Towards the Riemann-Roch Theorem

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Summary

The previous talk introduced some of the basics of K-theory needed in order to state the Grothendieck-Riemann-Roch theorem. This talk will introduce some other aspects of the Riemann-Roch theorem such as the cohomology of coherent sheaves and the Euler characteristic of such a sheaf on a variety. Since it is instructive to fully understand the lower-dimensional cases of the classical Riemann-Roch theorem, we will illustrate the theory for a nonsingular curve. At this level the introduction of the Chern character and the Chow ring only use the concept of a divisor class.

A variety is always quasi-projective over an algebraically closed field k.

Cohomology of sheaves

The category of abelian groups is denoted by Ab and the category of abelian sheaves on a topological space X is denoted by $\mathrm{Ab}(X)$. The global sections functor $\Gamma(X,-) : \mathrm{Ab}(X) \longrightarrow \mathrm{Ab}$ is given by $\Gamma(X,\mathcal{F}) = \mathcal{F}(X)$ for any $\mathcal{F} \in \mathrm{Ab}(X)$.

Proposition 1. The global sections functor is left exact.

Proof. The global sections functor is right adjoint to the constant sheaves functor. $\square$

Proposition 2. The category $\mathrm{Ab}(X)$ has enough injectives.

Proof. We admit the fact that Ab has enough injectives. Let $\mathcal{F} \in \mathrm{Ab}(X)$. Then for any $x \in X$, there is an injection of abelian groups $\mathcal{F}_x \hookrightarrow I_x$ where I_x is an injective abelian group. Let $j^x : \{x\} \hookrightarrow X$ be the embedding and let j^x_* be the direct image functor. Consider I_x as a sheaf on $\{x\}$ and define the sheaf

$$\mathcal{I} = \prod_{x\in X} j^x_*(I_x).$$

It is easy to see that we obtain a natural map $\mathcal{F} \longrightarrow \mathcal{I}$ from the above maps $\mathcal{F}_x \hookrightarrow I_x$. This natural map is clearly injective. Since the functor $\mathrm{Hom}(\cdot,\mathcal{I})$ is an exact functor, $\mathcal{I}$ is an injective object of $\mathrm{Ab}(X)$. $\square$

Remark 3. The global sections functor is not exact in general. Its right derived functors in the category of sheaves constitute the cohomology functors of X.

Definition 4. For any $i \in Z$, we define the *i-th cohomology functor* of X, denoted by $H^i(X,-)$, as the i-th right derived functor of $\Gamma(X,-)$ in the category of sheaves, i.e.

$$H^i(X,-) = R^i\Gamma(X,-).$$

For $\mathcal{F} \in \mathrm{Ab}(X)$, we define the *$i$-th cohomology group* of $\mathcal{F}$ by

$$H^i(X,\mathcal{F}) = H^i(X,-)\mathcal{F} = (R^i\Gamma(X,-))\mathcal{F}.$$

Let X be an n-dimensional projective variety over an algebraically closed field k. Let $\mathrm{Coh}(X)$ be the abelian category of coherent sheaves on X.

Theorem 5. For any coherent sheaf $\mathcal{F}$ on X and any $i \in \mathbf{Z}$, it holds that $H^i(X, \mathcal{F})$ is a finite dimensional k-vector space.

Proof. Let $\mathcal{F}$ be a coherent sheaf on X. Since the regular functions of a projective variety are constant, i.e. $\mathcal{O}_X(X) = k$, the cohomology groups of $\mathcal{F}$ are k-vector spaces. More precisely, the multiplication $\mathcal{F} \xrightarrow{s} \mathcal{F}$ by $s \in \mathcal{O}_X$ induces a morphism $H^i(X, \mathcal{F}) \longrightarrow H^i(X, \mathcal{F})$ which defines an $\mathcal{O}_X$-module structure on $\mathcal{F}$. The cohomology groups are finite dimensional by [Har, Theorem 5.2 (Serre), Chapter III.5]. □

