Chapter 5

Localization

Consider a category $\mathcal{C}$ and a family $\mathcal{S}$ of morphisms in $\mathcal{C}$. The aim of localization is to find a new category $\mathcal{C}_{\mathcal{S}}$ and a functor $Q : \mathcal{C} \to \mathcal{C}_{\mathcal{S}}$ which sends the morphisms belonging to $\mathcal{S}$ to isomorphisms in $\mathcal{C}_{\mathcal{S}}$, $(Q, \mathcal{C}_{\mathcal{S}})$ being "universal" for such a property.

In this chapter, we shall construct the localization of a category when $\mathcal{S}$ satisfies suitable conditions and the localization of functors. The reader shall be aware that in general, the localization of a $\mathcal{U}$-category $\mathcal{C}$ is no more a $\mathcal{U}$-category (see Remark 5.1.15).

Localization of categories appears in particular in the construction of derived categories.

A classical reference is [6].

5.1 Localization of categories

Let $\mathcal{C}$ be a category and let $\mathcal{S}$ be a family of morphisms in $\mathcal{C}$.

Definition 5.1.1. A localizaton of $\mathcal{C}$ by $\mathcal{S}$ is the data of a category $\mathcal{C}_{\mathcal{S}}$ and a functor $Q : \mathcal{C} \to \mathcal{C}_{\mathcal{S}}$ satisfying:

(a) for all $s \in \mathcal{S}$, $Q(s)$ is an isomorphism,

(b) for any functor $F : \mathcal{C} \to \mathcal{A}$ such that $F(s)$ is an isomorphism for all $s \in \mathcal{S}$, there exists a functor $F_{\mathcal{S}} : \mathcal{C}_{\mathcal{S}} \to \mathcal{A}$ and an isomorphism $F \simeq F_{\mathcal{S}} \circ Q$,

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "Q"'] & \mathcal{A} \\ \mathcal{C}_{\mathcal{S}} \arrow[ur, dotted, "F_{\mathcal{S}}"'] & \end{tikzcd}$$

(c) if G_1 and G_2 are two objects of $\mathrm{Fct}(\mathcal{C}_\mathcal{S}, \mathcal{A})$, then the natural map

$$\mathrm{Hom}_{\,\mathrm{Fct}(\mathcal{C}_\mathcal{S},\mathcal{A})}(G_1, G_2) \to \mathrm{Hom}_{\,\mathrm{Fct}(\mathcal{C},\mathcal{A})}(G_1 \circ Q, G_2 \circ Q) \tag{5.1}$$

is bijective.

Note that (c) means that the functor $\circ Q : \mathrm{Fct}(\mathcal{C}_\mathcal{S}, \mathcal{A}) \to \mathrm{Fct}(\mathcal{C}, \mathcal{A})$ is fully faithful. This implies that $F_\mathcal{S}$ in (b) is unique up to unique isomorphism.

Proposition 5.1.2. (i) *If $\mathcal{C}_\mathcal{S}$ exists, it is unique up to equivalence of categories.*

(ii) *If $\mathcal{C}_\mathcal{S}$ exists, then, denoting by $\mathcal{S}^{\mathrm{op}}$ the image of $\mathcal{S}$ in $\mathcal{C}^{\mathrm{op}}$ by the functor* op, $(\mathcal{C}^{\mathrm{op}})_{\mathcal{S}^{\mathrm{op}}}$ *exists and there is an equivalence of categories:*

$$(\mathcal{C}_\mathcal{S})^{\mathrm{op}} \simeq (\mathcal{C}^{\mathrm{op}})_{\mathcal{S}^{\mathrm{op}}}.$$

Proof. (i) is obvious.
(ii) Assume $\mathcal{C}_\mathcal{S}$ exists. Set $(\mathcal{C}^{\mathrm{op}})_{\mathcal{S}^{\mathrm{op}}} := (\mathcal{C}_\mathcal{S})^{\mathrm{op}}$ and define $Q^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to (\mathcal{C}^{\mathrm{op}})_{\mathcal{S}^{\mathrm{op}}}$ by $Q^{\mathrm{op}} = \mathrm{op} \circ Q \circ \mathrm{op}$. Then properties (a), (b) and (c) of Definition 5.1.1 are clearly satisfied. q.e.d.

Definition 5.1.3. One says that $\mathcal{S}$ is a right multiplicative system if it satisfies the axioms S1-S4 below.

S1 For all $X \in \mathcal{C}$, $\mathrm{id}_X \in \mathcal{S}$.

