

Intersection theory hidden in $K_0(X)$

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1 Summary

One aspect of the Riemann-Roch theorem when properly generalized to higher dimensions is the involvement of *intersection theory*. We shall motivate this by showing that the Grothendieck group $K_0(X)$ "hides" an intersection theory.

A variety is always quasi-projective and integral over an algebraically closed field k .

2 Introduction

Let X be a nonsingular variety. In the first talk we saw that the natural embedding $\text{Vect}(X) \hookrightarrow \text{Coh}(X)$ of categories induces an isomorphism of Grothendieck groups $K^0(X) \xrightarrow{\sim} K_0(X)$. We also stated that the group $K^0(X)$ is a commutative ring under the tensor product $\otimes_{\mathcal{O}_X}$. Thus $K_0(X)$ inherits a ringstructure.

Let's keep this in mind for now.

In the second talk we gave the Riemann-Roch theorem for a nonsingular projective curve.

Theorem 1. (Riemann-Roch) Let X be a nonsingular projective curve. Then the following diagram of groups

$$\begin{array}{ccc} K(X) & \xrightarrow{\text{ch} \cdot \text{Td}_X} & A(X)_{\mathbf{Q}} \\ & \searrow \chi(X, -) & \swarrow \deg \\ & \mathbf{Z} & \end{array}$$

is commutative.

Recall that $A(X) = \mathbf{Z} \oplus \text{Cl}(X) = A^0 X \oplus A^1 X$ is called the Chow ring and that its ringstructure is given by $(n, D) \cdot (m, E) = (nm, nE + mD)$. Now, in higher dimensions the above Theorem still holds, but the Chow ring now contains higher degree parts. This notably makes the ring structure on $A(X)$ more complicated to describe. For, the Chow ring is a so called *intersection ring* and we are forced to studying intersection theory. Now, in this talk we try to give a feeling for where exactly this intersection theory comes from. Namely, we will show that already there is some intersection theory "hidden" in the ringstructure of $K_0(X)$.

3 The geometry of $K_0(X)$

Let X be a variety, i.e. an integral quasi-projective scheme of finite type over an algebraically closed field. A subvariety of X will be a closed subset which is a variety. By integral we mean irreducible and reduced.

The class of a coherent sheaf $\mathcal{F}$ in $K_0(X)$ is denoted by $\text{cl}(\mathcal{F})$. We shall show that the group $K_0(X)$ is generated by a system of generators with a "geometric" origin.

Definition 2. A *cycle* on X is an element of the free abelian group $\mathbf{Z}X$ on the subvarieties of X , i.e. a finite formal sum $\sum n_Y Y$ where Y is a subvariety of X and n_Y is an integer.

Remark 3. The free group $ZX = Z \cdot X = \bigoplus_{r \in \mathbf{Z}} Z^r X$ is graded by codimension. If we let $Z_r X$ denote the free abelian group on the subvarieties of dimension r , then $Z_r X = Z^{n-r} X$. Therefore the grading by dimension of ZX is a renumbering of the grading by codimension. In the more general setting of schemes these gradings might not be renumberings of each other.

Remark 4. It is not hard to see that Z^r defines a flasque sheaf of abelian groups on X for any $r \in \mathbf{Z}$.

To a subvariety Y of X , we associate the coherent sheaf $\mathcal{O}_Y$. Here $\mathcal{O}_Y$ is (by abuse of notation) the extension to zero outside Y , i.e. $\mathcal{O}_Y(U) = \mathcal{O}_X(U \cap Y)$ for any open set U of X .

Theorem 5. The map $Z \cdot X \longrightarrow K_0(X)$ given by $Z \cdot X \ni \sum n_Y Y \mapsto \sum n_Y \text{cl}(\mathcal{O}_Y) \in K_0(X)$ is a surjective homomorphism.

Proof. We give the proof as in [Mur, Proposition 2.4., pp 103].