Theorem 6. For any coherent sheaf $\mathcal{F}$ on X, it holds that

$$H^n(X, \mathcal{F}) = H^{n+1}(X, \mathcal{F}) = \ldots = 0.$$

Proof. [Har, Theorem 2.7 (Grothendieck), Chapter III.2]. □

Definition 7. The *Euler characteristic* of a coherent sheaf $\mathcal{F}$ on X, denoted by $\chi(X, \mathcal{F})$, is defined as

$$\chi(X, \mathcal{F}) = \sum_{i=0}^{\infty} \dim_k H^i(X, \mathcal{F}) = \sum_{i=0}^{\dim X} \dim_k H^i(X, \mathcal{F}).$$

Proposition 8. For any exact sequence of coherent sheaves on a projective variety X

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0 \,,$$

it holds that $\chi(\mathcal{F}) = \chi(\mathcal{F}') + \chi(\mathcal{F}'')$.

Proof. By the δ-functoriality of right derived functors, a short exact sequence

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

gives rise to a long exact sequence of vector spaces

$$\begin{array}{l} 0 \longrightarrow \mathcal{F}'(X) \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{F}''(X) \longrightarrow \ . \\ \longrightarrow H^1(\mathcal{F}', X) \longrightarrow H^1(\mathcal{F}, X) \longrightarrow H^1(\mathcal{F}'', X) \longrightarrow \\ \longrightarrow H^2(\mathcal{F}', X) \longrightarrow \cdots\cdots \end{array}$$

The alternating sum of the dimensions is zero, i.e. $\chi(\mathcal{F}) = \chi(\mathcal{F}') + \chi(\mathcal{F}'')$. □

Corollary 9. The homomorphism $\chi(X, -) : K_0(X) \longrightarrow \mathbf{Z}$ is well-defined.

This finishes one aspect of the Riemann-Roch theorem. Namely, that of the Euler characteristic of a coherent sheaf on a projective variety. The next step is to link this number to a certain intersection number. Unfortunately, we will only be able to treat the one-dimensional case which does not involve any observable intersection theory.

The Chow ring

Let X be an integral variety, i.e. an integral quasi-projective variety over an algebraically closed field. A subvariety of X will be a closed irreducible subset which is a variety. We let $n = \dim X$.

Definition 10. Let r be an integer. A *cycle of codimension* r on X is an element of the free abelian group $Z^r X$ on the subvarieties of codimension r, i.e. a finite formal sum $\sum n_V V$ where V is a subvariety of codimension r and n_V is an integer. Note that $Z^i X = 0$ for all $i > n = \dim X$ and all $i < 0$.

Definition 11. A *Weil divisor* is a cycle of codimension 1. We write $Z^1 X = \mathrm{Div}(X)$ in this case.

Example 12. Note that $Z^0 X = \mathbf{Z}$.

Remark 13. The free group $Z^{\cdot} X = \bigoplus_r Z^r X$ is graded by codimension. If we let $Z_r X$ denote the free abelian group on the subvarieties of dimension r, then $Z_r X = Z^{n-r} X$. Therefore the grading by dimension of $Z^{\cdot} X$ is a renumbering of the grading by codimension. The grading by codimension is more useful though, since it is also applicable in the setting of (noetherian) schemes.

Let X be a non-singular, complete curve, i.e. a non-singular projective variety of dimension 1 over an algebraically closed field k. (On such schemes projective and complete are equivalent.)

We let subvarieties of X correspond to their generic points. Subvarieties of codimension 1 correspond to closed points of X whereas X itself corresponds to its generic point. Therefore, we write a divisor on a curve as a finite formal $D = \sum n_x x$ where n_x is an integer and x is a (closed) point of X.

Recall the following from commutative algebra. Let R be a one-dimensional integral domain with fraction field K. For $f \in K^*$, $f = \frac{a}{b}$ with $a, b \in R$ we put

$$\mathrm{ord}_R(f) := \dim_K(R/aR) - \dim_K(R/bR)$$

and call it the order of f. This defines a homomorphism $\mathrm{ord}_R : K^* \longrightarrow \mathbf{Z}$. When R is a one-dimensional regular local ring, then the order of any $f \in R$ coincides with the valuation of f. In this case, R is a discrete valuation ring.

