Grothendieck groups

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Summary

This talk intends to introduce one aspect of the Grothendieck-Riemann-Roch theorem: K-theory.

We define the Grothendieck group $K_0(X)$ associated to a projective variety X. We shall study some of its properties, such as its ringstructure, and give some elementary examples. There is another Grothendieck group which is easier to construct, namely $K^0(X)$. The main goal will consist of establishing a natural group isomorphism between the groups $K_0(X)$ and $K^0(X)$ when X is nonsingular, quasi-projective and irreducible. We will finish with a generalization of the Riemann-Roch theorem for nonsingular curves.

Varieties will always be quasi-projective over an algebraically closed field.

1 K-theory

Let $\mathcal{C}$ be an additive category embedded in an abelian category $\mathcal{A}$. Let $\mathrm{Ob}(\mathcal{C})$ denote the class of objects and let $\mathrm{Ob}(\mathcal{C})/\cong$ be the set of isomorphism classes[1]. Let $F(\mathcal{C})$ be the free abelian group on $\mathrm{Ob}(\mathcal{C})/\cong$, i.e. an element $T \in F(\mathcal{C})$ is a finite formal sum

$$\sum n_X[X],$$

where $[X]$ denotes the isomorphism class of $X \in \mathrm{Ob}(\mathcal{C})$ and n_X is an integer which is almost always zero.

Definition 1.1. To any sequence

$$(E) \qquad 0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

in $\mathcal{C}$, which is exact in $\mathcal{A}$, we associate the element $Q(E) = [A] - [A'] - [A'']$ in $F(\mathcal{C})$. Let $H(\mathcal{C})$ be the subgroup generated by the elements $Q(E)$ where E runs through all short exact sequences.

Definition 1.2. We define the *Grothendieck group*, denoted by $K(\mathcal{C})$, as the quotient group

$$K(\mathcal{C}) = F(\mathcal{C})/H(\mathcal{C}).$$

Remark 1.3. Note that $\mathcal{C}$ has finite direct sums. The fact that the sequence

$$0 \longrightarrow A \longrightarrow A \oplus B \longrightarrow B \longrightarrow 0$$

is exact (in $\mathcal{A}$) shows that the addition is given by $[A \oplus B] = [A] + [B]$.

Examples 1.4. Let R be a commutative ring.

1. Let $\mathcal{C}$ be the category of R-modules. (This is not a small category. To avoid this the reader may consider only countably generated R-modules.) Let us show that $K(\mathcal{C}) = (0)$. To this extent, let M be an R-module and note that $M \oplus \bigoplus_{n\in\mathbf{N}} M \cong \bigoplus_{n\in\mathbf{N}} M$. We see that $[M] + [\bigoplus_{n\in\mathbf{N}} M] = [\bigoplus_{n\in\mathbf{N}} M]$, which shows that $[M] = 0$ in $K(\mathcal{C})$.

[1] The reader should actually only consider small categories.

2. Let R be a principal ideal domain and $\mathcal{C}$ be the category of finitely generated R-modules. By the structure theorem of R-modules, any R-module is isomorphic to the direct sum of a free module and a torsion part which is the direct sum of cyclic modules. The rank of an R-module is defined as the rank of its free part. That gives us a surjective map $\mathrm{rk} : \mathrm{Ob}(\mathcal{C})/\cong \longrightarrow \mathbf{Z}$ which induces a surjective homomorphism from $F(\mathcal{C})$ to $\mathbf{Z}$. Since the rank is trivial on the elements $Q(E)$, it induces a group morphism $\widetilde{\mathrm{rk}}$ from $K(\mathcal{C})$ to $\mathbf{Z}$. Note that for any nonzero ideal $I = (x)$, we have a short exact sequence

$$0 \longrightarrow R \xrightarrow{\cdot x} R \longrightarrow R/I \longrightarrow 0\ ,$$

which shows that $[R/I] = 0$ in $K(\mathcal{C})$. Therefore the kernel of $\widetilde{\mathrm{rk}}$ is trivial. Furthermore, since any short exact sequence of free R-modules is split and $\mathrm{rk}([R]) = 1$, the rank induces an isomorphism from $K(\mathcal{C})$ to $\mathbf{Z}$.

3. The above example actually shows that if R is a ring and $\mathcal{C}$ is the category of finitely generated free R-modules, we have an isomorphism $\mathrm{rk} : K(\mathcal{C}) \xrightarrow{\cong} \mathbf{Z}$.

4. When $\mathcal{C}$ is the category of finitely generated projective R-modules, the reader may look at Chapter II of Weibel's book on K-theory.