S2 For all $f \in \mathcal{S}, g \in \mathcal{S}$, if $g \circ f$ exists then $g \circ f \in \mathcal{S}$.

S3 Given two morphisms, $f : X \to Y$ and $s : X \to X'$ with $s \in \mathcal{S}$, there exist $t : Y \to Y'$ and $g : X' \to Y'$ with $t \in \mathcal{S}$ and $g \circ s = t \circ f$. This can be visualized by the diagram:

$$\begin{tikzcd} X' & & & X' \arrow[r, dotted, "g"'] & Y' \\ X \arrow[u, "s"] \arrow[r, "f"'] & Y & \Rightarrow & X \arrow[u, "s"] \arrow[r, "f"'] & Y \arrow[u, dotted, "t"'] \end{tikzcd}$$

S4 Let $f, g : X \to Y$ be two parallel morphisms. If there exists $s \in \mathcal{S} : W \to X$ such that $f \circ s = g \circ s$ then there exists $t \in \mathcal{S} : Y \to Z$ such that $t \circ f = t \circ g$. This can be visualized by the diagram:

$$\begin{tikzcd} W \arrow[r, "s"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, dotted, "t"] & Z \end{tikzcd}$$

Notice that these axioms are quite natural if one wants to invert the elements of $\mathcal{S}$. In other words, if the element of $\mathcal{S}$ would be invertible, then these axioms would clearly be satisfied.

Remark 5.1.4. Axioms S1-S2 asserts that $\mathcal{S}$ is the family of morphisms of a subcategory $\widetilde{\mathcal{S}}$ of $\mathcal{C}$ with $\mathrm{Ob}(\widetilde{\mathcal{S}}) = \mathrm{Ob}(\mathcal{C})$.

Remark 5.1.5. One defines the notion of a left multiplicative system $\mathcal{S}$ by reversing the arrows. This means that the condition S3 is replaced by: given two morphisms, $f : X \to Y$ and $t : Y' \to Y$, with $t \in \mathcal{S}$, there exist $s : X' \to X$ and $g : X' \to Y'$ with $s \in \mathcal{S}$ and $t \circ g = f \circ s$. This can be visualized by the diagram:

$$\begin{tikzcd} & Y' \arrow[d, "t"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} X' \arrow[r, dotted, "g"] \arrow[d, dotted, "s"'] & Y' \arrow[d, "t"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

and S4 is replaced by: if there exists $t \in \mathcal{S} : Y \to Z$ such that $t \circ f = t \circ g$ then there exists $s \in \mathcal{S} : W \to X$ such that $f \circ s = g \circ s$. This is visualized by the diagram

$$\begin{tikzcd} W \arrow[r, dotted, "s"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "t"] & Z \end{tikzcd}$$

In the literature, one often calls a multiplicative system a system which is both right and left multiplicative.

Many multiplicative systems that we shall encounter satisfy a useful property that we introduce now.

Definition 5.1.6. Assume that $\mathcal{S}$ satisfies the axioms S1-S2 and let $X \in \mathcal{C}$. One defines the categories $\mathcal{S}_X$ and $\mathcal{S}^X$ as follows.

$$\begin{aligned} \mathrm{Ob}(\mathcal{S}^X) &= \{s : X \to X'; s \in \mathcal{S}\} \\ \mathrm{Hom}_{\mathcal{S}^X}((s : X \to X'), (s : X \to X'')) &= \{h : X' \to X''; h \circ s = s'\} \\ \mathrm{Ob}(\mathcal{S}_X) &= \{s : X' \to X; s \in \mathcal{S}\} \\ \mathrm{Hom}_{\mathcal{S}_X}((s : X' \to X), (s' : X'' \to X)) &= \{h : X' \to X''; s' \circ h = s\}. \end{aligned}$$

Proposition 5.1.7. *Assume that $\mathcal{S}$ is a right (resp. left) multiplicative system. Then the category $\mathcal{S}^X$ (resp. $\mathcal{S}_X^{\mathrm{op}}$) is filtrant.*

Proof. By reversing the arrows, both results are equivalent. We treat the case of $\mathcal{S}^X$.