Going back to geometry, let K be the function field of X. For any (closed) point $x \in X$, the local ring $\mathcal{O}_{X,x}$ is one-dimensional and regular (by our assumption on X) with quotient field K. Let ord_x denote the homomorphism $\mathrm{ord}_{\mathcal{O}_{X,x}} : K^* \longrightarrow \mathbf{Z}$ given above. Note that for any $f \in K$, there is only a finite number of points $x \in X$ such that $\mathrm{ord}_x(f) \neq 0$.

Definition 14. We define the *divisor* of a rational function $f \in K^*$ by

$$\mathrm{div}(f) = \sum \mathrm{ord}_x(f) x.$$

A divisor which is the divisor of a rational function is called a *principal* divisor.

Remark 15. The principal divisors form a subgroup of $\mathrm{Div}(X)$.

Definition 16. The *divisor class group* on X, denoted by $\mathrm{Cl}(X)$, is the quotient of $\mathrm{Div}(X)$ by the subgroup of all principal divisors.

Definition 17. We define the *degree*, denoted by deg, of a divisor $D = \sum n_x x$ on X by

$$\deg(D) = \sum n_V.$$

Clearly, $\deg : \mathrm{Div}(X) \longrightarrow \mathbf{Z}$ is a morphism of groups.

Lemma 18. The degree of a principal divisor (on a complete nonsingular curve) is zero.

This Lemma proves the following theorem.

Theorem 19. The degree function induces a surjective group morphism $\deg : \mathrm{Cl}(X) \longrightarrow \mathbf{Z}$.

Example 20. The degree function $\deg : \mathrm{Cl}(\mathbf{P}^1_k) \longrightarrow \mathbf{Z}$ is an isomorphism of groups. Let us show this.

Suppose that $D \in \mathrm{Cl}(X)$ is of degree zero. Then $D = D_1 - D_2$, where D_1 and D_2 are (classes of) effective divisors of the same degree d. A divisor $D \in \mathrm{Div}(X)$ is *effective* if all coefficients of D are ≥ 0. Since sums of principal divisors are principal, it suffices to show that D_1 and D_2 are principal divisors. Therefore, we may assume that D is effective. Since an effective divisor is the sum of points, we may assume that $D = P$ is a point.

We shall use [Har, Chapter I.3, Theorem 3.4.b].

Recall that the (closed) points of $\mathbf{P}^1_k$ are given by set of homogeneous coordinates $(a_0 : a_1)$ with $a_0, a_1 \in k$. Let us suppose that $P = (a_0 : a_1) \in \mathbf{P}^1_k$. The coordinate ring of $\mathbf{P}^1_k$ is $k[x, y]$ and its fraction field determines the function field. We claim that $P = \mathrm{div}(a_1 x - a_0 y)$. This follows from the fact that the ideal $(a_1 x - a_0 y) \subset k[x, y]$ is precisely the ideal generated by the set of homogeneous $f \in k[x, y]$ such that $f(P) = 0$.

Definition 21. The *Chow group* of X, denoted by $A(X)$, is defined as $A(X) = \mathbf{Z} \oplus \mathrm{Cl}(X)$.

Remark 22. We write $A^0(X) = \mathbf{Z}$, $A^1(X) = \mathrm{Cl}(X)$, $A^2(X) = A^3(X) = \ldots = 0$ and note that $A(X) = \bigoplus_{r=0}^{\infty} A^r(X)$ is a graded ring.

