Titel: The Polyakov measure and the modular height function.
Autor: Bas Edixhoven
Adresse: Mathematisch Instituut, Budapestln. 6, 3584 CD Utrecht, Nederland.

1. Rough sketch. (this section is very imprecise).

The Polyakov measure is a real volume form on $M_g(\mathbb{C})$, $(g \in \mathbb{Z}_{\geq 1})$, the complex moduli space of curves of genus g. This measure comes from string theory, see [Bo]. $M_g(\mathbb{C})$ = complex variety associated to M_g := modulispace $/\mathbb{Z}$ of stable curves of genus g over arbitrary base schemes. Let $P \in M_g(K)$, K a numberfield. This gives a K-vectorspace $T_{M_g,P}$ and its $\mathcal{O}_K$-submodule of integers ($\mathcal{O}_K$:= ring of integers of K).

This $\mathcal{O}_K$-module is a lattice in $\prod_{\tau \in K_\infty} T_{M_g,P_\tau}$ ← has a volume form, namely the Polyakov measure on each factor.

We will compute the volume of this lattice.

2. Description of the Polyakov measure.

Let $g \in \mathbb{Z}_{\geq 1}$, let $\mathcal{M}$ be the moduli stack over $\mathbb{Z}$ of stable curves of genus g and let $\mathcal{C} \xrightarrow{\pi} \mathcal{M}$ be the universal curve over $\mathcal{M}$. In the case $g=1$ we will always assume the presence of a given section "0", in order to eliminate infinitesimal automorphisms. We recall that sheaves on $\mathcal{M}$ are defined to be sheaves on the etale site $\mathcal{M}_{et}$ of $\mathcal{M}$ (c.f. [De-Mu], 4.10). This implies that one can define a sheaf on $\mathcal{M}$ by giving for every etale morphism $S \to \mathcal{M}$ (S is a scheme) a sheaf on S, together with compatible isomorphisms between different pullbacks. In this way we get $\Omega^1_{\mathcal{M}/\mathbb{Z}}$, $T^1_{\mathcal{M}/\mathbb{Z}}$, $(R^i\pi_*)(\omega_{\mathcal{C}/\mathcal{M}}^{\otimes n})$, $\mathcal{O}_{\mathcal{M}}(\Delta)$ and so on. Note that the sheaves $(R^i\pi_*)(\omega_{\mathcal{C}/\mathcal{M}}^{\otimes n})$ commute with all base changes. On $\mathcal{M}$ we have an isomorphism : $(\det\pi_*)\omega_{\mathcal{C}/\mathcal{M}}^{\otimes 2} \xrightarrow{\sim} (\det \Omega^1_{\mathcal{M}/\mathbb{Z}})(\Delta)$, induced by the Kodaira-Spencer map. (see [Ha-Mu] §2 p.49, [Sz 1] exp.3 p.47, 59, [De-Mu] p.82.)

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Mumford has shown the existence of an isomorphism: $((\det \pi_*)\omega_{\mathcal{C}/\mathcal{M}}^{\otimes 2})(\Delta) \xrightarrow{\sim} (\det \pi_* \omega_{\mathcal{C}/\mathcal{M}})^{\otimes 13}$ for $g \geq 2$ (c.f. [Fa 1] §6, [Mu 1] thm. 5.10, see also [Mu 2]). Explicit expressions for some constant multiple of this isomorphism over $\mathbb{C}$ can be found in [Be-Ma], see also [Fa 1] §6. Combining these two isomorphisms we get:

$$(\det \pi_* \omega_{\mathcal{C}/\mathcal{M}})^{\otimes 13} \xrightarrow{\sim} (\det \Omega^1_{\mathcal{M}/\mathbb{Z}})(2\Delta) \quad , \text{ if } g \geq 2, \qquad (1).$$

For $g=1$ the exponent 13 has to be replaced by 14, this is due to the nontriviality of $(R^1\pi_*)(\omega_{\mathcal{C}/\mathcal{M}}^{\otimes 2})$. In this case the Mumford isomorphism is described by the discriminant, i.e. the cusp form of weight 12. Note that all these isomorphisms (esp. (1)) are unique upto sign.
The line bundle $\det \pi_* \omega_{\mathcal{C}/\mathcal{M}}$ on $\mathcal{M}$ has a hermitian metric at infinity with a singularity along Δ_0: for $C \longrightarrow \mathrm{Spec}(\mathbb{C})$ and $\omega_1, \ldots, \omega_g, \eta_1, \ldots, \eta_g$ global sections of $\omega_{C/\mathbb{C}}$, $\langle \omega_1 \wedge \ldots \wedge \omega_g \mid \eta_1 \wedge \ldots \wedge \eta_g \rangle := \int_{\mathrm{Jac}(C)} (-1)^{\binom{g}{2}} \cdot (\frac{i}{2})^g \omega_1 \wedge \ldots \wedge \omega_g \wedge \overline{\eta}_1 \wedge \ldots \overline{\eta}_g$ $= \det(\int_C \frac{i}{2} \cdot \omega_i \wedge \overline{\eta}_j)$. Taking the 13 th power gives a metric on $(\det \pi_* \omega_{\mathcal{C}/\mathcal{M}})^{\otimes 13}$, applying isomorphism (1) gives one on $(\det \Omega^1_{\mathcal{M}/\mathbb{Z}})(2\Delta)$. Finally by duality we get a hermitian metric on $\det T^1_{\mathcal{M}/\mathbb{Z}}$, having a double pole at the boundary. The formulas (4) and (5) of [Be-Ma] imply that this metric on $\det T^1_{\mathcal{M}/\mathbb{Z}}$ is some constant (depending only on g) times the Polyakov measure. We normalize the Polyakov measure by taking this constant equal to 1.

