INJECTIVE MODULES AND THE INJECTIVE HULL OF A MODULE, November 27, 2009

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Abstract. In the first section we will define injective modules and we will prove some theorems. In the second section, we will define the concept of injective hull and show that any module has a 'unique' injective hull.

We will follow [LA], section 3A and 3D.

In the sections below we will fix a commutative ring R. For the theory R doesn't need to be commutative, and the generalizations follow easily.

1. Injective modules

1.1. Definition and some theory.

Definition 1.1. Let M be an R-module. Then M is called (R-)injective if for any monomorphism $f : N \to N'$ (of R-modules) and any morphism $g : N \to M$ there exists a morphism $h : N' \to M$ such that $h \circ f = g$. In a diagram this looks as follows:

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "f"] \arrow[d, "g"] & N' \arrow[dl, dashed, "\exists h"] \\ & M & \end{tikzcd}$$

Lemma 1.2. *Let M be a module. Then M is injective iff $\mathrm{Hom}_R(-, M)$ is exact.*

Proof. Let $0 \to N' \to N \to N'' \to 0$ be an exact sequence. In general it follows that $0 \to \mathrm{Hom}_R(N'', M) \to \mathrm{Hom}_R(N, M) \to \mathrm{Hom}_R(N', M)$ is exact. To make it right exact, we just need that $\mathrm{Hom}_R(N, M) \to \mathrm{Hom}_R(N', M)$ is surjective. This map is surjective for all exact sequences iff M is injective by definition. $\square$

Lemma 1.3. *We have:*

i. *$\prod_{i \in I} M_i$ is injective iff all the M_i are injective.*
ii. *A module I is injective iff any monomorphism $\varphi : I \to M$ splits.*

Proof. i. This follows directly from $\mathrm{Hom}(N, \prod_{i \in I} M_i) \cong \prod_{i \in I} \mathrm{Hom}(N, M_i)$ for any module N and the previous lemma.

ii. $\Longrightarrow$: Consider the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r, "\varphi"] \arrow[d, "\mathrm{id}"'] & M \arrow[dl, dashed, "h"] \\ & I & \end{tikzcd}$$

The morphism h gives a required splitting.

$\Longleftarrow$: Let $f : M \to N$ be a monomorphism and let $g : M \to I$ be a morphism. Define

$$M' := \frac{N \oplus I}{\{(f(m), -g(m)) : m \in M\}}.$$

We have natural maps from N and I to M', call them i_1 respectively i_2. First notice that by construction of M' it follows that $i_1 \circ f = i_2 \circ g$. We claim that i_2 is injective, indeed if $(0, i) = (f(m), g(m))$ for some $m \in M$, it follows that $m = 0$ and hence $i = 0$ (since f is a monomorphism). By our assumption we obtain a splitting $\varphi : M' \to I$, that is: $\varphi \circ i_2 = \mathrm{id}$. We have the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "f"] \arrow[d, "g"'] & N \arrow[d, "i_1"] \\ & I \arrow[r, bend left, "i_2"] & M' \arrow[l, bend left, "\varphi"] \end{tikzcd}$$

We obtain a map $\psi := \varphi \circ i_1 : N \to I$. We just calculate:

$$\begin{aligned} \psi \circ f &= \varphi \circ i_1 \circ f \\ &= \varphi \circ i_2 \circ g \\ &= g \end{aligned}$$

□

Theorem 1.4 (Baer's criterion)**.** *An R-module M is injective iff any morphism $I \to M$, where I is an ideal of R, can be extended to a morphism $R \to M$.*

Proof. $\Longrightarrow$: This follows directly from the definition of an injective module.
$\Longleftarrow$: Consider the following (exact) diagram:

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "f"] \arrow[d, "g"] & N' \\ & M & \end{tikzcd}$$

