PROJECTIVE MODULES OVER DEDEKIND DOMAINS, February 14, 2010

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Abstract. In these notes we will first define projective modules and prove some standard properties of those modules. Then we will classify finitely generated projective modules over Dedekind domains

Remark 0.1. All rings will be commutative with 1.

1. Projective modules

Definition 1.1. Let R be a ring and let M be an R-module. Then M is called projective if for all surjections $p : N \to N'$ and a map $f : M \to N'$ there is a map $g : M \to N$ such that $p \circ g = f$. In a diagram:

$$\begin{tikzcd} N \arrow[r, "p"] & N' \arrow[r] & 0 \\ & M \arrow[ul, "\exists g"] \arrow[u, "f"'] & \end{tikzcd}$$

Remark 1.2. Let $0 \to N' \to N \to N'' \to 0$ be an exact sequence of R-modules. Let M be an R-module. Then we can look at the following sequence:

$$0 \longrightarrow \mathrm{Hom}(M, N') \longrightarrow \mathrm{Hom}(M, N) \longrightarrow \mathrm{Hom}(M, N'') \longrightarrow 0$$

One can easily show that this sequence is already exact on the left (see [AT], Proposition 2.9ii). Then by definition we see that M is projective iff the functor $\mathrm{Hom}(M, -)$ is exact.

Lemma 1.3. *Let R be a ring and let M be an R-module. Then M is projective iff all exact sequences $0 \to N \to N' \to M \to 0$ split.*

Proof. $\implies$: Suppose we are given an exact sequence $0 \to N \to N' \to M \to 0$. Consider the following diagram:

$$\begin{tikzcd} N' \arrow[r] & M \arrow[r] & 0 \\ & M \arrow[ul, "\exists g"] \arrow[u, "\mathrm{id}"'] & \end{tikzcd}$$

The map g gives a required splitting.

$\impliedby$: Suppose we are given the following diagram with exact rows:

$$\begin{tikzcd} N \arrow[r, "f"] & N' \arrow[r] & 0 \\ & M \arrow[u, "g"'] & \end{tikzcd}$$

Now let $S = \{(x,y) \in N \oplus M : f(x) = g(y)\} \subset N \oplus M$. Notice that S is a submodule, which comes with projections $p_1 : S \to N$ and $p_2 : S \to M$. Notice that since f is surjective, the map p_2 is surjective. So we have the following exact diagram:

$$0 \longrightarrow \ker(p_2) \longrightarrow S \xrightarrow{p_2} M \longrightarrow 0$$

Hence we obtain a splitting $h : M \to S$ with $p_2 \circ h = \mathrm{Id}_M$. By construction of S we have $f \circ p_1 = g \circ p_2$. We have the following diagram:

$$\begin{tikzcd}
N \arrow[r, "f"] & N' \arrow[r] & 0 \\
S \arrow[u, "p_1"] \arrow[r, bend left, "p_2"] & M \arrow[u, "g"'] \arrow[l, bend left, "h"] &
\end{tikzcd}$$

We now claim that $p_1 \circ h : M \to N$ is the required map. Indeed:

$$\begin{aligned} f \circ p_1 \circ h &= g \circ p_2 \circ h \\ &= g \end{aligned}$$

□

Theorem 1.4. *Let M be an R-module. Then M is projective iff there exists an R-module N and some set I such that $M \oplus N \cong R^{(I)}$ (M is a free summand).*

Proof. $\Longrightarrow$: Suppose that M is projective and let S be the free R-module with basis e_m for $m \in M$. Consider the map

$$\begin{aligned} \varphi : S &\to M \\ e_m &\mapsto m \end{aligned}$$

Hence we have an exact diagram $0 \to \ker(\varphi) \to S \to M \to 0$. By Lemma 1.3 we conclude that it splits and hence $S \cong M \oplus \ker(\varphi)$.

