom}_A(B, \cdot)$ from A -algebras to sets commutes with direct limits. Explicitly, B is of fine presentation if and only if for any direct system $\{C_i, f_{ij}\}$, the natural map*

$$\varinjlim_{i \in I} \mathrm{Hom}_A(B, C_i) \rightarrow \mathrm{Hom}_A(B, \varinjlim_{i \in I} C_i)$$

is a bijection.

We want to comment briefly how the natural map in the definition is constructed. By universal property, in order to define a map from a direct limit it is sufficient (and necessary) to define maps from each component of the direct set which are compatible with the transition maps. In this case, these are the maps $\mathrm{Hom}_A(B, C_i) \rightarrow \mathrm{Hom}_A(B, \varinjlim_{i \in I} C_i)$ induced by composition with the maps $C_i \rightarrow \varinjlim_{i \in I} C_i$.

Let's start with a lemma, which is nothing but the first half of the proposition.

Lemma 12. *Let B a A -algebra of finite presentation. Then the functor $\mathrm{Hom}_A(B, \cdot)$ from A -algebras to sets commutes with direct limits.*

Proof. Let $\{C_i, f_{ij}\}$ be a direct system of A -algebras. Any A -algebra B of finite presentation is isomorphic to $A[x_1, \dots, x_n]/(p_1, \dots, p_m)$. In particular, giving a map $B \rightarrow \varinjlim C_i$ is equivalent to give an n -tuple of elements $[c_i], \dots, [c_n]$ such that $p_j([c_1], \dots, [c_n]) = 0$ for all i, j . We can set an index i such that all the representatives c_i are in C_i (now I'm using the finite generation property). Because of the A -algebra structure defined on the limit, we have then $0 = p_j([c_1], \dots, [c_n]) = [p_j(c_1, \dots, c_n)]$. Hence each $p_j(c_1, \dots, c_n)$ is zero at some level. Now let k be an index such that $f_{ik}(p_j(c_1, \dots, c_n)) = 0$ for all j (now I'm using the finite presentation property). Because the transition maps are A -morphisms, we conclude that $p_j(f_{ik}(c_1), \dots, f_{ik}(c_n)) = 0$. By definition of B then, we can define a unique map $B \rightarrow C_k$ which is represented by the n -tuple $(f_{ik}(c_1), \dots, f_{ik}(c_n))$, hence an element of $\varinjlim_{i \in I} \mathrm{Hom}_A(B, C_i)$. This splitting is unique. Indeed, let $[f], [g]$ two elements in the direct limit splitting the same map. We can assume that they are represented by two maps f_k, g_k in $\mathrm{Hom}(B, C_k)$, i.e. by two n -tuples $(x_{k,1}, \dots, x_{k,n})$, $(y_{k,1}, \dots, y_{k,n})$ of elements in C_k such that they are zeroes of the m polynomials. Because they both split the map to the direct limit, we get that each $[x_{k,i}]$ is equal to $[y_{k,i}]$, hence there exists an index r in which the two n -tuples coincide. This means that $f_r = g_r$, and so $[f] = [g]$. $\square$

Note that in order to prove the injectivity of the map, we only used the fact that B is finitely generated. Before proving the other half of our statement, let's see some applications of what we just proved.

Example 4. $\mathbb{Q}$ is not finitely presented.

Proof. We already know that $\mathbb{Q}$ is not even finitely generated because each of its finitely generated subring is cyclic. Let's use our alternative condition to show another proof of this fact. As already shown, $\mathbb{Q} = \varinjlim \mathbb{Z}[\frac{1}{n}]$ with respect to the inclusion maps $\mathbb{Z}[\frac{1}{n}] \rightarrow \mathbb{Z}[\frac{1}{m}]$ defined by the universal property of localization. Suppose $\mathbb{Q}$ is finitely presented. Then the identity map $id_{\mathbb{Q}} \in \mathrm{Hom}(\mathbb{Q}, \mathbb{Q})$ should split at some level $\mathbb{Q} \rightarrow \mathbb{Z}[\frac{1}{n}] \hookrightarrow \mathbb{Q}$. By universal property though, there could be no $\mathbb{Z}$ -linear map from $\mathbb{Q}$ to $\mathbb{Z}[\frac{1}{n}]$ because this is not a field. We end up with a contradiction. $\square$

Now let's see an example which shows that being finitely generated is not sufficient.

Example 5. $\mathbb{Z}$ is not finitely presented as a A -algebra where $A = \mathbb{Z}[x_i]_{i \in \mathbb{N}}$, with respect to the structure induced by $x_i \mapsto 0$.