Remark 23. The multiplication $(n, D)(m, E) = (nm, nE + mD)$ defines a graded ringstructure $\mathbf{Z} \oplus \mathrm{Div}(X)$ where the identity element is $(1, 0)$ and $(0, \mathrm{Div}(X))$ is an ideal of $\mathbf{Z} \oplus \mathrm{Div}(X)$ which squares to zero. Since the subgroup of principal divisors forms an ideal in $\mathbf{Z} \oplus \mathrm{Div}(X)$, there is a commutative ringstructure on $A(X)$ given by $(n, D)(m, E) = (nm, nE + mD)$. In particular, the identity element is $(1, 0)$ and the class group $\mathrm{Cl}(X)$ is an ideal of $A(X)$ which squares to zero.

Definition 24. The graded ring $A^{\cdot}X$ is called the *Chow ring* of X.

Example 25. It is easy to see that the Chow ring $A(\mathbf{P}^1_k) = \mathbf{Z}[P]/(P^2)$ where P is a point of $\mathbf{P}^1_k$. An explicit isomorphism is given by $A(\mathbf{P}^1_k) \ni (n, D) \mapsto n + \deg(D)P$.

Definition 26. We define the *degree* function on the Chow ring, denoted by deg, as the function $\deg : A(X) \longrightarrow \mathbf{Z}$ given by $\deg(n, D) = \deg(D)$. It is a surjective morphism of groups and consists of taking the projection on $\mathrm{Cl}(X)$ followed by the usual degree.

The Chern character

Let X be a nonsingular complete curve over an algebraically closed field k.

Recall that $K^0(X)$ denotes the Grothendieck group of vector bundles and that $K_0(X)$ denotes the Grothendieck group of coherent sheaves. Since X is a nonsingular curve, we have a natural isomorphism $K^0(X) \xrightarrow{\sim} K_0(X)$. Therefore, we shall usually write $K(X)$ for $K^0(X) = K_0(X)$.[1]

Definition 27. An *invertible sheaf* is a vector bundle of rank 1 on X. Let $\mathrm{Pic}(X) \subset K(X)$ be the subset of (isomorphism classes of) invertible sheaves.

Proposition 28. It holds that $\mathrm{Pic}(X)$ is an abelian subgroup of $K(X)^*$. We call it the *Picard group* of X.

[1] The notation $K(X)$ is used for the field of rational functions in [Har]. We use the notation K for the field of rational functions whereas [Har] also uses this to denote the canonical divisor.

Proof. The structure sheaf $\mathcal{O}_X$ is clearly an invertible sheaf. Furthermore, the tensor product of invertible sheaves is an invertible sheaf. Moreover, for any invertible sheaf $\mathcal{L}$ its dual sheaf $\mathcal{L}^\vee$ is an inverse, i.e. $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^\vee \cong \mathcal{O}_X$. $\square$

Proposition 29. *We have a homomorphism $c_1 : \mathrm{Pic}(X) \longrightarrow \mathrm{Cl}(X)$ of groups which is called the first Chern class.*

Proof. Let $\mathcal{L}$ be an invertible sheaf. Take a non-empty open subset $U \subset X$ such that there is a nonzero section $s \in \mathcal{L}(U)$. For $x \in X$, let (s_x) be a basis of $\mathcal{L}_x$ over $\mathcal{O}_{X,x}$. This determines an isomorphism $\mathcal{O}_{X,x} \longrightarrow \mathcal{L}_x$ of $\mathcal{O}_{X,x}$-modules given by $f \mapsto fs_x$. Let $U_x \subset U$ be an open subset such that s_x is a nonzero section of $\mathcal{L}(U_x)$. Then this determines an isomorphism $\mathcal{O}_X|_{U_x} \longrightarrow \mathcal{L}|_{U_x}$ given by $f \mapsto fs_x$. We see that there exists a unique nonzero regular function $f \in \mathcal{O}_X(U_x)$ such that $s|_{U_x} = fs_x$. We define

$$v_x(s) = \mathrm{ord}_x(f) \in \mathbf{Z}.$$

Let us show that this is well-defined.

