L-functions of the projective line

DEFINITION 0.0.1. Let k be a finite field. A character on the polynomial ring $k[X]$ is a map $\chi : k[X] \to \mathbf{C}$ such that there is a nonzero $f \in k[X]$ and a group homomorphism $\chi' : k[X]/(f) \to \mathbf{C}^*$ such that $\chi(g) = 0$ if $(g, f) \neq 1$ and $\chi(g) = \chi'(g \bmod f)$ otherwise. We call f a modulus of χ. If $\operatorname{im}\chi \subset \{0, 1\}$, we call χ principal.

DEFINITION 0.0.2. Let k be a finite field and let χ be a character of $k[X]$. Then we define the function

$$L(T, \chi) = \sum_{h \in k[X] \text{ monic}} \chi(h) T^{\deg h}$$

which is an element of $\mathbf{C}[[T]]$.

THEOREM 0.0.3. *Let k be a finite field with q elements and let χ be a non-principal character of $k[X]$ with irreducible modulus f. Write $\delta = 0$ if $\chi|_{k^*} = 1$ and $\delta = 1$ otherwise. Then there are $\alpha_1, \ldots, \alpha_m$, where $m = \deg f - 2 + \delta$, such that $|\alpha_i| = \sqrt{q}$ for all i and such that*

$$L(T, \chi) = \prod_{i=1}^{\deg f - 1} (1 - \alpha_i T) \cdot (1 - T)^{1-\delta}.$$

Something similar can be formulated if f is not irreducible, this would just make the formulas messier.

Although this looks like an ordinary case of determining roots of polynomials, the proof of this theorem turns out to be surprisingly complicated. The proof of the theorem transforms the characters of $k[X]$ into characters of the so called idele class group of the field $k(X)$ (to be defined). Then, making use of class field theory, the theorem can be deduced from the Riemann hypothesis for function fields.