$\Longleftarrow$: Let $N \to N'$ be a surjection. Then we need to check that $\mathrm{Hom}(M,N) \to \mathrm{Hom}(M,N')$ is surjective as well (see Remark 1.2). Suppose that $M \oplus S = R^{(I)}$. But then the sequence

$$\mathrm{Hom}(M,N) \oplus \mathrm{Hom}(S,N) = \mathrm{Hom}(R^{(I)},N) \to \mathrm{Hom}(R^{(I)},N') = \mathrm{Hom}(M,N') \oplus \mathrm{Hom}(S,N')$$

is surjective, and it follows that $\mathrm{Hom}(M,N) \to \mathrm{Hom}(M,N')$ is surjective as well. □

Lemma 1.5. *let R be a ring and M an R-module. If M is projective, then M is flat.*

Proof. First notice that free modules are obviously flat. Write $M \oplus M' \cong R^{(I)}$ (Theorem 1.4). Let $N \to N'$ be an inclusion. Then $(M \oplus M') \otimes_R N \to (M \oplus M') \otimes_R N'$ is an injection. This shows that $M \otimes_R N \to M \otimes_R N'$ is also an injection. Hence M is flat. □

2. Projective modules over local rings

Before we can consider projective modules over Dedekind domains, we will consider the case of projective modules over Noetherian local rings.

Lemma 2.1. *Let R be a local ring and let M, N be finitely generated R-modules. If $M \otimes_R N = 0$, then $M = 0$ or $N = 0$.*

Proof. Let $k = R/\mathfrak{m}$. From Nakayama's lemma ([AT], Proposition 2.8) it follows for a finitely generated module M' that $M' = 0$ iff $M'/\mathfrak{m}M' = M' \otimes_R k = 0$. We denote the k-module $M' \otimes_R k$ as M'_k. Now we have (use Exercise 2.15, page 27, [AT]):

$$\begin{aligned} M_k \otimes_k N_k &= (M \otimes_R k) \otimes_k (N \otimes_R k) \\ &= M \otimes_R k \otimes_k k \otimes_R N \\ &= M \otimes_R k \otimes_R N \\ &= (M \otimes_R N)_k \end{aligned}$$

Suppose that $M \otimes_R N = 0$. Then it follows that $M_k \otimes_k N_k = 0$. As these are just k vector spaces, it follows that $M_k = 0$ or $N_k = 0$. By Nakayama it then follows that $M = 0$ or $N = 0$. □

Lemma 2.2. *Let R be a Noetherian local ring, $\mathfrak{m}$ its maximal ideal and $k = R/\mathfrak{m}$. Let M be a finitely generated R-module. Then the following are equivalent:*

- i. *M is free;*
- ii. *M is projective;*
- iii. *M is flat;*
- iv. *the natural map $\mathfrak{m} \otimes_R M \to R \otimes_R M$ is injective;*
- v. $\mathrm{Tor}_1^R(k, M) = 0$.

Proof. i $\Longrightarrow$ ii: If M is free, then it is projective (Theorem 1.4).

ii $\Longrightarrow$ iii: A projective module M is flat by Lemma 1.5.

iii $\Longrightarrow$ iv: If M is flat, the inclusion $\mathfrak{m} \to R$ stays an inclusion after tensoring with M by definition of flatness.

iv $\Longrightarrow$ v: Consider the exact sequence $0 \to \mathfrak{m} \to R \to k \to 0$. Notice that $\mathrm{Tor}_1^R(R, M) = 0$ (as R itself is projective over R). Hence we obtain the following exact sequence:

$$0 = \mathrm{Tor}_1^R(R, M) \longrightarrow \mathrm{Tor}_1^R(k, M) \longrightarrow \mathfrak{m} \otimes_R M \longrightarrow R \otimes_R M \longrightarrow k \otimes_R M \longrightarrow 0$$

As the map $\mathfrak{m} \otimes_R M \to R \otimes_R M$ is injective, it follows that $\mathrm{Tor}_1^R(k, M) = 0$.