Suppose that we had chosen another basis (s'_x) for $\mathcal{L}_x$ over $\mathcal{O}_{X,x}$. Since $\mathcal{L}_x$ is of rank 1, there exists an invertible element $u \in \mathcal{O}^*_{X,x}$ such that $s'_x = us_x$. Then $s|_{U'_x} = f's'_x = (fu^{-1})(us_x)$ where $f' = fu^{-1}$. Since $\mathrm{ord}_x(f) = \mathrm{ord}_x(f')$, we see that $v_x(s)$ is independent of the chosen basis.

Now, we fix some rational section $s \in \mathcal{L}(U)$ and we define $c_1(\mathcal{L})$ to be the class of the divisor $\sum v_x(s)x \in \mathrm{Cl}(X)$. Now, suppose we take another rational section $s' \in \mathcal{L}(U)$. Then there is a rational function $f \in \mathcal{K}$ such that $s' = fs$ and one sees that $\sum v_x(s)x - \sum v_x(s')x$ is given by the principal divisors $\mathrm{div}(f)$. Therefore $c_1(\mathcal{L})$ is a well-defined element of $\mathrm{Cl}(X)$. Finally, note that $c_1(\mathcal{L})$ does not depend on the isomorphism class of $\mathcal{L}$ and therefore gives a well-defined map from $\mathrm{Pic}(X)$ to $\mathrm{Cl}(X)$. It is clearly a homomorphism. $\square$

Example 30. Since c_1 is a homomorphism, it holds that $c_1(\mathcal{O}_X) = 0$.

Remark 31. There is a unique morphism of rings $i : \mathbf{Z} \longrightarrow K(X)$ given by $n \mapsto n\mathcal{O}_X$. Furthermore, we also have a morphism of groups $\mathrm{rk} : K(X) \longrightarrow \mathbf{Z}$ from the additive group $K(X)$ to $\mathbf{Z}$ given by the rank. Note that $\mathrm{rk} \cdot i = \mathrm{id}_{\mathbf{Z}}$. This shows the surjectivity of rk and injectivity of i.

Remark 32. We have a morphism of groups $j : \mathrm{Pic}(X) \longrightarrow K(X)$ given by $\mathcal{L} \mapsto \mathcal{L} - \mathcal{O}_X$. We also have a morphism of groups $\det : K(X) \longrightarrow \mathrm{Pic}(X)$ from the additive group $K(X)$ to the multiplicatively written group $\mathrm{Pic}(X)$ given as follows: to a vector bundle $\mathcal{E}$ we associate the class of the element $\Lambda^r\mathcal{E}$, where $r = \mathrm{rk}\mathcal{E}$. It is easy to see that this is well-defined and induces a map from the Grothendieck group to the Picard group. One checks that for vector bundles $\mathcal{E}$ and $\widetilde{\mathcal{E}}$, it holds that $\det(\mathcal{E} \oplus \widetilde{\mathcal{E}}) = \det(\mathcal{E}) \otimes_{\mathcal{O}_X} \det(\widetilde{\mathcal{E}})$. Note that $\det \circ j = \mathrm{id}_{\mathrm{Pic}(X)}$. This shows the surjectivity of det and injectivity of j.

Definition 33. We define the *Chern character* $\mathrm{ch} : K(X) \longrightarrow A(X)$ by

$$\mathrm{ch}(\mathcal{F}) = (\mathrm{rk}(\mathcal{F}), c_1(\det(\mathcal{F})).$$

Proposition 34. *The Chern character is a morphism of rings.*

Proof. Clearly the Chern character

$$K(X) \xrightarrow{(\mathrm{rk},\det)} \mathbf{Z} \oplus \mathrm{Pic}(X) \xrightarrow{(\mathrm{id}_{\mathbf{Z}},c_1)} \mathbf{Z} \oplus \mathrm{Cl}(X)$$

factors through $\mathbf{Z} \oplus \mathrm{Pic}(X)$ and is given by the composition of $(\mathrm{rk}, \det)$ with (id, c_1). Therefore, the Chern character is a morphism of groups.

