

CLASSIFICATION OF VECTOR BUNDLES ON $\mathbb{P}^1$

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ABSTRACT. Last time we saw the classification of projective modules over Dedekind domains. We now consider the geometric analogue, classifying projective $\mathcal{O}_X$ -modules over smooth projective curves, in particular, when $X = \mathbb{P}^1$.

Let X/k be an irreducible smooth projective curve over the algebraically closed field k . We can produce an open affine cover of X by taking out a k -rational point from X . Denote this point $\infty \in X(k)$, and let $U = X - \{\infty\}$. Note that $U = \text{Spec } \mathcal{O}_X(U)$ and $\mathcal{O}_X(U)$ is a Dedekind domain. We want to classify locally free $\mathcal{O}_X$ -modules $\mathcal{E}$ of rank r , $r > 0$. There is an intrinsic way of decomposing $\mathcal{E}$ using line bundles (the analogue of invertible fractional ideals in the algebraic case) known as the Harder-Narasimhan filtration. Here, we discuss the simplified case of $X = \mathbb{P}^1$ and the classification of vector bundles over X via linear algebra.

Theorem 1 (Grothendieck). *A vector bundle of rank r over the projective line $\mathbb{P}^1$ can be decomposed into r line bundles uniquely up to isomorphism.*

If we let $\mathcal{E}$ be a vector bundle of rank r , with $\mathcal{O}_X$ the usual sheaf of functions on $X = \mathbb{P}^1$, then we can write our line bundles as the invertible sheaves $\mathcal{O}_X(n)$ with $n \in \mathbb{Z}$. Thus, the decomposition can be stated as

$$\mathcal{E} \cong \bigoplus_{i=1}^n \mathcal{O}(n_i) \quad n_1 \geq \dots \geq n_r.$$

If we use the usual open cover of $\mathbb{P}^1$ with two affine lines $U_0 = \mathbb{P}^1 - \{\infty\}$ and $U_1 = \mathbb{P}^1 - \{0\}$, note that $\mathcal{O}_{U_0 \cap U_1} = k[x, x^{-1}]$ (with $\mathcal{O}_{U_0} = k[x]$ and $\mathcal{O}_{U_1} = k[x^{-1}]$). A vector bundle (up to isomorphism) $\mathcal{E}$ of rank n is then a linear automorphism on $\mathcal{O}_{U_0 \cap U_1}^r$ modulo automorphisms of each $\mathcal{O}_{U_i}^r$ for $i = 0, 1$. (Using the definition given in Hartshorne II.5.18 where $A = k[x, x^{-1}]$, the linear automorphisms are $\psi_1^{-1} \circ \psi_0$ where $\psi_i : \mathcal{O}_{U_i}^r \rightarrow \mathcal{E}|_{U_i}$ are isomorphisms, and the definition of isomorphism of vector bundles allows us to change bases of $\mathcal{O}_{U_i}^r$.)

Thinking of this in linear algebra terms, these linear automorphisms on $\mathcal{O}_{U_0 \cap U_1}^r$ are elements of $GL_r(k[x, x^{-1}])$, and changing coordinates in $\mathcal{O}_{U_i}^r$ are elements of $GL_r(k[x])$ for $i = 0$ and $GL_r(k[x^{-1}])$ for $i = 1$. Thus up to isomorphism, the vector bundles of rank r on $\mathbb{P}^1$ are elements of the double quotient

$$GL_r(k[x^{-1}]) \backslash GL_r(k[x, x^{-1}]) / GL_r(k[x]).$$

The decomposition of vector bundles into line bundles should mean that these double cosets can be represented by diagonal matrices where the i th entry on the diagonal corresponds to the line bundle $\mathcal{O}(n_i)$ in the decomposition above.

Theorem 2. *For every matrix M in $GL_r(k[x, x^{-1}])$, there exists a matrix $P \in GL_r(k[x])$ and $Q \in GL_r(k[x^{-1}])$ such that*

$$QMP = \begin{pmatrix} x^{n_1} & & & 0 \\ & x^{n_2} & & \\ & & \ddots & \\ 0 & & & x^{n_r} \end{pmatrix} \quad n_1 \geq n_2 \geq \dots \geq n_r$$

[Note: For the affine case, taking the double quotient

$$GL_n(k[x]) \backslash M_{n,m}(k[x]) / GL_m(k[x])$$

gives the classification of vector bundles over $\mathbb{A}_k^1$ (and of course, when replacing $k[x]$ with an arbitrary PID, gives the usual structure theorem of finitely generated modules over PID).]

Proof. We prove this by induction. For $r = 1$, note that $GL_1(k[x]) = GL_1(k[x^{-1}]) = k^\times$, and $GL_1(k[x, x^{-1}]) = \{ux^n : u \in k^\times, n \in \mathbb{Z}\}$, thus take $Q = u^{-1}$.