v $\Longrightarrow$ i: Let $x_1, \dots, x_n$ be elements of M whose images in $M/\mathfrak{m}M$ form a k-basis of $M/\mathfrak{m}M$. By Nakayama's lemma ([AT], Proposition 2.8) it follows that the x_i generate M as an R-module. Let F be a free A-module with basis $e_1, \dots, e_n$ and define $\varphi : F \to M$ by $\varphi(e_i) = x_i$. Let $E = \mathrm{Ker}(\varphi)$. We then have an exact sequence $0 \to E \to F \to M \to 0$ and we obtain the following exact sequence:

$$0 = \mathrm{Tor}_1^R(k, M) \longrightarrow k \otimes_R E \longrightarrow k \otimes_R F \xrightarrow{1 \otimes \varphi} k \otimes_R M \longrightarrow 0$$

As both $k \otimes_R F$ and $k \otimes_R M$ have the same dimension over k, it follows that $1 \otimes \varphi$ is an isomorphism, and hence $k \otimes_R E = 0$. Notice that E is finitely generated, because it is a submodule of F and A is Noetherian ([EI], Proposition 1.4). By Lemma 2.1 it follows that $E = 0$ and hence $F \cong M$ and M is free. □

3. Projective modules over Dedekind domains

Lemma 3.1. *Let R be a domain and let M be an R-module. Then M is torsion free iff for all maximal ideals $\mathfrak{m} \subset R$ the module $M_{\mathfrak{m}}$ is a torsion free $R_{\mathfrak{m}}$-module.*

Proof. $\Longrightarrow$: Suppose that $\frac{r}{s}\frac{a}{s'} = 0$ where $s, s' \in R \setminus \mathfrak{m}$, $r \in R$ and $a \in M$. Then there is some $s'' \in R \setminus \mathfrak{m}$ such that $s''ra = 0$. If $a \neq 0$, it follows that $s''r = 0$. As R is a domain, it follows that $r = 0$ and we are done.

$\Longleftarrow$: Suppose that M is torsion. Let $x \in M$ be non-zero with $\mathrm{Ann}_R(x) \neq 0$. Suppose $\mathrm{Ann}_R(x) \subset \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ (use that x is non-zero here). Then $M_{\mathfrak{m}}$ is a torsion $R_{\mathfrak{m}}$-module. $\square$

Lemma 3.2. *Let A be a Noetherian ring, M a finitely generated A-module. Then M is a flat A-module iff $M_{\mathfrak{m}}$ is a free $A_{\mathfrak{m}}$-module for all maximal ideals $\mathfrak{m} \subset A$.*

Proof. Proposition 3.10 from [AT] says that flatness is a local property, so M is a flat A-module iff $M_{\mathfrak{m}}$ is a flat $A_{\mathfrak{m}}$-module for all maximal ideals $\mathfrak{m} \subset A$. Notice that $A_{\mathfrak{m}}$ is Noetherian and local, so we can apply Lemma 2.2 to see that $A_{\mathfrak{m}}$ is flat over $R_{\mathfrak{m}}$ iff $A_{\mathfrak{m}}$ is free $R_{\mathfrak{m}}$-module and the result follows. $\square$

Recall the definition of a Dedekind domain (proof can be found in [MAY]):

Theorem 3.3. *Let R be a domain which is not a field. Then the following are equivalent:*

i. *Every non-zero ideal can be written as a product of prime ideals;*
ii. *The ring R is Noetherian, and the localization at each maximal ideal is a discrete valuation ring;*
iii. *Every fractional ideal of R is invertible;*
iv. *The ring R is an integrally closed Noetherian domain with Krull dimension one.*

Definition 3.4. Let R be a domain which is not a field satisfying one of the equivalent statements of Theorem 3.3. Then R is called a Dedekind domain.