Clearly $\mathrm{ch}(\mathcal{O}_X) = (1, 0)$. To finish the proof, we check that $\mathrm{ch}(\mathcal{E}_1 \otimes \mathcal{E}_2) = \mathrm{ch}(\mathcal{E}_1)\mathrm{ch}(\mathcal{E}_2)$. But the latter follows from the fact that $\det(\mathcal{E}_1 \otimes \mathcal{E}_2) = (\det \mathcal{E}_1)^{\otimes r_2} \otimes (\det \mathcal{E}_2)^{\otimes r_1}$ where $\mathrm{rk}(\mathcal{E}_1) = r_1$ and $\mathrm{rk}(\mathcal{E}_2) = r_2$.

More explicitly,

$$\begin{aligned}\mathrm{ch}(\mathcal{E}_1 \otimes \mathcal{E}_2) &= (\mathrm{rk}(\mathcal{E}_1 \otimes \mathcal{E}_2), c_1(\det \mathcal{E}_1 \otimes \mathcal{E}_2)) = (\mathrm{rk}\mathcal{E}_1 \mathrm{rk}\mathcal{E}_2, c_1((\det \mathcal{E}_1)^{\otimes r_2} \otimes (\det \mathcal{E}_2)^{\otimes r_1})) \\ &= (\mathrm{rk}\mathcal{E}_1 \mathrm{rk}\mathcal{E}_2, r_2 c_1(\det \mathcal{E}_1) + r_1 c_1(\det \mathcal{E}_2)) = (r_1, c_1(\mathcal{E}_1))(r_2, c_1(\mathcal{E}_2)).\end{aligned}$$

This proves that the Chern character is a morphism of rings. $\square$

Remark 35. Let Ω_X be the sheaf of relative differentials of X (over k). It is an invertible sheaf and equals the canonical sheaf ω_X. Furthermore, its dual is by definition the tangent sheaf $\mathcal{T}_X$ of X. In the following definition we let $A(X)_{\mathbf{Q}}$ denote $A(X) \otimes_{\mathbf{Z}} \mathbf{Q}$.

Definition 36. The *Todd genus* of X, denoted by Td_X, is the element of the Chow ring defined as $\mathrm{Td}(X) = (1, \frac{1}{2}\mathcal{T}_X) = (1, -\frac{1}{2}c_1(\omega_X)) \in A(X)_{\mathbf{Q}}$.

Proposition 37. We have a morphism of rings $\mathrm{ch} \cdot \mathrm{Td}_X : K(X) \longrightarrow A(X)_{\mathbf{Q}}$ given by

$$\mathcal{F} \mapsto \mathrm{ch}(\mathcal{F}) \cdot \mathrm{Td}_X = (\mathrm{rk}(\mathcal{F}), c_1(\det \mathcal{F}) - \frac{1}{2}\mathrm{rk}(\mathcal{F}) c_1(\omega_X)).$$

The Riemann-Roch theorem

In this section X is a non-singular complete curve.

Theorem 38. (Riemann-Roch) The following diagram of groups

$$\begin{tikzcd} K(X) \arrow[rr, "\mathrm{ch}\cdot\mathrm{Td}_X"] \arrow[dr, "\chi(X,-)"'] & & A(X)_{\mathbf{Q}} \arrow[dl, "\deg"] \\ & \mathbf{Z} & \end{tikzcd}$$

is commutative. That is, for any $\alpha \in K(X)$ of rank r, it holds that

$$\chi(X, \alpha) = \deg(\mathrm{ch}(\alpha)\mathrm{Td}_X) = \deg(c_1(\det \alpha)) - \frac{1}{2} r \deg(c_1(\omega_X)).$$

The following Lemma will allow us to reformulate the above Riemann-Roch theorem in terms of divisor classes.

Lemma 39. The first Chern class is an isomorphism of groups. To $D \in \mathrm{Cl}(X)$ we associate the line bundle $\mathcal{O}_X(D) := c_1^{-1}(D)$.