Proof. Again, this statement is obvious by the fact that the kernel of the morphism is not a finitely generated ideal, and being of finite presentation does not depend on the presentation given. Let's see another proof of that, using our last lemma. We claim that $L := \varinjlim_n A/(x_i)_{i \leq n} = \mathbb{Z}$. Indeed, we have by universal property a map $L \rightarrow \mathbb{Z}$ which is obviously surjective. Now let $[f(x_{N+1}, \dots, x_{N+M})]$ be in the kernel of the map, where $f \in A/(x_i)_{i \leq N}$. It means that has zero as the constant factor. Then f is mapped to zero through the projection to $A/(x_i)_{i \leq N+M}$, hence $[f] = 0$. Now if $\mathbb{Z}$ were of finite

presentation, then the identity map $id_{\mathbb{Z}} \in \text{Hom}_A(\mathbb{Z}, \mathbb{Z})$ would split through $\mathbb{Z} \rightarrow A/(x_i)_{i \leq n} \rightarrow \mathbb{Z}$ for some n , but this is absurd. Indeed, there could be no A -linear map from $\mathbb{Z}$ to $A/(x_i)_{i \leq n}$ because, for example, x_{n+1} is zero in $\mathbb{Z}$ but non-zero in $A/(x_i)_{i \leq n}$. $\square$

The tricks seen in these two counterexamples are the model by which the other part of the proposition can be finally proved.

Proof of Prop.11. This proof is split into two parts: we will show initially that a A -algebra B that satisfies the commutativity condition has to be finitely generated, then we will show that the kernel of the presentation is finitely generated as well. By Example 1, we know that B can be expressed as the direct limit of its finitely generated sub- A -algebras. By hypothesis then, there exists a splitting of the identity map $id_B \in \text{Hom}_A(B, B)$ in $B \rightarrow A[b_1, \dots, b_n] \hookrightarrow B$, where $A[b_1, \dots, b_n]$ is a finitely generated sub- A -algebra of B . We conclude that the inclusion map $A[b_1, \dots, b_n] \hookrightarrow B$ is surjective, hence an identity. B is finitely generated.

Let I be the kernel of a presentation $A[x] := A[x_1, \dots, x_n] \rightarrow B$. Again by Exercise 1, it is the direct limit of its finitely generated sub- A -algebras $\{I_i\}$. Because the direct limit is right exact (by the adjointness property for example), we can conclude that B is the direct limit of the direct system $\{A[x]/I_i, p_{ij}: A[x]/I_i \rightarrow A[x]/I_j\}$. In particular, the identity map of B splits $B \rightarrow A[x]/I_i \rightarrow B$ for some index i . Let's give a name to all the maps involved. We will let g_i be the induced map $g_i: B \rightarrow A[x]/I_i$, and we will refer to the projections with the following notations $p_i: A[x]/I_i \rightarrow B$, $\pi: A \rightarrow B$ and $\pi_i: A[x] \rightarrow A[x]/I_i$. We know that $\pi = p_i \pi_i$ for any i , and that $p_k p_{ik} = p_i$ for all $k \geq i$. In particular, calling g_k the map $p_{ik} g_i$ for any index $k \geq i$, we have another splitting of the identity $id_B = p_k g_k: B \rightarrow A[x]/I_k \rightarrow B$.

The map we obtain composing $g_i \pi$ needs not to be the same projection map π_i . However, claim that there exists a suitable index $k \geq i$ such that the map $g_k \pi$ is indeed the same projection map π_k . Because $p_i g_i = id_B$, we know that $p_i (g_i p_i \pi_i - \pi_i)(x_j) = 0$ for all $j = 1, \dots, n$. Because there are just finitely many j 's, we can hence suppose that all the elements $(g_i p_i \pi_i - \pi_i)(x_j)$ lie in I_k/I_i for some index $k \geq i$. This is equivalent to say that the two maps $p_{ik} g_i p_i \pi_i = g_k p_k$ and $\pi_{ik} \pi_i = \pi_k$ are indeed the same map, as claimed. Now we can write a commutative square

$$\begin{array}{ccc} A[x] & \xrightarrow{\pi} & B \\ \parallel & & \downarrow g_k \\ A[x] & \xrightarrow{\pi_k} & A[x]/I_k \end{array}$$

that fits into the following commutative diagram, in which the dotted arrow is induced by the universal property of the kernel.