Now consider $r > 1$. In the double quotient above, note that matrices are equivalent under row operations with elements of $k[x^{-1}]$ and column operations with elements of $k[x]$.

Multiply M by x^n , $n \in \mathbb{Z}_{\geq 0}$ such that $M' = x^n M$ has entries in $k[x]$. Note that by doing column operations on M' with elements of $k[x]$, we can sequentially eliminate entries of the first row, being left with the only nonzero entry as the g.c.d. in $m'_{1,1}$. Note that the determinant of M is a unit in $k[x, x^{-1}]$, hence it must be ux^s with $u \in k^\times$. Thus, we can get a matrix of the form

$$\begin{pmatrix} x^k & 0 & \dots & 0 \\ m'_{2,1} & m'_{2,2} & \dots & m'_{2,r} \\ \vdots & \vdots & \ddots & \vdots \\ m'_{r,1} & m'_{r,2} & \dots & m'_{r,r} \end{pmatrix}$$

which is equivalent to M' .

Consider the $(r-1) \times (r-1)$ minor of M' not containing $m'_{1,1}$. We know by induction that there exists P_{r-1} and Q_{r-1} such that

$$P_{r-1} M'_{r-1} Q_{r-1} = \begin{pmatrix} x^{k_2} & & & 0 \\ & x^{k_3} & & \\ & & \ddots & \\ 0 & & & x^{k_r} \end{pmatrix}$$

where $P_{r-1} \in GL_{r-1}(k[x])$ and $Q_{r-1} \in GL_{r-1}(k[x^{-1}])$. Note that because we multiplied M originally by a suitable power of x , $k_2, \dots, k_r \in \mathbb{Z}_{\geq 0}$. Thus, acting on M' gives P_{r-1} and Q_{r-1} gives

$$\begin{pmatrix} 1 & 0 \\ 0 & Q_{r-1} \end{pmatrix} M' \begin{pmatrix} 1 & 0 \\ 0 & P_{r-1} \end{pmatrix} = \begin{pmatrix} x^k & & & 0 \\ m'_{2,1} & x^{k_2} & & \\ m'_{3,1} & & x^{k_3} & \\ \vdots & 0 & & \ddots \\ m'_{r,1} & & & & x^{k_r} \end{pmatrix}$$

where $m'_{2,1}, \dots, m'_{r,1} \in k[x, x^{-1}]$. We can subtract $k[x^{-1}]$ -multiples of the first row from the 2nd thru r th row, hence we can force $m'_{2,1}, \dots, m'_{r,1}$ to be elements in $x^{k+1}k[x]$. Furthermore, subtracting $k[x]$ -multiples of the 2nd thru r th columns from the first column forces $m'_{2,i}$ to have degree less than k_i for all $i \in 2, \dots, r$. Thus, if $k \geq k_i$ for all i , we can eliminate each $m'_{2,1}, \dots, m'_{r,1}$ and get the diagonal matrix as needed.

Assume $k < k_i$ for some i . As noted earlier, we can find an element $c(x^{-1})$ of $k[x^{-1}]$ such that

$$m'_{i,1} - c(x^{-1})x^k = x^{k+1}d(x) \in x^{k+1}k[x].$$

Switch the i th row with the first row. Note that the gcd of the two elements in the new first row has degree strictly greater than k . As in the first step, we replace the first row, first column entry with the gcd and eliminate the other non-zero entry in the first row. We have a matrix now where $m'_{1,1}$ is a power of x which has strictly increased. However, the degree of $m'_{1,1}$ is bounded by the degree of the determinant of $x^n M$. Thus, consider the set of all matrices with only nonzero elements in the first column and diagonal that are equivalent to M' . Note that the determinant of M' is a unit times a nonnegative power of x , thus k is bounded above. Furthermore, since $k > 0$, we can find a matrix in this form which has maximal k , i.e. the degree of $m'_{1,1}$ cannot be increased. With this matrix, there cannot exist an element $c(x^{-1})$ such that $m'_{i,1} - c(x^{-1})x^k = x^{k+1}d(x) \in x^{k+1}k[x]$, otherwise we can do the above trick. Thus, $k \geq k_i$ for all i and we

can eliminate each $m'_{2,i}, \dots, m'_{r,i}$. Thus, we have found a matrix

$$M' \sim \begin{pmatrix} x^{k_1} & & 0 \\ & x^{k_2} & \\ & & \ddots \\ 0 & & & x^{k_r} \end{pmatrix} \implies M \sim \begin{pmatrix} x^{k_1-n} & & 0 \\ & x^{k_2-n} & \\ & & \ddots \\ 0 & & & x^{k_r-n} \end{pmatrix}.$$

We can rearrange such that $n_1 = k_1 - n \geq \dots \geq n_r = k_r - n$. □