A *filtration* of an abelian group A is a collection of subgroups $(A^i)_{i=0}^\infty$ of A such that

$$A^0 \subset A^1 \subset \dots$$

The graded group associated to this filtration, denoted by $\text{Gr}A$, is defined as

$$\text{Gr}A = \bigoplus_{i=0}^\infty A^i / A^{i+1}.$$

We define a (topological) filtration

$$K_0(X) = F^0 X \supset F^1 X \supset \dots \supset F^{\dim X} X \supset F^{\dim X+1} X = 0$$

on $K_0(X)$ as follows. For $i \in \mathbf{Z}_{\geq 0}$, we define

$$F^i X = \{\text{cl}(\mathcal{F}) \in K_0(X) \mid \text{codimSupp}(\mathcal{F}) \geq i\}.$$

Here $\text{Supp}(\mathcal{F})$ is the closed subset of points $x \in X$ such that $\mathcal{F}_x \neq 0$. For example, for a subvariety Y of codimension i , the support of $\mathcal{O}_Y$ (extension by zero) is Y . Therefore $\mathcal{O}_Y \in F^i(X)$. Thus, since ZX is graded by codimension, it suffices to see that the map $Z^i X \longrightarrow F^i X / F^{i+1} X$ is surjective. To this extent, suppose that $\mathcal{F}$ is a coherent sheaf such that $\text{codimSupp}(\mathcal{F}) = i$, i.e. $\text{cl}(\mathcal{F}) \in F^i X$. Consider the cycle

$$Z(\mathcal{F}) = \sum_{Y \subset X} l_{\mathcal{O}_{X,y}}(\mathcal{F}_y) \cdot Y.$$

Here Y runs through the subvarieties of codimension i in X , y is the generic point of Y , $\mathcal{F}_y$ is the stalk of $\mathcal{F}$ in y and we let $l_R(M)$ denote the length of an R -module M . It is an easy verification that the image of $Z(\mathcal{F})$ in $F^i X / F^{i+1} X$ is given by $\mathcal{F} \bmod F^{i+1} X$. $\square$

Remark 6. The above map induces a surjective morphism $Z \cdot X \longrightarrow \text{Gr}K_0(X)$ of groups. The kernel of this homomorphism modulo torsion is given by the cycles rationally equivalent to zero. (The definition of rational equivalence will be given later.) That is, this map factors through a surjective morphism of group $A \cdot X \longrightarrow \text{Gr}K_0(X)$ which becomes an isomorphism after tensoring with $\mathbf{Q}$. Here $A(X)$ denotes the Chow group, i.e., the group of cycles modulo rational equivalence. It goes even further, namely, this morphism of groups is actually a morphism of rings and we get a functorial isomorphism of rings $A \cdot X \otimes_{\mathbf{Z}} \mathbf{Q} \longrightarrow \text{Gr}K_0(X) \otimes_{\mathbf{Z}} \mathbf{Q}$.

Remark 7. The topological filtration on $K_0(X)$ is called the coniveau filtration. We can define a similar filtration on $K^0(X)$. Namely, for $i \geq 0$,

$$F^i X = \text{Image}(K^0(X)^{\geq i} \longrightarrow K^0(X)).$$

Let us define the scheme-theoretic intersection of two subvarieties.

Remark 8. Consider affine space $X = \mathbf{A}^n$ and its coordinate ring $A = k[x_1, \dots, x_n]$. By Hilbert's Nullstellensatz, there is an inclusion-reversing bijection from the set of radical ideals $I \subset A$ to the set of Zariski closed subsets $Y \subset X$ and similarly, from the set of prime ideals to the set of subvarieties $Y \subset X$. Recall that the map from left to right consists of taking the zero locus.

A closed subset Y of X defines a coherent sheaf of ideals $\mathcal{J}_Y \subset \mathcal{O}_X$ by

$$\mathcal{J}_Y(U) = \{f \in \mathcal{O}_X(U) \mid f(x) = 0 \text{ for all } x \in U \cap Y\}.$$

For example, $\mathcal{J}_X$ is the zero sheaf and $\mathcal{J}_\emptyset = \mathcal{O}_X$. When Y is a subvariety, we have an exact sequence

$$0 \longrightarrow \mathcal{J}_Y \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

of sheaves on X . (In particular, the sheaf of ideals $\mathcal{J}_Y$ is given by the kernel of $i^\# : \mathcal{O}_X \longrightarrow i_*\mathcal{O}_Y$ where $i : Y \longrightarrow X$ is the natural inclusion.)