We need to find a map from $h : N' \to M$. Consider the set of pairs (N'', h) such that $N \subset N'' \subset N'$, $h : N'' \to M$ with the property that $h|_N = g$. This set is non-empty, since it contains (N, g). We order this set by the relations that $(N_1, h_1) \leq (N_2, h_2)$ if $N_1 \subset N_2$ and $h_2|_{N_1} = h_1$. A non-empty chain (S_i, h_i) has an upperbound, namely the 'union' defined as (S, h) where $S = \bigcup S_i$ and for $x \in S_i$ define $h(x) := h_i(x)$. Zorn's lemma now gives a maximal element (N'', h), we claim that $N'' = N'$ and hence h will be an extension of g. Suppose that $N'' \neq N'$ and let $x \in N' \setminus N''$. Let $I := \{r \in R : rx \in N''\} \subset R$, then I is an ideal of R. Consider the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d, "i \mapsto h(ix)"'] & R \\ & M & \end{tikzcd}$$

For $i \in I$ we have that $ix \in N''$ and hence $h(ix)$ is defined and this obviously is R-linear. By the assumption in the theorem, we obtain a map $\varphi : R \to M$ such that the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & I \arrow[r] \arrow[d, "i \mapsto h(ix)"'] & R \arrow[dl, "\varphi"] \\ & M & \end{tikzcd}$$

As $x = 1 \cdot x$ it seems natural to define the following map:

$$\begin{array}{rcl} \varphi' : Rx + N'' & \to & M \\ rx + n'' & \mapsto & r\varphi(1) + h(n'') \end{array}$$

for $r \in R$ and $n'' \in N$. We check that this map is well-defined. For this suppose that $rx = n$ where $r \in R$ and $n \in N''$. But this follows since $r\varphi(1) = \varphi(r) = h(rx) = h(n)$. Hence $(Rx + N'', \varphi)$ is a proper extension of (N'', h), contradicting the maximality of (N'', h). Hence $N'' = N'$ and we are done.

□

Example 1.5. For example $\mathbf{Q}/\mathbf{Z}$ is $\mathbf{Z}$-injective. This follows easily from Baer's criterion (it shows that a group is injective iff the group is divisble).

With this criterion one can also for example prove that any local Artinian ring with principal maximal ideal is injective over itself.

Example 1.6. Let R be a domain. We claim that its quotient field, $Q(R)$, is injective over R. We check this using Baer's criterion. Let $\varphi : I \to Q(R)$ be an R-linear map where I is an ideal of R. If $I = 0$ extend by the zero map. Otherwise let $0 \neq i \in I$ and define the following map:

$$\begin{array}{rcl} \psi : R & \to & Q(R) \\ r & \mapsto & r\dfrac{\varphi(i)}{i} \end{array}$$

This map is obviously R-linear and if $j \in I$:

$$\psi(j) = j\frac{\varphi(i)}{i} = i\frac{\varphi(j)}{i} = \varphi(j)$$

1.2. **Enough injectives.** We will now prove that any R-module M can be embedded into an injective module. We will first prove this for $\mathbf{Z}$-modules:

Lemma 1.7. *Let A be a $\mathbf{Z}$-module. Then there exists an injective module I and a monomorphism $\varphi : M \to I$.*

Proof. Recall that $\mathbf{Q}/\mathbf{Z}$ is injective. For a $\mathbf{Z}$-module B define $B^\vee := \mathrm{Hom}_{\mathbf{Z}}(B, \mathbf{Q}/\mathbf{Z})$. We now have a natural map as follows:

$$\begin{array}{rcl} \psi : A & \to & A^{\vee\vee} \\ a & \mapsto & (\varphi \mapsto \varphi(a)) \end{array}$$

One can easily see that this map is injective since $\mathbf{Q}/\mathbf{Z}$ is injective. Now let $\bigoplus_{j \in J} \mathbf{Z} \to A^\vee$ be a surjection, then we get an embedding $A^{\vee\vee} = \mathrm{Hom}_{\mathbf{Z}}(A^\vee, \mathbf{Q}/\mathbf{Z}) \to \mathrm{Hom}_{\mathbf{Z}}(\bigoplus_{j \in J} \mathbf{Z}, \mathbf{Q}/\mathbf{Z}) \cong (\mathbf{Q}/\mathbf{Z})^J$. Hence we have an embedding $A \to (\mathbf{Q}/\mathbf{Z})^J$. By Lemma 1.3 this last module is injective, and hence we are done. □