Lemma 3.5. *Let R be a ring and let M, N be R-modules. Let $f, g : M \to N$. Suppose that for all maximal ideals $\mathfrak{m} \subset R$ we have that $f_{\mathfrak{m}} = g_{\mathfrak{m}}$. Then $f = g$.*

Proof. Take $a \in M$. Then it is given that for all maximal ideals $\mathfrak{m}$ there is an $s \notin \mathfrak{m}$ with $s(f(a) - g(a)) = 0$. Hence $\mathrm{Ann}(f(a) - g(a)) = R$ and $f(a) = g(a)$. $\square$

Theorem 3.6. *Let R be a Dedekind domain and let M be a finitely generated R-module. Then the following statements are equivalent:*

i. *M is torsion free;*
ii. *M is flat;*
iii. *M is projective.*

Proof. i $\Longleftrightarrow$ ii: Notice that by Lemma 3.1 M is torsion free if for all maximal ideals $\mathfrak{m} \subset R$ the module $M_{\mathfrak{m}}$ is torsion free as an $R_{\mathfrak{m}}$-module. By Lemma 3.2 we need to show that $M_{\mathfrak{m}}$ is torsion free iff $M_{\mathfrak{m}}$ is free. Now notice that $R_{\mathfrak{m}}$ is a discrete valuation ring and hence we are done if we prove the that a finitely generated module N over a discrete valuation ring S is torsion free iff N is free. If N is free, then N is also torsion free (as S is a domain). For the other implication use that a discrete valuation ring is a principal ideal domain, and apply the Structure theorem of finitely generated modules over a principal ideal domain.

iii $\implies$ ii: See Lemma 1.5.

i $\implies$ iii: Note that M is projective iff for all exact sequences $0 \to N \to N' \to M \xrightarrow{f} 0$ there is a splitting $g : M \to N'$ such that $f \circ g = \mathrm{Id}_M$ (Lemma 1.3). Now we localize at a maximal ideal $\mathfrak{m}$ and since this is flat we obtain an exact sequence $0 \to N_{\mathfrak{m}} \to N'_{\mathfrak{m}} \to M_{\mathfrak{m}} \to 0$. As M is flat from i $\implies$ ii, it follows that $M_{\mathfrak{m}}$ is projective by Lemma 2.2. Hence we obtain a splitting $g_{\mathfrak{m}} : M_{\mathfrak{m}} \to N'_{\mathfrak{m}}$ with $f_{\mathfrak{m}} \circ g_{\mathfrak{m}} = \mathrm{Id}_{M_{\mathfrak{m}}}$. Now as $M_{\mathfrak{m}}$ is finitely generated it follows that there is a $c_{\mathfrak{m}} \in R \backslash \mathfrak{m}$ such that $c_{\mathfrak{m}} g_{\mathfrak{m}}(M_{\mathfrak{m}}) \subset N'$ (we clear denominators). As the ideal generated by the $c_{\mathfrak{m}}$ is not contained in any maximal ideal, there are maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ and $x_1, \ldots, x_n \in R$ such that $\sum_{i=1}^n x_i c_{\mathfrak{m}_i} = 1$. Let $i_{\mathfrak{m}} : M \to M_{\mathfrak{m}}$ be the natural map and we define:

$$\begin{aligned} g : M &\to N' \\ a &\mapsto \sum_{i=1}^n x_i c_{\mathfrak{m}_i} g_{\mathfrak{m}_i} \circ i_{\mathfrak{m}_i}(a) \end{aligned}$$

We now claim that $f \circ g = \mathrm{Id}_M$. Indeed, pick a maximal ideal $\mathfrak{m}$ and let $\frac{a}{b} \in M_{\mathfrak{m}}$. Then notice that for $x \in M$ and any maximal ideal $\mathfrak{m}'$ we have $f_{\mathfrak{m}'}(x) = f(x)$:

$$\begin{aligned} (f \circ g)_{\mathfrak{m}} \left(\frac{a}{b}\right) &= f_{\mathfrak{m}} \circ g_{\mathfrak{m}} \left(\frac{a}{b}\right) \\ &= \frac{1}{b} f_{\mathfrak{m}} \circ (\sum_{i=1}^n x_i c_{\mathfrak{m}_i} g_{\mathfrak{m}_i}(a)) \\ &= \frac{1}{b} \sum_{i=1}^n f_{\mathfrak{m}} \circ x_i c_{\mathfrak{m}_i} g_{\mathfrak{m}_i}(a) \\ &= \frac{1}{b} \sum_{i=1}^n f_{\mathfrak{m}_i} \circ x_i c_{\mathfrak{m}_i} g_{\mathfrak{m}_i}(a) \\ &= \frac{1}{b} \sum_{i=1}^n x_i c_{\mathfrak{m}_i} a \\ &= \frac{a}{b} \end{aligned}$$

Apply Lemma 3.5 to see that $f \circ g = \mathrm{Id}_M$. □

We will now state the structure theorem of finitely generated modules over a Dedekind domain, just for completeness.

Theorem 3.7. *Let R be a Dedekind domain and let M be a finitely generated R-module. Then $M \cong T(M) \oplus P$ where $T(M)$ is the torsion submodule of M and where P is torsion free. Furthermore, $T(M) \cong \bigoplus_{i=1}^n R/\mathfrak{p}_i^{n_i}$ for some prime ideals $\mathfrak{p}_i \neq 0$ and $n_i \in \mathbf{Z}_{\geq 1}$. These $(\mathfrak{p}_i, n_i)$ are uniquely determined up to permutation.*

Proof. Notice that $M/T(M)$ is torsion free, hence projective by Theorem 3.6. Hence the map $M \to M/T(M)$ splits by Lemma 1.3 and we see that $M = T(M) \oplus P$ for some projective module P (which is isomorphic to $M/T(M)$).

Now we will prove the statements about the structure of $T(M)$. As M is finitely generated, it follows that $\mathrm{Ann}(T(M)) = Q_1^{s_1} \ldots Q_t^{s_n}$ for some distinct primes $Q_1, \ldots, Q_t$ and integers $s_i, t \in \mathbf{Z}_{\geq 1}$. Furthermore, from Exercise 3.19v ([AT]) it follows that $\mathrm{Supp}(T(M)) = \{Q_1, \ldots, Q_t\}$, notice that $Q_i \neq 0$. Suppose that

$T(M)_{\mathfrak{p}} \neq 0$, then by the structure theorem for principal ideal domains we obtain (and by flatness of localization):

$$\begin{aligned} T(M)_{\mathfrak{p}} &= \bigoplus_{i=1}^{t} R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}^{m_i} \\ &= \bigoplus_{i=1}^{t} (R/\mathfrak{p}^{m_i})_{\mathfrak{p}} \end{aligned}$$

for some $m_i \in \mathbf{Z}_{\geq 1}$. Now notice that $(R/\mathfrak{p}^{m_i})_{\mathfrak{p}} = R/\mathfrak{p}^{m_i}$ as the ring is already local (where $\mathfrak{p} \neq 0$). If $Q \neq \mathfrak{p}$ is another nonzero prime, then we claim that $(R/\mathfrak{p}^{m_i})_Q = 0$. Indeed, take an $s \in \mathfrak{p} \setminus Q$. Then $s^{m_i} R/\mathfrak{p}^{m_i} = 0$, hence $(R/\mathfrak{p}^{m_i})_Q = 0$. We see that $(T(M)_{\mathfrak{p}})_{\mathfrak{p}} = T(M)_{\mathfrak{p}}$ and $(T(M)_{\mathfrak{p}})_Q = 0$ for nonzero primes $\mathfrak{p} \neq Q$. This gives us the following isomorphism, since locally this map is an isomorphism (Proposition 3.8, [AT]):

$$T(M) \longrightarrow \bigoplus_{\mathfrak{p} \neq 0} T(M)_{\mathfrak{p}} \longrightarrow \bigoplus_{i=1}^{n} T(M)_{\mathfrak{p}_i}$$

This shows that $T(M) \cong \bigoplus_{i=1}^{n} R/\mathfrak{p}_i^{n_i}$ for some prime ideals $\mathfrak{p}_i \neq 0$ and $n_i \in \mathbf{Z}_{\geq 1}$. The uniqueness follows from the uniqueness statement in the structure theorem for principal domains after localizing. □

4. One-dimensional projective modules over Dedekind domains

Definition 4.1. Let R be a domain and let A be a finitely generated R-module. Then we define $\mathrm{rank}_R(A) = \dim_{Q(R)}(A \otimes_R Q(R))$.