(a) Let $s : X \to X'$ and $s' : X \to X''$ belong to $\mathcal{S}$. By S3, there exists $t : X' \to X'''$ and $t' : X'' \to X'''$ such that $t' \circ s' = t \circ s$, and $t \in \mathcal{S}$. Hence, $t \circ s \in \mathcal{S}$ by S2 and $(X \to X''')$ belongs to $\mathcal{S}^X$.
(b) Let $s : X \to X'$ and $s' : X \to X''$ belong to $\mathcal{S}$, and consider two morphisms $f, g : X' \to X''$, with $f \circ s = g \circ s = s'$. By S4 there exists $t : X'' \to W, t \in \mathcal{S}$ such that $t \circ f = t \circ g$. Hence $t \circ s' : X \to W$ belongs to $\mathcal{S}^X$. q.e.d.

One defines the functors:

$$\begin{aligned} \alpha_X : \mathcal{S}^X \to \mathcal{C} \qquad & (s : X \to X') \mapsto X', \\ \beta_X : \mathcal{S}_X^{\mathrm{op}} \to \mathcal{C} \qquad & (s : X' \to X) \mapsto X'. \end{aligned}$$

We shall concentrate on right multiplicative system.

Definition 5.1.8. Let $\mathcal{S}$ be a right multiplicative system, and let $X, Y \in \mathrm{Ob}(\mathcal{C})$. We set

$$\mathrm{Hom}_{\mathcal{C}^r_{\mathcal{S}}}(X, Y) = \varinjlim_{(Y \to Y') \in \mathcal{S}^Y} \mathrm{Hom}_{\mathcal{C}}(X, Y').$$

Lemma 5.1.9. *Assume that $\mathcal{S}$ is a right multiplicative system. Let $Y \in \mathcal{C}$ and let $s : X \to X' \in \mathcal{S}$. Then s induces an isomorphism*

$$\mathrm{Hom}_{\mathcal{C}^r_{\mathcal{S}}}(X', Y) \xrightarrow[\circ s]{\sim} \mathrm{Hom}_{\mathcal{C}^r_{\mathcal{S}}}(X, Y).$$

Proof. (i) The map $\circ s$ is surjective. This follows from S3, as visualized by the diagram in which $s, t, t' \in \mathcal{S}$:

$$\begin{tikzcd} X' \arrow[r, dotted] & Y'' & \\ X \arrow[u, "s"] \arrow[r, "f"'] & Y' \arrow[u, dotted, "t'"'] & Y \arrow[l, "t"'] \end{tikzcd}$$

(ii) The map $\circ s$ is injective. This follows from S4, as visualized by the diagram in which $s, t, t' \in \mathcal{S}$:

$$\begin{tikzcd} X \arrow[r, "s"] & X' \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y' \arrow[r, dotted, "t'"] & Y'' \\ & & Y \arrow[u, "t"] & \end{tikzcd}$$

q.e.d.

Using Lemma 5.1.9, we define the composition

$$\operatorname{Hom}_{\mathcal{C}^r_{\mathcal{S}}}(X,Y)\times\operatorname{Hom}_{\mathcal{C}^r_{\mathcal{S}}}(Y,Z)\to\operatorname{Hom}_{\mathcal{C}^r_{\mathcal{S}}}(X,Z) \tag{5.2}$$

as

$$\begin{aligned}
&\varinjlim_{Y\to Y'}\operatorname{Hom}_{\mathcal{C}}(X,Y')\times\varinjlim_{Z\to Z'}\operatorname{Hom}_{\mathcal{C}}(Y,Z')\\
&\qquad\simeq\varinjlim_{Y\to Y'}\Big(\operatorname{Hom}_{\mathcal{C}}(X,Y')\times\varinjlim_{Z\to Z'}\operatorname{Hom}_{\mathcal{C}}(Y,Z')\Big)\\
&\qquad\xleftarrow{\ \sim\ }\varinjlim_{Y\to Y'}\Big(\operatorname{Hom}_{\mathcal{C}}(X,Y')\times\varinjlim_{Z\to Z'}\operatorname{Hom}_{\mathcal{C}}(Y',Z')\Big)\\
&\qquad\to\varinjlim_{Y\to Y'}\varinjlim_{Z\to Z'}\operatorname{Hom}_{\mathcal{C}}(X,Z')\\
&\qquad\simeq\varinjlim_{Z\to Z'}\operatorname{Hom}_{\mathcal{C}}(X,Z')
\end{aligned}$$

Lemma 5.1.10. *The composition* (5.2) *is associative.*

The verification is left to the reader.

Hence we get a category $\mathcal{C}^r_{\mathcal{S}}$ whose objects are those of $\mathcal{C}$ and morphisms are given by Definition 5.1.8.

