

SEMI-SIMPLE MODULES, RINGS AND ITS APPLICATIONS TO REPRESENTATION THEORY, March 28, 2010

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ABSTRACT. In these notes, we will shortly discuss the theory of semi-simple rings and modules. We will then give applications in representation theory and calculate the irreducible representations of S_4 .

1. SEMI-SIMPLE MODULES

Fix a ring R (not necessarily commutative). In previous lecture notes, one can find the theory of injective and projective modules. These modules had to do with splitting of certain exact sequences. We will define a semi-simple module along the same lines.

Definition 1.1. An R -module M is called semi-simple if any exact sequence of R -modules $0 \rightarrow N \rightarrow M \rightarrow N' \rightarrow 0$ splits.

Recall the definition of a simple R -module.

Definition 1.2. An R -module M is called simple if it has exactly two R -submodules (so $0 \neq M$).

One can then prove the following theorem.

Theorem 1.3. *Let M be an R -module. Then the following are equivalent.*

- i. M is semi-simple.
- ii. *There is a collection $(S_i)_{i \in I}$ of simple R -modules together with an R -isomorphism $M \cong \bigoplus_{i \in I} S_i$.*
- iii. *There is a collection $(S_i)_{i \in I}$ of simple R -modules together with an R -linear surjection $\bigoplus_{i \in I} S_i \rightarrow M$.*

Proof. The proof is a nice exercise. See for example [MI], Theorem 2.3. □

Example 1.4. For example, let $R = \mathbf{Z}$. What is a semi-simple $\mathbf{Z}$ -module? We first look for the simple R -modules. These are isomorphic to $\mathbf{Z}$ modulo a maximal ideal of $\mathbf{Z}$. So a simple module looks like $\mathbf{Z}/p\mathbf{Z}$ for some prime $p \in \mathbf{Z}_{\geq 0}$. By the previous theorem, a semi-simple module looks like $\bigoplus_{i \in I} \mathbf{Z}/p_i\mathbf{Z}$ for some set I .

2. SEMI-SIMPLE RINGS

Definition 2.1. Let R be a ring. Then R is called semi-simple if every (left) R -module is semi-simple.

Theorem 2.2. *The following are equivalent for a ring R .*

- i. R is semi-simple.
- ii. R is semi-simple as (left) R -module.

iii. *Every short exact sequence of R -modules splits.*

Proof. See [DA], Proposition 9.5. $\square$

There is a nice classification result for semi-simple rings, which shows that left semi-simplicity is actually the same as right semi-simplicity. This classification says that every semi-simple ring is isomorphic to a finite product of matrix rings over division rings (uniquely, and the converse also holds; see for example [MI], Theorem 2.13). As we will focus on representations, we will state a slightly more specific version ([DA], Theorem 9.15).

Theorem 2.3. *Let $k = \bar{k}$ be an algebraically closed field. Let R be a ring with $k \subseteq Z(R)$. Assume that $\dim_k(R) < \infty$. Then $R \cong \prod_{i=1}^t M(n_i, k)$ for some t and n_i positive integers. Furthermore, every simple R -module is isomorphic to k^{n_i} (where just one factor $M(n_i, k)$ acts on) where i is uniquely determined.*

3. REPRESENTATION THEORY

In this section, let G be a finite group. We will also fix a field $k = \bar{k}$ with $\text{char}(k) = 0$.

Definition 3.1. A representation of G is a group morphism $\rho : G \rightarrow \text{GL}(n, k)$ for some $n \geq 0$. This n is called the dimension of ρ .

Lemma 3.2. *There is a bijection between the morphisms $G \rightarrow \text{GL}(n, k)$ and the set of $k[G]$ -module structures on k^n .*

Proof. Let $\rho : G \rightarrow \text{GL}(n, k)$ be a morphism. Then we give a k^n a $k[G]$ structure by putting, for $\sigma \in G$ and $v \in k^n$, $\sigma \cdot v = \rho(\sigma)(v)$ and we extend this k -linearly. If on the other hand we have a $k[G]$ -module, then we define the map

$$\begin{aligned} \rho : G &\rightarrow \text{GL}(n, k) \\ \sigma &\mapsto (v \mapsto \sigma \cdot v). \end{aligned}$$

These maps are obviously inverses of each other. $\square$

The following theorem makes the connection with the previous section clear.

Theorem 3.3. *The group ring $k[G]$ is semi-simple.*

Proof. The proof is rather nice and can be found in [MI], Theorem 2.8. Here we use that $\text{char}(k) = 0$. $\square$

We will call a representation simple or irreducible if the corresponding $k[G]$ -module is semi-simple. Differently stated, $\rho : G \rightarrow \text{GL}(n, k)$ is irreducible if the only sub vector-spaces $V \subset k^n$ satisfying $\rho(g)(V) = V$ for any $g \in G$ are 0 and k^n . As any module over a semi-simple ring is the direct sum of simple modules (Theorem 2.2, for finitely generated modules this is in a sense unique), one sees that it is enough to classify the irreducible representations. We can now use Theorem 2.3 to find all simple representations. First notice that an isomorphism between representations ρ, ρ' of k^n is given by $A \in \text{GL}(n, k)$ such that $\rho'(g) = A^{-1}\rho(g)A$ for all $g \in G$. Now remark that all 1-dimensional representations are irreducible and non-isomorphic.

Lemma 3.4. *The following statements hold.*

- i. *There is a bijection between the 1-dimensional representations of G and the set $\text{Hom}(G, k^*)$ of size $\#G/[G, G]$.*
- ii. *The number of irreducible representations of G is equal to the number of conjugacy classes of G .*

Proof. For complete proofs, see [DA], Stelling 9.16 and Stelling 9.17.