3. Computation of the volume.

In this section we work with arithmetic surfaces as in [Fa 1], [Sz 2] exp II §6. Let K be a numberfield, $\mathcal{O}_K$ its ring of integers, $S := \mathrm{Spec}(\mathcal{O}_K)$, $C \xrightarrow{\pi} S$ a stable curve of genus $g \geq 1$ with smooth generic fibre.

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Let $P \in \mathcal{M}(S)$ be the S-valued point of $\mathcal{M}$ corresponding to $C \xrightarrow{\pi} S$:

$$\begin{tikzcd} C \arrow[r] \arrow[d, "\pi"'] \arrow[dr, phantom, "\square"] & \mathcal{C} \arrow[d, "\pi"] \\ S \arrow[r, "P"] & \mathcal{M} \end{tikzcd}$$

On S we get the locally free $\mathcal{O}_S$-module $P^* T^1_{\mathcal{M}/\mathbb{Z}}$. According to section 2 the Polyakov measure gives hermitian metrics at infinity on $\det P^* T^1_{\mathcal{M}/\mathbb{Z}}$. The $\mathcal{O}_K$-module $P^* T^1_{\mathcal{M}/\mathbb{Z}}$ is a lattice in $(P^* T^1_{\mathcal{M}/\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{R} = \prod_{\sigma \in S_\infty} P_\sigma^* T^1_{\mathcal{M}/\mathbb{Z}}$ and the metrics on $\det P^* T^1_{\mathcal{M}/\mathbb{Z}}$ give a volume form on the last $\mathbb{R}$-vector space. Let M be $P^* T^1_{\mathcal{M}/\mathbb{Z}}$. We want to compute the volume of $(M \otimes_{\mathbb{Z}} \mathbb{R})/M$. The Riemann-Roch thm. on S reads: $\chi(M) = \deg M + \operatorname{rank}(M) \cdot \chi(\mathcal{O}_S)$ ([Sz 2] exp II, (6.12.6), (6.12.7)), where $\chi(M) = -\log \operatorname{vol}(M \otimes_{\mathbb{Z}} \mathbb{R}/M)$, $\deg M = \deg \det M$.

Now we can write: $\log \operatorname{vol}(M \otimes_{\mathbb{Z}} \mathbb{R}/M) = -\deg M - (3g-3)\log(2^{r_2} \cdot D^{-1/2})$,

$-\deg M = -\deg P^* \det T^1_{\mathcal{M}/\mathbb{Z}} = (\text{by isom.}(1)) = \deg P^*\big((\det \pi_* \omega_{\mathcal{C}/\mathcal{M}})^{\otimes 13}(-2\Delta)\big) =$
$= 13 \cdot \deg(\det \pi_* \omega_{C/S}) - 2 \cdot \deg \mathcal{O}_S(\Delta)$.

We recall that the modular height of C_K is defined as:

$h(C_K) = \frac{1}{[K:\mathbb{Q}]} \cdot \deg(\det \pi_* \omega_{C/S})$, (c.f. [Fa 2] §3 first def., see also §2 remarks).

Since $\mathcal{O}_S(\Delta)$ has trivial metrics, $\deg \mathcal{O}_S(\Delta) = \log \#(\mathcal{O}_S(\Delta)/\mathcal{O}_S \cdot 1) = \log(N_{K/\mathbb{Q}} \Delta_{C/S})$.

For $\tilde{\mathcal{C}}/S$ the minimal model of C_K we can write:

$\deg \mathcal{O}_S(\Delta) = \sum_{x \in S_f} (\#\operatorname{Sing} \tilde{\mathcal{C}}_{\bar{x}}) \cdot \log(\# k(x))$. Putting everything together:

$$\frac{1}{[K:\mathbb{Q}]} \cdot \log \operatorname{vol}\Big(\big(\prod_{\sigma \in S_\infty} P_\sigma^* T^1_{\mathcal{M}/\mathbb{Z}}\big) / P^* T^1_{\mathcal{M}/\mathbb{Z}}\Big) = 13 \cdot h(C_K) - \frac{2}{[K:\mathbb{Q}]} \cdot \sum_{x \in S_f} (\#\operatorname{Sing} \tilde{\mathcal{C}}_{\bar{x}}) \cdot \log(\# k(x))$$

$$+ \frac{3g-3}{[K:\mathbb{Q}]} \cdot \log(2^{-r_2} D^{1/2}), \quad \text{for } g \geq 2, \ \tilde{\mathcal{C}}/S \text{ minimal regular.}$$

For $g=1$ the 13 should be replaced by 14 and $3 \cdot g - 3$ by 1.

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