Let us denote by $Q_{\mathcal{S}}:\mathcal{C}\to\mathcal{C}^r_{\mathcal{S}}$ the natural functor associated with

$$\operatorname{Hom}_{\mathcal{C}}(X,Y)\to\varinjlim_{(Y\to Y')\in\mathcal{S}^Y}\operatorname{Hom}_{\mathcal{C}}(X,Y').$$

If there is no risk of confusion, we denote this functor simply by Q.

Lemma 5.1.11. *If $s:X\to Y$ belongs to $\mathcal{S}$, then $Q(s)$ is invertible.*

Proof. For any $Z\in\mathcal{C}^r_{\mathcal{S}}$, the map $\operatorname{Hom}_{\mathcal{C}^r_{\mathcal{S}}}(Y,Z)\to\operatorname{Hom}_{\mathcal{C}^r_{\mathcal{S}}}(X,Z)$ is bijective by Lemma 5.1.9. q.e.d.

A morphism $f:X\to Y$ in $\mathcal{C}^r_{\mathcal{S}}$ is thus given by an equivalence class of triplets (Y',t,f') with $t:Y\to Y', t\in\mathcal{S}$ and $f':X\to Y'$, that is:

$$X\xrightarrow[f']{}Y'\xleftarrow[t]{}Y$$

the equivalence relation being defined as follows: $(Y',t,f')\sim(Y'',t',f'')$ if there exists (Y''',t'',f''') $(t,t',t''\in S)$ and a commutative diagram:

$$\begin{tikzcd}
 & Y' \arrow[d, dotted] & \\
X \arrow[ur, "f'"] \arrow[r, dotted, "f'''"] \arrow[dr, "f''"'] & Y''' & Y \arrow[ul, "t"'] \arrow[l, dotted, "t''"'] \arrow[dl, "t'"] \\
 & Y'' \arrow[u, dotted] &
\end{tikzcd} \tag{5.3}$$

Note that the morphism (Y', t, f') in $\mathcal{C}^r_{\mathcal{S}}$ is $Q(t)^{-1} \circ Q(f')$, that is,

$$f = Q(t)^{-1} \circ Q(f'). \tag{5.4}$$

For two parallel arrows $f, g : X \rightrightarrows Y$ in $\mathcal{C}$ we have the equivalence

$$Q(f) = Q(g) \in \mathcal{C}^r_{\mathcal{S}} \Leftrightarrow \text{ there exits } s : Y \to Y', s \in \mathcal{S} \text{ with } s \circ f = s \circ g. \tag{5.5}$$

The composition of two morphisms $(Y', t, f') : X \to Y$ and $(Z', s, g') : Y \to Z$ is defined by the diagram below in which $t, s, s' \in \mathcal{S}$:

$$\begin{tikzcd}
& & W & & \\
X \arrow[r, "f'"'] & Y' \arrow[ur, dotted, "h"] & Y \arrow[l, "t"'] \arrow[r, "g'"'] & Z' \arrow[ul, dotted, "s'"'] & Z \arrow[l, "s"']
\end{tikzcd}$$

Theorem 5.1.12. *Assume that $\mathcal{S}$ is a right multiplicative system.*

(i) *The category $\mathcal{C}^r_{\mathcal{S}}$ and the functor Q define a localization of $\mathcal{C}$ by $\mathcal{S}$.*

(ii) *For a morphism $f : X \to Y$, $Q(f)$ is an isomorphism in $\mathcal{C}^r_{\mathcal{S}}$ if and only if there exist $g : Y \to Z$ and $h : Z \to W$ such that $g \circ f \in \mathcal{S}$ and $h \circ g \in \mathcal{S}$.*

Notation 5.1.13. From now on, we shall write $\mathcal{C}_{\mathcal{S}}$ instead of $\mathcal{C}^r_{\mathcal{S}}$. This is justified by Theorem 5.1.12.

Remark 5.1.14. (i) In the above construction, we have used the property of $\mathcal{S}$ of being a right multiplicative system. If $\mathcal{S}$ is a left multiplicative system, one sets

$$\mathrm{Hom}_{\mathcal{C}^l_{\mathcal{S}}}(X, Y) = \varinjlim_{(X' \to X) \in \mathcal{S}_X} \mathrm{Hom}_{\mathcal{C}}(X', Y).$$

By Proposition 5.1.2 (i), the two constructions give equivalent categories.
(ii) If $\mathcal{S}$ is both a right and left multiplicative system,

$$\mathrm{Hom}_{\mathcal{C}_{\mathcal{S}}}(X, Y) \simeq \varinjlim_{(X' \to X) \in \mathcal{S}_X, (Y \to Y') \in \mathcal{S}^Y} \mathrm{Hom}_{\mathcal{C}}(X', Y').$$