2 $\mathcal{O}_X$-modules

The reader is referred to Chapter II, paragraph 5 of [HAG] for the theory of coherent sheafs on affine and projective varieties.

Let X be a quasi-projective variety and let $\mathcal{O}_X$ be its structure sheaf. Recall that the affine open subsets of X form a basis for the topology on X and that $\mathcal{O}_X$ is determined by the rule

$$\mathcal{O}_X(U) = \Gamma(U, \mathcal{O}_X) = \Gamma(U, \mathcal{O}_U) = k[T_1, \ldots, T_n]/I,$$

if $U \subset X$ is isomorphic to the affine variety determined by the prime ideal $I \subset k[T_1, \ldots, T_n]$.

Definition 2.1. A *coherent sheaf* is a sheaf of abelian groups $\mathcal{F}$ on X endowed with a multiplication $\mathcal{O}_X \times \mathcal{F} \longrightarrow \mathcal{F}$ such that the following properties hold.

$\mathcal{O}_X$-module structure: For each open $U \subset X$, the abelian group of sections $\mathcal{F}(U)$ becomes a module over $\mathcal{O}_X(U)$.

Quasi-coherence: For every open affine subsets $U \subset V \subset X$, $\mathcal{F}(U) = \mathcal{F}(V) \otimes_{\mathcal{O}_X(V)} \mathcal{O}_X(U)$.

Coherence: For each open affine $U \subset X$ the module $\mathcal{F}(U)$ is finitely generated over $\mathcal{O}_X(U)$.

Let $\mathrm{Coh}(X)$ be the category of coherent sheaves on X. (A morphism of coherent sheaves is a morphism of sheaves which respects the module structure.)

Remark 2.2. The category $\mathrm{Coh}(X)$ is abelian.

Definition 2.3. A *vector bundle (of rank r)* is a coherent sheaf $\mathcal{F}$ where every point $x \in X$ has an affine neighborhood $U \subset X$ such that $\mathcal{F}(U)$ is a free $\mathcal{O}_X(U)$-module of rank r. A *line bundle* is a vector bundle of rank 1. Let $\mathrm{Vect}(X)$ be the category of vector bundles on X.

Remark 2.4. The category $\mathrm{Vect}(X)$ is additive and embedded in $\mathrm{Coh}(X)$. It is not an abelian category in general. The following theorem states why this is the case.

Theorem 2.5. For $X = \mathrm{Spec}(A)$ and A noetherian, $\mathrm{Vect}(X)$ is equivalent to the category of projective finitely generated A-modules and $\mathrm{Coh}(X)$ is equivalent to the category of finitely generated A-modules.

3 K-theory of a variety

Let X be a quasi-projective variety.

Definition 3.1. We define the *Grothendieck group of vector bundles* on X, denoted by $K^0(X)$, as

$$K^0(X) = K(\mathrm{Vect}(X)).$$

Proposition 3.2. The tensor product (over $\mathcal{O}_X$) defines a commutative ringstructure on $F(\mathrm{Vect}(X))$.

Proof. The tensor product of vector bundles is a vector bundle. The tensor product is associative and commutative as follows from its universal property and $\mathcal{O}_X$ is clearly the identity element. The tensor product is also distributive with respect to the direct sum. $\square$

Proposition 3.3. The tensor product defines a commutative ring structure on $K^0(X)$.

Proof. We need to show that the subgroup $H(\mathrm{Vect}(X))$ is an ideal of $F(\mathrm{Vect}(X))$. But this follows from the fact that any vector bundle is flat. $\square$

Definition 3.4. The *Grothendieck group of coherent sheaves*, denoted by $K_0(X)$, is defined as

$$K_0(X) = K(\mathrm{Coh}(X)).$$

Remark 3.5. The embedding $\mathrm{Vect}(X) \longrightarrow \mathrm{Coh}(X)$ of categories induces a natural homomorphism $K^0(X) \longrightarrow K_0(X)$.

Theorem 3.6. If X is nonsingular, quasi-projective and irreducible, the canonical homomorphism $K^0(X) \longrightarrow K_0(X)$ is an isomorphism of groups.