$$\begin{array}{ccccccc} 0 & \longrightarrow & I & \longrightarrow & A[x] & \longrightarrow & B \longrightarrow 0 \\ & & \downarrow \text{dotted} & & \parallel & & \downarrow g_k \\ 0 & \longrightarrow & I_k & \longrightarrow & A[x] & \longrightarrow & A[x]/I_k \longrightarrow 0 \\ & & \downarrow & & \parallel & & \downarrow p_k \\ 0 & \longrightarrow & I & \longrightarrow & A[x] & \longrightarrow & B \longrightarrow 0 \end{array}$$

We obtain then the following diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & I & \longrightarrow & A[x] & \longrightarrow & B \longrightarrow 0 \\ & & \downarrow & & \parallel & & \parallel \\ 0 & \longrightarrow & I & \longrightarrow & A[x] & \longrightarrow & B \longrightarrow 0 \end{array}$$

in which the map $I \rightarrow I$ must be an isomorphism. The injection $I_k \hookrightarrow I$ must then be also surjective, hence it is an identity. We conclude that B is of finite presentation. $\square$

We denote by h_X the functor on rings represented by a scheme X , i.e. $h_X(A) := \text{Hom}(\text{Spec } A, X)$. We will use the same symbol in case we consider schemes over Y .

Corollary 13. *Let f be a morphism of schemes $f : \text{Spec } B \rightarrow Y$. Let also $(\text{Spec} \downarrow Y)$ be the category of rings A and maps $\text{Spec } A \rightarrow Y$, where an arrow is a morphism of rings $A \rightarrow B$ that induces a morphism of schemes $\text{Spec } B \rightarrow \text{Spec } A$ over Y . Then f is locally of finite presentation provided that the functor $h_{\text{Spec } B}|_{(\text{Spec} \downarrow Y)}$ commutes with direct limits. Explicitly, f is locally of finite presentation if for any direct system $\{C_i, f_{ij}\}$ of rings over Y , the natural map*

$$\varinjlim_{i \in I} \text{Hom}_Y(\text{Spec } C_i, \text{Spec } B) \rightarrow \text{Hom}_Y(\text{Spec}(\varinjlim_{i \in I} C_i), \text{Spec } B)$$

is a bijection.

Proof. Being locally of finite presentation is local on Y , so we can suppose that Y is affine. In this case, the corollary follows from Proposition 11. $\square$

The previous result is indeed a special case of a more general one, which we are not able to prove. It gives a fully categorical characterisation of locally of finite presentation morphisms of schemes.

Theorem 14 ([Gro66], IV.8.14.2). *Let f be a morphism of schemes $f : X \rightarrow Y$. Let also $(\text{Spec} \downarrow Y)$ be the category of rings A and maps $\text{Spec } A \rightarrow Y$, where an arrow is a morphism of rings $A \rightarrow B$ that induces a morphism of schemes $\text{Spec } B \rightarrow \text{Spec } A$ over Y . f is locally of finite presentation if and only if the functor $h_X|_{(\text{Spec} \downarrow Y)}$ commutes with direct limits. Explicitly, f is locally of finite presentation if and only if for any direct system $\{C_i, f_{ij}\}$ of rings over Y , the natural map*

$$\varinjlim_{i \in I} \text{Hom}_Y(\text{Spec } C_i, X) \rightarrow \text{Hom}_Y(\text{Spec}(\varinjlim_{i \in I} C_i), X)$$

is a bijection.

4 Back to the theorem

The link between the last conditions of the main theorem is a classical result of Grothendieck.

Theorem 15 ([Gro67], IV.17.6.2). *A morphism $f : X \rightarrow Y$ is étale if and only if it is flat, unramified ([Gro67], IV.17.3.1) and locally of finite presentation.*

This characterisation is more helpful in order to prove the implication $(ii) \Rightarrow (i)$. We are not able to show it completely, but we can give a sketch of part of it. We will show how to prove that a flat locally of finite presentation map is open.

Theorem 16. *Let $f : X \rightarrow Y$ be a flat monomorphism locally of finite presentation. Then it is a homeomorphism onto its image.*

Sketch of Proof. Monomorphisms are injective, so we are left to prove that such a map is open. Being open is a local property on X and Y , so we can think of them as affine schemes, let's say $X = \text{Spec } B$, $Y = \text{Spec } A$. Fix a point $x = \mathfrak{q}$ of X and its image $f(x) = \mathfrak{p}$ in Y . Consider the flat local map induced at stalks $\mathcal{O}_{Y,y} = A_{\mathfrak{p}} \rightarrow \mathcal{O}_{X,x} = B_{\mathfrak{q}}$. We claim that it induces a surjective map on spectra, i.e. every ideal in $A_{\mathfrak{p}}$ is a contracted ideal. Let $\mathfrak{a}$ be an ideal of $A_{\mathfrak{p}}$. Consider the map of $A_{\mathfrak{p}}$ -modules

$$A_{\mathfrak{p}}/\mathfrak{a} \rightarrow A_{\mathfrak{p}}/\mathfrak{a} \otimes_{A_{\mathfrak{p}}} B_{\mathfrak{q}} = B_{\mathfrak{q}}/\mathfrak{a}B_{\mathfrak{q}}, \quad x \mapsto x \otimes 1.$$