Remark 5.1.15. In general, $\mathcal{C}_{\mathcal{S}}$ is no more a $\mathcal{U}$-category. However, if one assumes that for any $X \in \mathcal{C}$ the category $\mathcal{S}^X$ is small (or more generally, cofinally small, which means that there exists a small category cofinal to it), then $\mathcal{C}_{\mathcal{S}}$ is a $\mathcal{U}$-category, and there is a similar result with the $\mathcal{S}_X$'s.

5.2 Localization of subcategories

Proposition 5.2.1. *Let $\mathcal{C}$ be a category, $\mathcal{I}$ a full subcategory, $\mathcal{S}$ a right multiplicative system in $\mathcal{C}$, $\mathcal{T}$ the family of morphisms in $\mathcal{I}$ which belong to $\mathcal{S}$.*

(i) *Assume that $\mathcal{T}$ is a right multiplicative system in $\mathcal{I}$. Then $\mathcal{I}_{\mathcal{T}} \to \mathcal{C}_{\mathcal{S}}$ is well-defined.*

(ii) *Assume that for every $f : Y \to X$, $f \in \mathcal{S}$, $Y \in \mathcal{I}$, there exists $g : X \to W$, $W \in \mathcal{I}$, with $g \circ f \in \mathcal{S}$. Then $\mathcal{T}$ is a right multilplicative system and $\mathcal{I}_{\mathcal{T}} \to \mathcal{C}_{\mathcal{S}}$ is fully faithful.*

Proof. (i) is obvious.
(ii) It is left to the reader to check that $\mathcal{T}$ is a right multpiplicative system. For $X \in \mathcal{I}$, $\mathcal{T}^X$ is the full subcategory of $\mathcal{S}^X$ whose objects are the morphisms $s : X \to Y$ with $Y \in \mathcal{I}$. By Proposition 5.1.7 and the hypothesis, the functor $\mathcal{T}^X \to \mathcal{S}^X$ is cofinal, and the result follows from Definition 5.1.8. q.e.d.

Corollary 5.2.2. *Let $\mathcal{C}$ be a category, $\mathcal{I}$ a full subcategory, $\mathcal{S}$ a right multiplicative system in $\mathcal{C}$, $\mathcal{T}$ the family of morphisms in $\mathcal{I}$ which belong to $\mathcal{S}$. Assume that for any $X \in \mathcal{C}$ there exists $s : X \to W$ with $W \in \mathcal{I}$ and $s \in \mathcal{S}$.*

Then $\mathcal{T}$ is a right multpiplicative system and $\mathcal{I}_{\mathcal{T}}$ is equivalent to $\mathcal{C}_{\mathcal{S}}$.

Proof. The natural functor $\mathcal{I}_{\mathcal{T}} \to \mathcal{C}_{\mathcal{S}}$ is fully faithful by Proposition 5.2.1 and is essentially surjective by the assumption. q.e.d.

5.3 Localization of functors

Let $\mathcal{C}$ be a category, $\mathcal{S}$ a right multiplicative system in $\mathcal{C}$ and $F : \mathcal{C} \to \mathcal{A}$ a functor. In general, F does not send morphisms in $\mathcal{S}$ to isomorphisms in $\mathcal{A}$. In other words, F does not factorize through $\mathcal{C}_{\mathcal{S}}$. It is however possible in some cases to define a localization of F as follows.

Definition 5.3.1. A right localization of F (if it exists) is a functor $F_{\mathcal{S}} : \mathcal{C}_{\mathcal{S}} \to \mathcal{A}$ and a morphism of functors $\tau : F \to F_{\mathcal{S}} \circ Q$ such that for any functor $G : \mathcal{C}_{\mathcal{S}} \to \mathcal{A}$ the map

$$\mathrm{Hom}_{\,\mathrm{Fct}(\mathcal{C}_{\mathcal{S}},\mathcal{A})}(F_{\mathcal{S}}, G) \to \mathrm{Hom}_{\,\mathrm{Fct}(\mathcal{C},\mathcal{A})}(F, G \circ Q) \tag{5.6}$$

is bijective. (This map is obtained as the composition $\mathrm{Hom}_{\,\mathrm{Fct}(\mathcal{C}_{\mathcal{S}},\mathcal{A})}(F_{\mathcal{S}}, G) \to \mathrm{Hom}_{\,\mathrm{Fct}(\mathcal{C},\mathcal{A})}(F_{\mathcal{S}} \circ Q, G \circ Q) \xrightarrow{\tau} \mathrm{Hom}_{\,\mathrm{Fct}(\mathcal{C},\mathcal{A})}(F, G \circ Q)$.)