Remark 40. Recall that the genus g of X is defined as $\dim_k H^0(X, \omega_X)$. By Serre duality, it holds that $g = \dim_k H^1(X, \mathcal{O}_X)$. Since $\chi(X, \mathcal{O}_X) = k$, it holds that $\chi(X, \mathcal{O}_X) = 1 - g$.

Theorem 41. The Riemann-Roch theorem holds if and only if for every divisor D on X, we have that

$$\chi(X, \mathcal{O}_X(D)) = \deg D + 1 - g.$$

Proof. Let us suppose that the Riemann-Roch theorem holds. For $\alpha = \mathcal{O}_X$, this gives us

$$1 - g = \chi(X, \mathcal{O}_X) = -\frac{1}{2}\deg(c_1(\omega_X)).$$

Thus for any $D \in \mathrm{Cl}(X)$, it holds that $\chi(X, \mathcal{O}_X(D)) = \deg D + 1 - g$ since $c_1(\mathcal{O}_X(D)) = D$.

Suppose that $\chi(X, \mathcal{O}_X(D)) = \deg D + 1 - g$ for all $D \in \mathrm{Cl}(X)$. By Lemma 39, the Riemann-Roch theorem holds for (the class of) a line bundle $\mathcal{L}$. To prove the Riemann-Roch theorem, it suffices to do so for the class of a vector bundle $\mathcal{E}$ of rank $r > 1$. (What about $r = 0$?)

Suppose that we have a short exact sequence

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0,$$

with $\mathcal{E}'$ a line bundle and $\mathcal{E}''$ a vector bundle of rank $r-1$. Then, by the additivity of the Euler characteristic and the Riemann-Roch theorem for line bundles, the theorem follows by induction on r.

Let us show that we always have such a short exact sequence. Choose $n \gg 0$ such that $\mathcal{E}(n)$ has a global section s which is nowhere zero. This is possible since X is of dimension 1. Note that $\mathcal{E}(n)/s\mathcal{O}_X$ is a vector bundle of rank $r-1$ and that we have a short exact sequence

$$0 \longrightarrow s\mathcal{O}_X \longrightarrow \mathcal{E}(n) \longrightarrow \mathcal{E}(n)/s\mathcal{O}_X \longrightarrow 0.$$

Now tensor with $\mathcal{O}_X(-n)$ (which is a vector bundle) to get the desired short exact sequence. □

Theorem 42. For $D \in \mathrm{Cl}(X)$, it holds that

$$\chi(X, \mathcal{O}_X(D)) = \deg D + 1 - g.$$

Proof. We already noted that the Theorem holds for $D = 0$. Hence it suffices to show that the theorem holds for D if and only if it holds for $D + P$ where P is some point in X. We have the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-P) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_P \longrightarrow 0\ .$$

Tensoring this with $\mathcal{O}_X(D+P)$ gives us the result. Here $\mathcal{O}_P = i_*\mathcal{O}_X$ is the extension by zero with $i : \{P\} \longrightarrow X$ the inclusion. □

Let us give some final results.

Proposition 43. We have a surjective homomorphism $\mathbf{Z} \oplus \mathrm{Div}(X) \longrightarrow K(X)$ given by $(1, P) \mapsto 1 + \mathcal{O}_P$.

Corollary 44. The Chern character $\mathrm{ch} : K_0(X) \longrightarrow A(X)$ is an isomorphism of rings.

Remark 45. The concept of the Chow ring, the degree on the Chow ring, the Chern character and the Todd genus generalize to projective varieties of any dimension with similar properties as described above. The Hirzebruch-Riemann-Roch theorem then states that for any nonsingular projective variety X, the following diagram

$$\begin{tikzcd} K(X) \arrow[rr, "\mathrm{ch}\cdot\mathrm{Td}_X"] \arrow[dr, "\chi(X,-)"'] & & A(X) \arrow[dl, "\deg"] \\ & \mathbf{Z} & \end{tikzcd}$$

is commutative.

The next talk will introduce intersection theory on a projective variety.

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