Reversely, a coherent sheaf of ideals $\mathcal{J}$ determines a closed subset (actually closed subscheme) $Y \subset X$ by

$$\begin{aligned} Y &= \text{Supp}(\mathcal{O}_X/\mathcal{J}) = \{x \in X \mid \mathcal{J}_x \neq \mathcal{O}_{X,x}\} \\ &= \{x \in X \mid \text{for all open } U \subset X \text{ such that } x \in U \text{ and } f \in \mathcal{J}(U) \text{ we have } f(x) = 0\} \end{aligned}$$

with structure sheaf $\mathcal{O}_Y = j^{-1}(\mathcal{O}_X/\mathcal{J})$ where $j : Y \longrightarrow X$ is the natural inclusion. Note that Y need not be irreducible. Furthermore, we always take the reduced structure on Y , i.e. we divide the structure sheaf out by its sheaf of nilpotents. This corresponds to taking radical ideals in the affine setting.

Remark 9. Let us make a remark on schemes. Basically, one would like to be able to speak about "multiplicities" in geometry. For example, two planar curves of degree m and n should intersect in precisely mn points when counted properly (k algebraically closed). Thus we want the variety defined by the equation $x^2 = 0$ to be different of the variety defined by $x = 0$. In [Har, Chapter I], this is not possible since we take radical ideals. Schemes do allow us to distinguish a nonradical ideal from its radical geometrically. But, unfortunately, since we take the "reduced closed subscheme structure" we are actually forgetting this again.

Let $i : Y \longrightarrow X$ be a closed subscheme. The closed subscheme defined by $\mathcal{J}_Y$ is just Y . Firstly, one checks this on affine schemes (use the equivalence of categories). Then the general statement follows from the fact that closed immersions are affine. Reversely, the same argument applies. Thus, we get a bijection from the set of (reduced) closed subschemes to the set of coherent sheaves of (radical?) ideals. Furthermore, we also get a bijection from the set of subvarieties to the set of coherent sheaves of prime ideals. (Right?)

Example 10. Let X be a nonsingular variety. To a subvariety D of codimension 1 (Weil divisor) we can associate the line bundle $\mathcal{O}_X(D)$. Note that the dual $\mathcal{O}_X(D)^\vee$ is the sheaf of ideals $\mathcal{J}_D$. Thus the subvariety associated to $\mathcal{O}_X(D)^\vee$ is D . (More precisely, for every $p \in X$ there is an open neighbourhood of p such that $D \cap U$ is given by a single equation. This gives us an open covering $\{U_i : i \in I\}$ and rational functions f_i on U_i such that on any intersection $U_i \cap U_j$ the rational function $\frac{f_i}{f_j}$ is invertible (no zeroes nor poles). Thus D defines a Cartier divisor and $\mathcal{O}_X(D)$ is the invertible sheaf associated to this Cartier divisor.)

Let Y_1 (resp. Y_2) be a subvariety of X and let $\mathcal{J}_1$ (resp. $\mathcal{J}_2$) be the sheaf of ideals associated to Y_1 (resp. Y_2). We let $Y_1 \cap Y_2$ be the subvariety associated to the sheaf of ideals $\mathcal{J}_1 + \mathcal{J}_2$. By definition, $\mathcal{O}_{Y_1 \cap Y_2} = \mathcal{O}_X/(\mathcal{J}_1 + \mathcal{J}_2)$. This can be "geometrically" interpreted as the following.

The set of solutions of the union of two systems of equations is the intersection of the set of solutions of each of them.

From the identity of rings

$$A/I \otimes_A A/J \cong A/(I + J),$$

it is easy to see that $\mathcal{O}_{Y_1 \cap Y_2} \cong \mathcal{O}_{Y_1} \otimes_{\mathcal{O}_X} \mathcal{O}_{Y_2}$.

Let $W_1, \dots, W_s$ be the irreducible components of a closed subscheme Y and let w_i denote the generic point of W_i . Then we define $[Y]$ as the image of the cycle $\sum_{i=1}^s l_{\mathcal{O}_{X, w_i}}(\mathcal{O}_{Y, w_i})W_i$ under the map in Theorem 5, i.e. $[Y] = \sum_{i=1}^s l_{\mathcal{O}_{X, w_i}}(\mathcal{O}_{Y, w_i})\text{cl}(\mathcal{O}_{W_i})$. If Y is a subvariety, it is easy to see that $[Y] = \text{cl}(\mathcal{O}_Y) \in K_0(X)$. (Question: Is $[Y_1 \cap Y_2] = \text{cl}(\mathcal{O}_{Y_1 \cap Y_2})$?)