Lemma 1.8. *Let R be an S algebra. Let A be an injective S-module and P a projective R-module. Then* $\mathrm{Hom}_S(P, A)$ *is an injective R-module.*

Proof. We need to show that $\mathrm{Hom}_R(-, \mathrm{Hom}_S(P, A))$ is exact. First notice that $\mathrm{Hom}_R(-, \mathrm{Hom}_S(P, A)) \cong \mathrm{Hom}_S(- \otimes_R P, A)$ (universal property of tensor product). Now notice that the functor $- \otimes_R P$ is exact since P is projective. As A is injective, it follows that $\mathrm{Hom}_S(-, A)$ is exact. Combine both to obtain the result. □

Theorem 1.9. *Let M be an R-module. Then there is an injective module I and a monomorphism $\varphi : M \to I$.*

Proof. First consider M as $\mathbf{Z}$-module and by Lemma 1.7 there is a $\mathbf{Z}$-injective module I_1 such that we have a monomorphism $\varphi_1 : M \to I_1$. By the previous lemma, since R is projective over R, $\mathrm{Hom}_{\mathbf{Z}}(R, I_1)$ is injective. Consider the following map:

$$\begin{array}{rcl} \varphi : M & \to & \mathrm{Hom}_{\mathbf{Z}}(R, I_1) \\ m & \mapsto & (r \mapsto \varphi_1(rm)) \end{array}$$

One can easily show that φ is R-linear and that φ is injective. Indeed, if $\varphi(m) = 0$, then $\varphi_1(m) = \varphi_1(1c\dot{m}) = 0$ in I, hence $m = 0$. □

2. Injective hulls

2.1. **Essential extensions.**

Definition 2.1. Let M be a module. A module $E \supset M$ is called an essential extension of M if every non-zero submodule of E intersect M non-trivially. We denote this as $E \supset_e M$. Such an essential extension is called maximal if no module properly containing E is an essential extension of M.

Remarks 2.2.

i. If $E_2 \supset_e E_1$ and $E_1 \supset_e M$, then $E_2 \supset_e M$ (follows directly).
ii. Let $E \supset M$. Then E is an essential extension of M if for any $0 \neq a \in E$ we have $Ra \cap M \neq 0$.

Lemma 2.3. *A module M is injective iff M has no proper essential extensions.*

Proof. $\Longrightarrow$: Suppose that M is injective and let $E \supset_e M$ be an essential extension. Apply Lemma 1.3 ii, to see that $0 \to M \to E$ splits, that is, $E = M \oplus E'$ for some submodule $E' \subset E$. But then $E' \cap M = 0$, and hence $E' = 0$ and $M = E$.

$\Longleftarrow$: Now suppose that M has no proper essential extension. Embed M into an injective module I and let S be a maximal submodule such that $S \cap M = 0$ (Zorn). Then I/S is an essential extension of I, hence $M = I/S$, hence $I = M \oplus S$. Now apply Lemma 1.3 i to see that M itself is injective. □

Lemma 2.4. *Any module M has a maximal essential extension.*

Proof. Embed M into an injective module I. We claim that there are maximal essential extensions of M in I. We order the set of essential extensions of M in I by inclusion. The union of a chain of essential extensions is again essential (use Remark 2.2), and by Zorn's lemma there are maximal essential extensions of M in I. We claim that such an extension is a maximal essential extension (in general). Let E be such a maximal essential extension inside I and suppose that $E' \supsetneq_e E \supset_e M$. Since $E \to E'$ is an inclusion and I is injective, we can extend the inclusion $E \to I$ to a map $\varphi : E' \to I$. Since $\mathrm{Ker}(\varphi) \cap M = 0$ (by construction), it follows that φ

is injective ($E' \supset_e M$ is essential), but this contradicts the maximality of E inside I. $\square$

2.2. **Injective hulls.**

Theorem 2.5. *For modules $M \subset I$, the following are equivalent:*

i. *I is a maximal essential extension of M.*
ii. *I is injective, and is essential over M.*
iii. *I is minimal injective over M.*