We shall say that F is right localizable if it admits a right localization.

One defines similarly the left localization. Since we mainly consider right localization, we shall sometimes omit the word "right" as far as there is no risk of confusion.

If $(\tau, F_{\mathcal{S}})$ exists, it is unique up to unique isomorphisms. Indeed, $F_{\mathcal{S}}$ is a representative of the functor

$$G \mapsto \operatorname{Hom}_{\operatorname{Fct}(\mathcal{C},\mathcal{A})}(F, G \circ Q).$$

(This last functor is defined on the category $\operatorname{Fct}(\mathcal{C}_{\mathcal{S}}, \mathcal{A})$ with values in **Set**.)

Proposition 5.3.2. *Let $\mathcal{C}$ be a category, $\mathcal{I}$ a full subcategory, $\mathcal{S}$ a right multiplicative system in $\mathcal{C}$, $\mathcal{T}$ the family of morphisms in $\mathcal{I}$ which belong to $\mathcal{S}$. Let $F : \mathcal{C} \to \mathcal{A}$ be a functor. Assume that*

(i) *for any $X \in \mathcal{C}$ there exists $s : X \to W$ with $W \in \mathcal{I}$ and $s \in \mathcal{S}$,*

(ii) *for any $t \in \mathcal{T}$, $F(t)$ is an isomorphism.*

Then F is right localizable.

Proof. We shall apply Corollary 5.2.2.

Denote by $\iota : \mathcal{I} \to \mathcal{C}$ the natural functor. By the hypothesis, the localization $F_{\mathcal{T}}$ of $F \circ \iota$ exists. Consider the diagram:

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "Q_{\mathcal{S}}"] & & \mathcal{C}_{\mathcal{S}} \arrow[dd, dotted, "F_{\mathcal{S}}"] \\
\mathcal{I} \arrow[u, "\iota"] \arrow[r, "Q_{\mathcal{T}}"] \arrow[drr, "F\circ\iota"'] & \mathcal{I}_{\mathcal{T}} \arrow[ur, "\sim", "\iota_Q"'] \arrow[dr, "F_{\mathcal{T}}"] & \\
 & & \mathcal{A}
\end{tikzcd}$$

Denote by ι_Q^{-1} a quasi-inverse of ι_Q and set $F_{\mathcal{S}} := F_{\mathcal{T}} \circ \iota_Q^{-1}$. Let us show that $F_{\mathcal{S}}$ is the localization of F. Let $G : \mathcal{C}_{\mathcal{S}} \to \mathcal{A}$ be a functor. We have the chain of morphisms:

$$\begin{aligned}
\operatorname{Hom}_{\operatorname{Fct}(\mathcal{C},\mathcal{A})}(F, G \circ Q_{\mathcal{S}}) &\xrightarrow{\lambda} \operatorname{Hom}_{\operatorname{Fct}(\mathcal{I},\mathcal{A})}(F \circ \iota, G \circ Q_{\mathcal{S}} \circ \iota) \\
&\simeq \operatorname{Hom}_{\operatorname{Fct}(\mathcal{I},\mathcal{A})}(F_{\mathcal{T}} \circ Q_{\mathcal{T}}, G \circ \iota_Q \circ Q_{\mathcal{T}}) \\
&\simeq \operatorname{Hom}_{\operatorname{Fct}(\mathcal{I}_{\mathcal{T}},\mathcal{A})}(F_{\mathcal{T}}, G \circ \iota_Q) \\
&\simeq \operatorname{Hom}_{\operatorname{Fct}(\mathcal{C}_{\mathcal{S}},\mathcal{A})}(F_{\mathcal{T}} \circ \iota_Q^{-1}, G) \\
&\simeq \operatorname{Hom}_{\operatorname{Fct}(\mathcal{C}_{\mathcal{S}},\mathcal{A})}(F_{\mathcal{S}}, G).
\end{aligned}$$

We shall not prove here that λ is an isomorphism. The first isomomorphism above (after λ) follows from the fact that $Q_{\mathcal{T}}$ is a localization functor (see Definition 5.1.1 (c)). The other isomorphisms are obvious. q.e.d.