Lemma 4.2. *Let R be a domain and let I be a nonzero fractional R-ideal. Then I has rank 1 over R.*

Proof. Pick a non-zero $x \in R$ such that $xI \subset R$ and $I \cong xI$. Hence we may assume that $I \subset R$. If we tensor this with $Q(R)$, we get, by flatness of localization, $I \otimes_R Q(R) \subseteq R \otimes_R Q(R) = Q(R)$. Hence $I \otimes_R Q(R)$ has dimension 0 or 1 over $Q(R)$. Notice that $I \otimes_R Q(R) = S^{-1}I$ for some multiplicatively closed subset without 0, hence it is nonzero. □

Lemma 4.3. *Let R be a domain and let I be a nonzero fractional ideal. Then I is projective iff I is invertible.*

Proof. $\Longrightarrow$: Suppose that I is projective. By Theorem 1.4 we see that there is a module S such that $I \oplus S \cong R^{(J)}$ for some set J. For our convenience, assume that $I \oplus S = R^{(J)}$. Now consider the following diagram, where π_j is the projection on the j-th coordinate:

$$I \xrightarrow{i_1} I \oplus S = R^{(J)} \xrightarrow{\pi_j} R$$

Call this composition $g_j : I \to R$. Let $p_1 : R^{(J)} \to I$ be the projection. Notice that $p_1 \circ i_1 = \mathrm{Id}_I$. Pick a nonzero $a \in I$ and for $j \in J$ let $b_j = a^{-1} g_j(a) \in Q(R)$. Let K be the module generated by the b_j. We will show that $K = I^{-1}$. First notice that $aK \subset R$, hence we have a fractional R-ideal. Now take $x \in I$. Then:

$$\begin{aligned} xb_j &= a^{-1} x g_j(a) \\ &= a^{-1} g_j(xa) \\ &= g_j(x) \in R \end{aligned}$$

Hence $KI \subset R$. Furthermore we have, where e_j is the j-th standard basis vector of $R^{(J)}$ (where all sums we consider are in fact finite):

$$\begin{aligned} 1 &= a^{-1}p_1 \circ i_1(a) \\ &= a^{-1}p_1(\sum_{j\in J} g_j(a)e_j) \\ &= \sum_{j\in J} \left(a^{-1}g_j(a)\right) p_1(e_j) \in KI \end{aligned}$$

Hence $KI = R$ and we are done.

$\Longleftarrow$: Suppose that I is invertible and suppose we are given a surjection $f : M \to I$. We need to construct a map $g : I \to M$ with $f \circ g = \mathrm{Id}_I$ by Lemma 1.3. As I is invertible, there are $x_i \in I$ and $y_i \in I^{-1}$ such that $1 = \sum_{i=1}^n x_i y_i$. As f is surjective, we can pick $e_i \in M$ with $f(e_i) = x_i$. Now define the following map:

$$\begin{aligned} g : I &\to M \\ x &\mapsto \sum_{i=1}^n (xy_i)e_i \end{aligned}$$

We then calculate for $x \in I$:

$$\begin{aligned} f \circ g(x) &= f(\sum_{i=1}^n (xy_i)e_i) \\ &= \sum_{i=1}^n (xy_i)f(e_i) \\ &= x\sum_{i=1}^n (x_i y_i) \\ &= x \end{aligned}$$

□

Lemma 4.4. *Let R be a Dedekind domain and let I, J be fractional ideals. Then $I \cong J$ iff there is a nonzero $x \in Q(R)$ with $xI = J$.*

Proof. $\Longrightarrow$: Suppose that $J \cong I$. Then $R = II^{-1} \cong JI^{-1}$. Hence it follows that $xR = JI^{-1}$ for some $x \in Q(R)$ and $J = xI$.