Proof. i $\implies$ ii: It follows from Remark 2.2 that I is maximal essential, hence by Lemma 2.3 I it injective.

ii $\implies$ iii: Suppose that $M \subset I' \subset I$ is injective. Then $I = I' \oplus J$ for some submodule J (Lemma 1.3 ii). As $M \subset I'$, it follows that $J \cap M = 0$, since $I \supset_e M$, it follows that $J = 0$ and hence $I = I'$.

iii $\implies$ i: From the proof of Lemma 2.4 it follows that there is a maximal essential extension E of M contained in I. By i $\implies$ ii we see that E is injective. Since I was a minimal injective module containing M, we have $E = I$. $\square$

Definition 2.6. If $M \subset I$ satisfy the equivalent properties of the above theorem (Theorem 2.5), then I is called an injective hull of M (we have proved the existence in Lemma 2.4).

Lemma 2.7. *Let I, I' be injective hulls of M. Then there exists an isomorphism $g : I' \to I$ which is the identity on M.*

Proof. The map $M \to I'$ can be extended, by injectivity of I, to a map $g : I \to I'$. The map is the identity on M and as before since $(\ker g) \cap M = 0$, it follows by essentiality that g is injective. Since I' was minimal injective, it follows that g is surjectie as well. (Note that the isomorphism is not necessarily unique). $\square$

Notation 2.8. 'The' injective hull of M is denoted by $E(M)$.

Lemma 2.9.

i. *If I is an injective module containing M, then I contains a copy of $E(M)$.*
ii. *If $M \subset_e N$, then N can be enlarged to a copy of $E(M)$ and $E(M) = E(N)$.*

Proof.
i. Follows from the proof of Lemma 2.4.
ii. It follows that $E(N) \supset_e N \supset_e M$. Hence $E(N) \supset_e N$ and it is still a maximal essential extension. It follows that $E(M) = E(N)$.

$\square$

Lemma 2.10. *Let $M_j \subset E_j$ for all $j \in J$ be modules over R. Then $\bigoplus_{j\in J} M_j \subset_e \bigoplus_{j \in J} E_j$ iff for all $j \in J : M_j \subset_e E_j$.*

Proof. $\implies$: Trivial.

$\impliedby$: Trivial. $\square$

Lemma 2.11. *Let M_j for $1 \leq j \leq n$ be R-modules. Then $E(\bigoplus_{j=1}^n M_j) = \bigoplus_{j=1}^n E(M_j)$.*

Proof. Note that $\bigoplus_{j=1}^n E(M_j)$ is injective (Lemma 1.3) and by the previous lemma it is essentiel over $\bigoplus_{j=1}^n M_j$, hence we are done. $\square$

2.3. **Examples.**

Example 2.12. Let R be a domain. Then we know that $Q(R)$ is injective (Example 1.6), and $Q(R)$ is essential over R. Hence $E(R) = Q(R)$.

Example 2.13. Let C_n denote the cyclic group of order n. Define $C_{p^\infty} = \bigcup_{i \in \mathbf{Z}_{\geq 1}} C_{p^i}$. One can easily check that this group is divisble, hence injective over $\mathbf{Z}$. It is easy to see that C_{p^∞} is essential over C_{p^i} for $i \in \mathbf{Z}_{\geq 1}$. Therefore $E(C_{p^i}) = C_{p^\infty}$ for $i \in \mathbf{Z}_{\geq 1}$.

Example 2.14. Let k be a field, then k is injective over k (see Example 1.6). Let R be a finite algebra over k. Let $\hat{R} := \mathrm{Hom}_k(R, k)$. We have seen in Lemma 1.8 that $\hat{R}$ is injective. Let $S \subset \hat{R}$ be the module generated by all simple submodules of $\hat{R}$. Since any module contains a simple submodule, it follows that $E(S) = \hat{R}$. One can show that $S \cong R/\mathrm{rad}R$ where $\mathrm{rad}R$ is the Jacobson radical of R (the intersection of the maximal ideals).

References

[LA] T.Y. Lam, Lectures on modules and rings, Springer-Verlag New York, Inc., 1999