Examples 11. Firstly, $[\emptyset] = 0$ and $[X] = 1$ in $K_0(X)$. If Y is a subvariety of X , it holds that $[Y] = 1 - \text{cl}(\mathcal{J}_Y)$ since we have the exact sequence

$$0 \longrightarrow \mathcal{J}_Y \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0.$$

Thus, when $D = Y$ is of codimension 1, we have that $[D] = 1 - \text{cl}(\mathcal{O}_X(D)^\vee) = 1 - \text{cl}(\mathcal{O}_X(D))^{-1}$. Now, let D and E be two curves on a nonsingular surface X with no common irreducible component. Note that D and E intersect in a finite number of (closed) points. For, the intersection $D \cap E$ does not contain an irreducible component of D nor E . Therefore, the irreducible components of $D \cap E$ must be closed points. In this case, for $x \in D \cap E$, the intersection multiplicity of D and E at x , denoted by $i_x(D, E)$, is precisely the coefficient of x in the above cycle, i.e.

$$i_x(D, E) = l_{\mathcal{O}_{X, x}} \mathcal{O}_{X, x} / (\mathcal{O}_X(-D)_x + \mathcal{O}_X(-E)_x).$$

We are now ready to state the main theorem.

We assume X to be nonsingular and identify $K_0(X)$ with $K^0(X)$. We already stated that $K_0(X)$ inherits a ringstructure from $K^0(X)$. For $\alpha, \beta \in K_0(X)$, we let $\alpha \cdot \beta$ denote their product in $K_0(X)$. The main theorem reads as follows and will be proven in Section 5.

Theorem 12. Let Y_1 and Y_2 be subvarieties of X . Suppose that Y_1 and Y_2 are in "general position". Then

$$[Y_1] \cdot [Y_2] = [Y_1 \cap Y_2]$$

in $K_0(X)$.

Remark 13. When Y_1 and Y_2 are not in "general position" the equality should be replaced by the equivalence

$$[Y_1] \cdot [Y_2] \equiv [Y_1 \cap Y_2] \pmod{F^{\text{codim} Y_1 + \text{codim} Y_2 + 1} X}.$$

In order to define what subvarieties in "general position" are we need the notion of regular sequences on a ring and to prove the Theorem we need to study Koszul complexes associated to such sequences.

4 Algebraic intermezzo: Koszul complexes

Let A be a (commutative) ring.

Definition 14. A sequence $(x_1, \dots, x_n)$ of elements $x_1, \dots, x_n \in A$ is said to be a *regular sequence* if x_1 is a nonzerodivisor and the image of x_i in $A/(x_1, \dots, x_{i-1})$ is a nonzerodivisor for $2 \leq i \leq n$. (Do we need that $(x_1, \dots, x_n)A \neq A$?)

Example 15. An element $x \in A$ defines a regular sequence (x) if and only if the complex

$$0 \longrightarrow A \xrightarrow{x} A \longrightarrow 0$$

is exact.

Example 16. A set of linear forms in $A = k[X_1, \dots, X_n]$ defines a regular sequence if and only if it is linearly independent.

Let $x_1, \dots, x_n$ be (distinct?) elements in A and let $E = A^n$ be the free A -module of rank n with basis $(e_1, \dots, e_n)$. The Koszul complex associated to $(x_1, \dots, x_n)$, denoted by $K^A(x_1, \dots, x_n)$, is the complex

$$0 \longrightarrow \Lambda^n E \xrightarrow{d} \Lambda^{n-1} E \xrightarrow{d} \dots \xrightarrow{d} \Lambda^1 E = E \xrightarrow{d} \Lambda^0 E = A \longrightarrow 0.$$

We define the boundary map $d : \Lambda^p E \longrightarrow \Lambda^{p-1} E$ by

$$d(e_{i_1} \wedge \dots \wedge e_{i_p}) = \sum_{j=1}^p (-1)^{j-1} x_{i_j} e_{i_1} \wedge \dots \wedge \hat{e}_{i_j} \wedge \dots \wedge e_{i_p}.$$

The reader may verify that $d^2 = 0$.

Example 17. The Koszul complex associated to an element $x \in A$ is the complex

$$0 \longrightarrow A \xrightarrow{x} A \longrightarrow 0$$

given in Example 15.

Example 18. The Koszul complex associated to $x_1, x_2 \in A$ is the complex

$$0 \longrightarrow A \xrightarrow{f} A^2 \xrightarrow{g} A \longrightarrow 0.$$

Here $f : a \mapsto (ax_1, -ax_2)$ and $g : (a, b) \mapsto ax_1 - bx_2$. (Might want to check this.)