$\Longleftarrow$: If $xI = J$ for some non-zero $x \in Q(R)$, then obviously $I \cong J$. □

5. Classification of projective modules over Dedekind domains

We now want to classify projective modules of rank more than one, and we first need to prove some lemmas. We will first prove some approximation lemma.

Lemma 5.1. *Let R be a Dedekind domain, let X be a finite set of primes and suppose we are given an integer $n_\mathfrak{p}$ for each $\mathfrak{p} \in X$. Then there is an $x \in Q(R)$ with*

$$\begin{aligned} \mathrm{ord}_\mathfrak{p}(x) = n_\mathfrak{p} &\qquad \text{if } \mathfrak{p} \in X \\ \mathrm{ord}_\mathfrak{p}(x) \geq 0 &\qquad \text{if } \mathfrak{p} \notin X. \end{aligned}$$

Proof. Let $X = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ and let $n_1, \ldots, n_n$ be the corresponding integers. First assume that $n_1, \ldots, n_n \geq 0$. Notice that $\mathfrak{p}_1^{n_1+1}, \ldots, \mathfrak{p}_n^{n_n+1}$ are coprime and hence we have a surjection $\varphi : R \to R/\mathfrak{p}_1^{n_1+1} \times \ldots \times R/\mathfrak{p}_n^{n_n}$. Now take an $x_i \in \mathfrak{p}^{n_i} \setminus \mathfrak{p}^{n_i+1}$ (we can do this by the unique factorization) and let x be a preimage under φ of $([x_1], \ldots, [x_n])$. Then notice that $\mathrm{ord}_{\mathfrak{p}_i}(x) = n_i$ and for all other primes we get a non-negative valuation.

Now assume that the n_i may also be negative. Write $n_i = n_i^+ - n_i^-$ where $n_i^+, n_i^- \in \mathbf{Z}_{\geq 0}$. Now pick an $x_1 \in R$ with $\mathrm{ord}_{\mathfrak{p}_i}(x) = n_i^-$. Now there are finitely many other prime $Q_1, \ldots, Q_m$ with $\mathrm{ord}_{Q_i}(x) = m_i \geq 1$. Now take an $x_2 \in R$ with $\mathrm{ord}_{\mathfrak{p}_i}(x_2) = n_i^+$ and $\mathrm{ord}_{Q_i}(x_2) = m_i$. Then $\frac{x_2}{x_1}$ satisfies the required properties. $\square$

Remark 5.2. Let R be a Dedekind domain and suppose that I, J are ideals. Then I and J are coprime iff for all prime ideals $P \subset R$ we have $\mathrm{ord}_P(I) \cdot \mathrm{ord}_P(J) = 0$. This follows directly from the way in which we decompose ideals in products of powers of prime ideals.

Lemma 5.3. *Let R be a Dedekind domain and let I, J be nonzero fractional ideals. Then there are nonzero $x, y \in Q(R)$ such that xI and yJ are coprime ideals.*

Proof. After multiplying by certain elements of $Q(R)^*$ we may assume that $I, J \subseteq R$. Let $I = \prod_{i=1}^n \mathfrak{p}_i^{n_i}$ and let $J = \prod_{i=1}^n \mathfrak{p}_i^{m_i}$ where $n_i, m_i \geq 0$. Now by Lemma 5.1 we can pick an $a \in I$ with $\mathrm{ord}_{\mathfrak{p}_i}(a) = n_i$ and $\mathrm{ord}_Q(a) \geq 0$ for $Q \neq \mathfrak{p}_i$. Now pick a $b \in I^{-1}a$ with $\mathrm{ord}_{\mathfrak{p}_i}(b) = 0$ for $i = 1, \ldots, n$ (again Lemma 5.1). Notice that $\frac{b}{a}I \subseteq R$ and by construction we have $\mathrm{ord}_{\mathfrak{p}_i}(\frac{b}{a}I) = 0$. This shows that the constructed ideal is coprime with J. $\square$