- i. The first part of the statement is obvious. For the second part, use the structure theorem of finitely generated Abelian groups and the fact that k^* is Abelian.
- ii. Consider Theorem 2.3. One can show that $k = Z(\text{Gl}(n, k))$, and hence t , the number of irreducible representations, is equal to $\dim_k(Z(k[G]))$. A simple calculation then gives the result.

□

In terms of Theorem 2.3, this means that t is equal to the number of conjugacy classes of G and the number of n_i where $n_i = 1$ is equal to $\#G/[G, G]$.

Definition 3.5. Let $\rho : G \rightarrow \text{GL}(n, k)$ be a representation. Then we can define, for $\sigma \in G$, $\text{Tr}_\rho(\sigma) = \text{Tr}(\rho(\sigma))$. As for matrices A, B we have $\text{Tr}(AB) = \text{Tr}(BA)$, it follows that the trace of an element only depends on the conjugacy class of σ inside G . Furthermore, we see that the trace only depends on the isomorphy class of ρ .

One can actually show that the square matrix $[\text{Tr}_s(\sigma)]_{s \in S, \sigma \in G/\sim}$, where S is the set of irreducible representations and $\sim$ is the conjugacy relation, is invertible.

4. EXAMPLE

We will determine all irreducible representations of S_4 , the symmetric group on the set $\{1, 2, 3, 4\}$ of size 24. First we apply Lemma 3.4. Notice that $[S_4, S_4] = A_4$, hence $\#S_4/[S_4, S_4] = 2$. Every cycle type gives a conjugacy class, and the types are (1) , (12) , (123) , (1234) , $(12)(34)$, hence $t = 5$. By Theorem 2.3 (comparing dimensions), we have to write $24 = n_1^2 + \dots + n_5^2$. We know that we can pick $n_1 = n_2 = 1$ and that $n_3, n_4, n_5 \geq 2$. The only solution is $24 = 1^2 + 1^2 + 2^2 + 3^2 + 3^2$. Hence we see that we have 2 1-dimensional, 1 2-dimensional and finally 2 3-dimensional representations of S_4 .

First consider the 1-dimensional representations. They come from elements of $\text{Hom}(\pm 1, k^*)$. If one translates everything back, we get:

$$\begin{aligned} \rho_0 : S_4 &\rightarrow \text{GL}(1, k) \\ \sigma &\mapsto (1) \end{aligned}$$

and

$$\begin{aligned} \rho_1 : S_4 &\rightarrow \text{GL}(1, k) \\ \sigma &\mapsto (\text{sign}(\sigma)). \end{aligned}$$

We will now try to get more representations. We will first look for representations coming from ‘smaller’ groups. The only normal subgroups of S_4 are

$$\{\{(1)\}, \{(1), (12)(34), (13)(24), (14)(23)\}, A_4, S_4\}.$$

A representation $\rho : G/H \rightarrow \text{GL}(n, k)$ where H is a normal subgroup of G gives a representation $\rho' = \rho \circ \pi$ where $\pi : G \rightarrow G/H$ is the canonical map. An irreducible

representation stays irreducible (there are still no non-trivial fixed subspaces). If we take $H = S_4$, we get ρ_0 back and we have already considered A_4 .

We will now try the V_4 inside. Notice that $\#S_4/V_4 = 6$, and as S_4 has no elements of order 6, it follows that $S_4/V_4 \cong S_3$. Hence we need to study representations of S_3 . Again we have the sign and the identity (but we have seen these already). As S_3 has 3 conjugacy classes, we find one representation of dimension 2. One can get this representation as follows. Consider the space k^3 with basis e_1, e_2, e_3 . Now we can let S_3 act on this space by permuting the e_i and extend by k -linearity. As $k(e_1 + e_2 + e_3)$ is fixed under this action, our presentation is not irreducible. Now consider $k^3/(k(e_1 + e_2 + e_3))$ with the induced action. We claim that this space is irreducible. Indeed, if it isn't, it is a sum of the representations given by the sign or identity. But the stabilizer of a non-zero vector in our space is always $0 \subset S_3$, and in our previous spaces this was not the case. Call this representation ρ_2 .

Now we still need to find the irreducible representations of dimension 3. The first one is obtained in the same fashion as the one for the S_3 , call this one ρ_3 . One can also define the tensor of two representations M and N as $M \otimes_k N$ where $\sigma(m \otimes n) = \sigma(m) \otimes \sigma(n)$. Now consider the representation given by $\rho_3 \otimes \rho_1$. We claim that this gives us our final irreducible representation, call it ρ_4 . One can again see that this is not a sum of representations of smaller degree. We still need to prove that it is not equal to ρ_3 (as in the case of S_3 happens). To see that they are really different, we will calculate the trace matrix.

We find the following matrix:

repr, conj	(1)	(12)	(123)	(1234)	(12)(34)
ρ_0	1	1	1	1	1
ρ_1	1	-1	1	-1	1
ρ_2	2	0	-1	0	2
ρ_3	3	1	0	-1	-1
ρ_4	3	-1	0	1	-1

This shows that $\rho_4 \not\cong \rho_3$ and we have found all irreducible representations.

REFERENCES

- [DA] Daems and Palenstijn, Representatietheory, 2003, <http://websites.math.leidenuniv.nl/algebra/rpt.pdf> (in Dutch)
- [MI] Joost, Semisimplicity and finite groups, 2008, <http://www.math.leidenuniv.nl/scripties/MichielsenBachelor.pdf>