Let K be its kernel. We have $K = \mathfrak{a}^{ec}/\mathfrak{a}$. In order to prove that $\mathfrak{a}$ is contracted, it is then sufficient to prove that $K = 0$. Because of the flatness property, we obtain the exact sequence

$$0 \rightarrow K \otimes_{A_{\mathfrak{p}}} B_{\mathfrak{q}} \rightarrow A_{\mathfrak{p}}/\mathfrak{a} \otimes_{A_{\mathfrak{p}}} B_{\mathfrak{q}} \rightarrow A_{\mathfrak{p}}/\mathfrak{a} \otimes_{A_{\mathfrak{p}}} B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} B_{\mathfrak{q}}$$

but the last map is an injection, hence $K \otimes_{A_{\mathfrak{p}}} B_{\mathfrak{q}} = 0$. We claim that this implies $K = 0$. Indeed, suppose there exists $0 \neq x \in K$, and consider the sub- $A_{\mathfrak{p}}$ -module $A_{\mathfrak{p}}x$. Because of the flatness property, we have that $A_{\mathfrak{p}}x \otimes_{A_{\mathfrak{p}}} B_{\mathfrak{q}} \rightarrow K \otimes_{A_{\mathfrak{p}}} B_{\mathfrak{q}}$ is still an injection. Then can just prove that $A_{\mathfrak{p}}x \otimes_{A_{\mathfrak{p}}} B_{\mathfrak{q}}$ can't be zero. But this module is $B_{\mathfrak{q}}/\text{Ann}(x)^e$, and because $\text{Ann}(x) \subset \mathfrak{p}A_{\mathfrak{p}}$, $(\mathfrak{p}A_{\mathfrak{p}})^e \subset \mathfrak{q}B_{\mathfrak{q}}$, we can conclude that is not zero. Hence we finally proved that the map of spectra is surjective.

As a topological space, $\text{Spec } B_{\mathfrak{q}}$ is the intersection of all open neighbourhoods of x , and similarly $\text{Spec } A_{\mathfrak{p}}$ is the intersection of all open neighbourhoods of $f(x)$. Indeed, $\text{Spec } A_{\mathfrak{p}} = \{\mathfrak{p}' \in \text{Spec } A : \mathfrak{p}' \subset$

$\mathfrak{p}\} = \bigcap_{g \notin \mathfrak{p}} \{\mathfrak{p}' : g \notin \mathfrak{p}'\} = \bigcap_{g \notin \mathfrak{p}} D(g)$. We have then proved that the intersection of all open sets containing $f(x)$ is contained in all the images of open sets containing x . We claim that this implies that $f(x)$ is in the interior of $f(U)$ for any open neighbourhood U of x , hence that the map is open at x . Fix an affine open neighbourhood U of x . Because f is locally of finite presentation, by Chevalley Theorem and by [Gro64], I.1.9.5 (viii)-(ix), we can conclude that there exists a map $\text{Spec } C \rightarrow Y$ such that its image is $Y \setminus f(U)$. Because $\text{Spec } A_{\mathfrak{p}}$ lies inside $f(U)$, we conclude that $\text{Spec } A_{\mathfrak{p}} \times_Y \text{Spec } C = \text{Spec } A_{\mathfrak{p}} \otimes_A C$ is empty. This means $0 = \text{Spec } A_{\mathfrak{p}} \otimes_A C = (\varinjlim_{g \notin \mathfrak{p}} A_g) \otimes_A C$. Because tensor products commute with colimits (left adjointness), we conclude $\varinjlim_{g \notin \mathfrak{p}} (A_g \otimes_A C) = 0$. But if a direct limit of A algebras is zero then $[0] = [1]$, hence one of the elements in the direct system is zero. We conclude that there exists a $g \notin \mathfrak{p}$ such that $(A_g \otimes_A C) = 0$, which means that $\text{Spec } A_g = D(g)$ is included in $f(U)$. $\square$

The monomorphism hypothesis has a central role in proving that such a map is indeed an open immersion, and not just a homeomorphism on its image.

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