Theorem 19. Let $(x_1, \dots, x_n)$ be a regular sequence in A . Then the augmented Koszul complex

$$0 \longrightarrow \Lambda^n E \xrightarrow{d} \Lambda^{n-1} E \xrightarrow{d} \dots \xrightarrow{d} E \xrightarrow{d} A \longrightarrow A/(x_1, \dots, x_n) \longrightarrow 0$$

is exact, i.e. the Koszul complex provides us with a free resolution of $A/(x_1, \dots, x_n)$.

Proof. See [Lang][Chapter XXI, Theorem 4.6, p. 856]. □

Remark 20. In the first talk we looked at $A = k[x, y]/(xy)$ and the nonregular sequence (x, y) . We constructed an infinite resolution and saw that A had infinite Tor-dimension. This shows that regularity of our sequence is required.

Remark 21. In our applications A will be $k[x_1, \dots, x_n]/I$ where I is an ideal. Question: Can we give an easy proof of the Theorem in this case?

Corollary 22. Suppose that $(x_1, \dots, x_r)$ and $(y_1, \dots, y_s)$ are regular sequences and let us suppose that $(x_1, \dots, x_r, y_1, \dots, y_s)$ is also a regular sequence in A . Then

$$\mathrm{Tor}_i^A(A/(x_1, \dots, x_r), A/(y_1, \dots, y_s)) = \begin{cases} A/(x_1, \dots, x_r, y_1, \dots, y_s) & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases}.$$

Proof. Let $I = (x_1, \dots, x_r)$ and let $J = (y_1, \dots, y_s)$. Then $I + J = (x_1, \dots, x_r, y_1, \dots, y_s)$. By Theorem 19, the Koszul complex

$$0 \longrightarrow \Lambda^n E \xrightarrow{d} \Lambda^{n-1} E \xrightarrow{d} \dots \xrightarrow{d} E \xrightarrow{d} A \longrightarrow 0$$

is a free resolution for A/I . Tensoring this with A/J gives us the complex $K^A(x_1, \dots, x_n) \otimes_A A/J \cong K^{A/J}(x_1 + J, \dots, x_r + J)$. By our assumption, $(x_1 + J, \dots, x_r + J)$ is a regular sequence in A/J . Thus $K^{A/J}(x_1 + J, \dots, x_r + J)$ is a free resolution for $A/(I + J)$ and it follows that

$$\mathrm{Tor}_i^A(A/I, A/J) = \begin{cases} A/(I + J) & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases}. \quad \square$$

5 The main theorem

Let X be a variety. Let Y_1 (resp. Y_2) be a subvariety of X and let $\mathcal{J}_1$ (resp. $\mathcal{J}_2$) be the sheaf of ideals associated to Y_1 (resp. Y_2).

Definition 23. We say that Y_1 and Y_2 are in *general position* or *intersect properly* if for every $x \in Y_1 \cap Y_2$ there is an affine open subset $\text{Spec} A = U \subset X$ and elements $f_1, \dots, f_r, g_1, \dots, g_s \in A$ with

$$\mathcal{J}_1(U) = (f_1, \dots, f_r), \quad \mathcal{J}_2(U) = (g_1, \dots, g_s)$$

such that $(f_1, \dots, f_r)$ is a regular sequence in $\mathcal{J}_1(U)$, $(g_1, \dots, g_s)$ is a regular sequence in $\mathcal{J}_2(U)$ and $(f_1, \dots, f_r, g_1, \dots, g_s)$ is a regular sequence in $\mathcal{J}_1(U) + \mathcal{J}_2(U)$.

Example 24. If $Y_1 \cap Y_2 = \emptyset$, the condition is empty and therefore Y_1 and Y_2 are in general position.

Remark 25. Can the last condition " $(f_1, \dots, f_r, g_1, \dots, g_s)$ is a regular sequence" be relaxed when X is nonsingular?

Example 26. Suppose that Y_1 and Y_2 are (locally) defined by linear forms. Then Y_1 and Y_2 intersect properly if $Y_1 \cap Y_2 = \emptyset$ or $\text{codim} Y_1 \cap Y_2 = \text{codim} Y_1 + \text{codim} Y_2$. (How does one show this in general? Is it still true?) See Example 16.

Let X be nonsingular.