Lemma 5.4. *Let R be a Dedekind domain and let $I_1, \ldots, I_n$ be fractional ideals. Then:*

$$I_1 \oplus \ldots \oplus I_n \cong R^{n-1} \oplus I_1 \ldots I_n$$

Proof. We will first proof the following statement: Let R be a Dedekind domain and let I, J be nonzero fractional ideals. Then $I \oplus J \cong R \oplus IJ$.

First assume that $I, J \subseteq R$. Then we have the following exact sequence:

$$0 \longrightarrow I \cap J \longrightarrow I \oplus J \longrightarrow I + J \longrightarrow 0$$

Now assume that I and J are coprime, then $I \cap J = IJ$ and $I + J = R$. As R is projective (it is free over R) it follows that this sequence splits and hence $I \oplus J \cong R \oplus IJ$.

Now let I, J be fractional ideals. Pick nonzero $x, y \in Q(R)$ such that xI and yJ are coprime (Lemma 5.3). Notice that $xI \cong I$ and $yJ \cong J$ by Lemma 4.4. But then:

$$\begin{aligned} I \oplus J &\cong xI \oplus yJ \\ &\cong R \oplus xyIJ \\ &\cong R \oplus IJ \end{aligned}$$

The general statement now easily follows from induction. $\square$

We now want to know when two expressions as in Lemma 5.4 are isomorphic.

Lemma 5.5. *Let R be a Dedekind domain and let I, J be a fractional ideals. Then $R^{n-1} \oplus I \cong R^{m-1} \oplus J$ iff $n = m$ and $[I] = [J] \in \mathrm{Cl}(Q(R))$.*

Proof. $\Longrightarrow$: Suppose that $R^{n-1} \oplus I \cong R^{m-1} \oplus J$. Looking at the rank yields that $n = m$. Now consider take the $n+1$-th exterior power (the determinant) which transfers direct sums into tensor products (see [BO] Bourbaki, Algebra, 7.7). This gives us an isomorphism $I \to J$, an by Lemma 4.4 we are done.

$\Longleftarrow$: If $[I] = [J]$, then I and J are isomorphic (Lemma 4.4) and the result follows directly. $\square$

We can finally describe the structure finitely generated projective modules over a Dedekind domain.

Theorem 5.6. *Let R be a Dedekind domain and let P be a finitely generated projective R-module of rank n. Then $P \cong R^{n-1} \oplus I$ where I is a nonzero fractional ideal. This $[I] \in \mathrm{Cl}(Q(R))$ is uniquely determined.*

Proof. By induction we show that we can write P as the direct sum of n fractional ideals. We do this by induction, the case $n = 1$ following from the fact that $P = P \otimes_R R \subseteq P \otimes_R Q(R) \cong Q(R)$ in that case (as P is flat, Lemma 1.5) and Lemma 4.3. Now we continue with induction and assume that P has rank n. Then pick a submodule P' of rank $n-1$, then $0 \to P' \to P \to P/P' \to 0$ is an exact sequence and as tensoring with $Q(R)$ is exact, it follows that P/P' has rank 1. Notice that all rank 1 modules are projective by Lemma 4.3 (since all ideals are invertible). Hence $P \cong P' \oplus P/P'$ and we are done by induction.

By Lemma 5.4 and Lemma 5.5 the rest of the lemma follows. $\square$

References

[AT] M. F Atiyah, I.G. MacDonald, Introduction to Commutative Algebra, Addison-Wesley Publishing Company

[BO] N. Bourbaki, Elements of Mathematics Algebra I, Springer-Verlag Berlin

[EI] D. Eisenbud, Commutative Algebra: with a View Toward Algebraic Geometry, Springer, 1995

[MAY] J. P. May, Notes on Dedekind rings