Proof. From standard considerations on projective varieties it follows that any coherent sheaf $\mathcal{F}$ is the quotient of some vector bundle: for $n \gg 0$, the twisted sheaf $\mathcal{F}(n)$ is generated by its global sections. Since X is quasi-compact, we may cover X with a finite number of open affine subsets U_i $(i = 1, \ldots, d)$. On each U_i, $\mathcal{F}(n)(U_i)$ is generated by a finite number of global sections and therefore there exist a finite number of global sections $s_1, \ldots, s_r \in \mathcal{F}(n)(X)$ which generate $\mathcal{F}(n)$ on every open U_i. Therefore there is a surjective morphism $\mathcal{O}_X^r \longrightarrow \mathcal{F}(n)$. Since the tensor product is right exact, tensoring this with $\mathcal{O}_X(-n)$ gives a surjective morphism $\mathcal{O}_X^r(-n) \longrightarrow \mathcal{F}$. Replacing a quasi-projective variety by its closure in some projective and extending our sheaf to this closure shows that any coherent sheaf on X is the quotient of a vector bundle. This allows one to always construct a (not necessarily finite) resolution of vector bundles for a coherent sheaf.

Let $n = \dim(X)$. Since X is nonsingular projective, any coherent sheaf $\mathcal{F}$ has a finite resolution of vector bundles $\mathcal{E}_0, \ldots, \mathcal{E}_n$. That is, we have a complex

$$\cdots \longrightarrow 0 \longrightarrow \mathcal{E}_n \longrightarrow \cdots \longrightarrow \mathcal{E}_0 \longrightarrow 0$$

such that the augmented complex

$$0 \longrightarrow \mathcal{E}_n \longrightarrow \cdots \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{F} \longrightarrow 0$$

is exact. This means that we can define an inverse to the above map by $[\mathcal{F}] \mapsto \sum_{i=1}^{\dim X} (-1)^i [\mathcal{E}_i]$. One can show that this map is well-defined, i.e. independent of the chosen resolution and that it an additive map. (See Lemma 11 and Lemma 12 in [BorSer].) $\square$

Let us illustrate the importance of nonsingularity.

Example 3.7. Let k be a field, $A = k[x]$ and $I = (x) \subset A$. (Picture) The A-module $k = A/I$ has a finite resolution of free A-modules

$$0 \longrightarrow A \xrightarrow{f} A \longrightarrow k \longrightarrow 0\ .$$

Here $f : s \mapsto sx$. We see that the (Krull) dimension of A is equal to the length of this (minimal) resolution.

Example 3.8. Let k be a field, $A = k[x, y]$ and $I = (x, y) \subset A$. (Picture) The ring A is regular and the A-module $k = A/I$ has a finite resolution of free A-modules

$$0 \longrightarrow A \xrightarrow{g} A^2 \xrightarrow{f} A \longrightarrow k \longrightarrow 0\ .$$

Here $g : s \mapsto (-sy, sx)$ and $f : (s, t) \mapsto sx + ty$. Again the dimension of A is equal to the length of this (minimal) resolution.

Example 3.9. Let k be a field, $A = k[x, y]/(xy)$ and $I = (x, y) \subset A$. (Picture) Note that A is nonregular. Consider the infinite resolution of free A-modules for $k = A/I$ given by

$$\cdots \xrightarrow{g} A^2 \xrightarrow{h} A^2 \xrightarrow{g} A^2 \xrightarrow{h} A^2 \xrightarrow{g} A^2 \xrightarrow{f} A \longrightarrow k \longrightarrow 0\ .$$

Here

$$f : (s, t) \mapsto sx + ty,\ \ g : (s, t) \mapsto (sy, tx) \text{ and } h : (s, t) \mapsto (sx, ty).$$

It is easy to see that

$$Tor_i^A(k, k) = \begin{cases} k & \text{if} \quad i = 0 \\ k^2 & \text{if} \quad i > 0 \end{cases}\ .$$

To this extent it suffices to note that after tensoring the above resolution with $k \otimes_A -$ all the maps are zero. This shows that there can not be a finite (projective) resolution of A-modules for k. (Since then the $Tor_i^A(k, -)$ functors would be identically zero for $i \gg 0$.) Now, in contrast with the above examples, we see that the dimension of A is strictly smaller than the length of any flat resolution, i.e., the flat dimension of k is bigger than the dimension of A. The reader may took a look at Chapter 4 of Weibel's Homological Algebra for a readable account on dimension theory for rings.

Corollary 3.10. Using the isomorphism in Theorem 3.6 we get a ringstructure on $K_0(X)$. We can show that the product of two coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ equals $\mathcal{F} \cdot \mathcal{G} = \sum_{i=0}^{\dim X} (-1)^i\, Tor_i^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ in $K_0(X)$.

The next talk will be on a generalization of the Riemann-Roch theorem for non-singular curves.

References

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