Theorem 27. Suppose that Y_1 and Y_2 are in general position. Then

$$[Y_1] \cdot [Y_2] = [Y_1 \cap Y_2]$$

in $K_0(X)$.

Proof. This is hard without knowing how the product on $K_0(X)$ looks like explicitly. So let us recall the formula. For coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ on X , it holds that

$$\text{cl}(\mathcal{F}) \cdot \text{cl}(\mathcal{G}) = \sum_{i=0}^{\dim X} (-1)^i \text{cl}(\text{Tor}_i^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}))$$

in $K_0(X)$.

We have already given this in our first talk. It is a direct consequence of the universality and additivity of the Tor functors. More precisely, one takes a resolution $\mathcal{E}_i$ of vector bundles for $\mathcal{F}$ (or $\mathcal{G}$) and notes that the homology of the complex $\mathcal{E}_i \otimes_{\mathcal{O}_X} \mathcal{G}$ is given by the Tor functors but also by the product of $\mathcal{F}$ and $\mathcal{G}$.

Note that, even if Y_1 and Y_2 are not in general position, it holds that

$$\mathcal{O}_{Y_1} \otimes_{\mathcal{O}_X} \mathcal{O}_{Y_2} \cong \mathcal{O}_{Y_1 \cap Y_2}.$$

This can be checked locally and then boils down to the identity $A/I \otimes_A A/J \cong A/(I + J)$.

Thus it suffices to show that $i \geq 1$ and Y_1 and Y_2 in general position

$$\text{Tor}_i^{\mathcal{O}_X}(\mathcal{O}_{Y_1}, \mathcal{O}_{Y_2}) = 0.$$

We show that for any $x \in X$, the stalk $\left(\text{Tor}_i^{\mathcal{O}_X}(\mathcal{O}_{Y_1}, \mathcal{O}_{Y_2})\right)_x = 0$.

If $x \notin Y_1 \cap Y_2$, it is clear that $\left(\text{Tor}_i^{\mathcal{O}_X}(\mathcal{O}_{Y_1}, \mathcal{O}_{Y_2})\right)_x = 0$ for any $i \geq 0$ (not just $i \geq 1$). This follows easily from the fact that either $(\mathcal{O}_{Y_1})_x = 0$ or $(\mathcal{O}_{Y_2})_x = 0$.

Now, suppose that $x \in Y_1 \cap Y_2$. Using the fact that Y_1 and Y_2 are in general position we can clearly reduce to the affine case. The assertion now follows from Corollary 22. $\square$

Corollary 28. Let Y be a nonsingular hyperplane section of X . Then $[Y]^{\dim X+1} = 0$ in $K_0(X)$.

Proof. Let $k = \dim X$. We take $k+1$ nonsingular hyperplane sections $Y_1, \dots, Y_{k+1}$ such that their intersection is empty, i.e.

$$Y_1 \cap \dots \cap Y_{k+1} = \emptyset,$$

and such that Y_i is in "general position" with the subvariety $Y_{i-1} \cap \dots \cap Y_1$ ($i = 1, \dots, k+1$). (Why is this possible?) Since $\text{Cl}(\mathbf{P}^k) = \mathbf{Z}$, we have that $[Y_i] = [Y]$ (right?). By the previous theorem, it holds that $[Y]^{k+1} = 0$. $\square$

The key steps in defining intersection theory on a nonsingular quasi-projective variety over a field are:

- Defining intersection multiplicities.
- The Moving Lemma.

In defining intersection multiplicities, one method is "Serre's Tor formula". This formula can be compared to the product we studied, but one needs to take supports into account. The reader may look at the lecture notes by Gillet available on his website.

References

- [BorSer] A. Borel, J.P. Serre *Le théorème de Riemann-Roch* Bull. Soc. math. France, 86, 1985, p.97-136.
- [Lang] Lang *Algebra* Addison-Wesley
- [Ful] Fulton *Intersection Theory* Springer
- [Har] R. Hartshorne *Algebraic geometry* Springer Science 2006.
- [FAC] J.P. Serre *Faisceaux algébriques cohérents* The Annals of Mathematics, 2nd Ser., Vol. 61, No. 2. (Mar., 1955), pp. 197-278.
- [Mur] J. Murre *Algebraic cycles and algebraic aspects of cohomology and k-theory* Lecture Notes in Mathematics, 1594 Algebraic Cycles and Hodge Theory, Torino, 1993, pp. 93-152.