Bas Edixhoven, Luminy, 2011, 05, 19

(Counter)example to semi-ab. relative Manin-Mumford. 1

Ref. Daniel Bertrand, Special points and Poincaré bi-extensions, arxiv, april 2011. (+ Appendix by me). (Too bad that Richard Pink and Daniel Bertrand are not here).

Context: extend Masser-Zannier result to semi-abelian case: $(P,Q)\left(\begin{array}{c} E\times_S E \\ \downarrow \\ S = \text{curve}/\mathbb{C}\end{array}\right.$ P, Q $\mathbb{Z}$-lin. indep. Then $\{s\in S \mid P(s)$ and $Q(s)$ torsion$\}$ is finite.

work in progress by M-Z-B-P ~~Anand~~ Anand Pillay..

There was a problem.. B found a counterexample.

1. Present the counterexample
2. Show that it is an example for Pink's conjecture.
   So: the proper context for S.A.R.M.M. is MHS.

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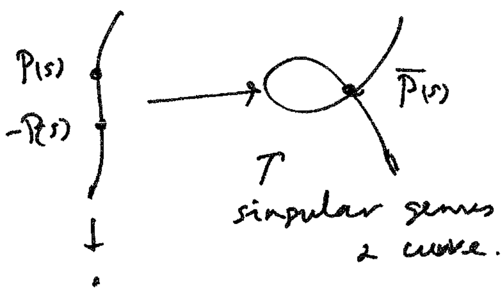
1. Let $E/\mathbb{C}$ be an ell. curve with CM.

$S := E$ – fin. many points to be determined equaliser

$$P := \Delta\left(\begin{array}{c} E_S = E\times S \\ \downarrow \\ S\end{array}\right. \qquad S \underset{-P}{\overset{P}{\rightrightarrows}} E_S \xrightarrow{\text{quot}} C \; : \text{glue}, \forall s\in S, \; P(s)\sim -P(s)$$

$S \xrightarrow{\text{id}} S$, $E_S \to S$, $C \to S$



$J := \mathrm{Pic}^{\circ}_{C/S}$

represents $(T\to S) \mapsto \mathrm{Pic}^{\circ}(C_T)/\mathrm{Pic}(T)$

($C\to S$ has a section: $\overline{P}$)

$\forall s\in S$: $J_s = \{\mathrm{Div}^{\circ}(C_s)/\text{prime divisors}$ – $\mathrm{div}(f)$, $f\in\mathbb{C}(E)^{\times}$, f reg. at $P(s)$ and $-P(s)$, $0\neq f(P(s)) = f(-P(s))$

D on C_s support away from $\overline{P}(s)$, degree 0,

Let $a\in\mathrm{End}(E)$ s.t. $\overline{a}-a\neq 0$. (Bertrand calls this a Ribet section)

Put $\beta_a := \left[a^*(P-(-P)) - a_*(P-(-P))\right] \in J(S)$.

(remove the $s\in S$ s.t. not disjoint from $P(s)$, $-P(s)$.)

$\mathbb{G}_{m,S} \hookrightarrow J \overset{\text{quot}^*}{\twoheadrightarrow} E_S$

so, for $P(s)$ not torsion, the ext. is not split. even $2P(s)\in E$

but, as group scheme,

Thm: 1. $\overline{\left(\bigcup_{n\in\mathbb{Z}} n\cdot\beta_a(S)\right)}^{zar} = J$

2. $\forall s\in S$ s.t. $P(s)\in E$ torsion of order n prime to $2\cdot\deg(a^2-1)\cdot(\bar{a}-a)$)
$\beta_a(s) \in J_s$ is of order $\mid n^2$, and $=n^2$ does occur.

1. ~~[illegible]~~ quot $\circ\beta_a = \left[a^*(P-(-P)) - a_*(P-(-P))\right] \in E(S)$ $(\bar{a}-a\neq 0!)$

$$\| $$

$2\bar{a}P_{an} - 2aP = 2(\bar{a}-a)\cdot P$, not torsion.
+ lack of subgroups in J. (take e.g. $s\in S$ s.t. $P(s)$ not torsion)

2. Let s, n be as stated. ~~[illegible]~~

$\exists f: n\cdot(P-(-P)) = \mathrm{div}(f)$ on E
$\cap$
$\mathbb{C}(E)^\times$

$\mathrm{div}(f\circ a) = \mathrm{div}(a^*f) = a^*\mathrm{div}\, f = a^*(n\cdot P - n\cdot(-P))$

$\mathrm{div}(\mathrm{Norm}_a f) = \mathrm{div}(a_*f) = a_*(n\cdot P - n\cdot(-P))$

$g_a := \dfrac{f\circ a}{\mathrm{Norm}_a f} \in \mathbb{C}(E)^\times$. Then $\mathrm{div}(g_a) = a^*(n\cdot P - n\cdot(-P)) - a_*(n\cdot P - n\cdot(-P))$

So: $\cancel{n}\,\beta_a(s) = \dfrac{g_a(P(s))}{g_a(-P(s))} \in \mathbb{C}^\times$

progress since Pisa partly due to Lenny Taelman and Daniel B.

Now comes a nice argument: Weil reciprocity

$$\left(\frac{g_a(P(s))}{g_a(-P(s))}\right)^n = g_a(\mathrm{div}(f)) = f(\mathrm{div}(g_a)) =$$

$$= f\left(\mathrm{div}(f\circ a) - \mathrm{div}(\mathrm{Norm}_a f)\right)$$

$$= \frac{f(\mathrm{div}(f\circ a))}{f(\mathrm{div}\,\mathrm{Norm}_a f)} = \frac{(f\circ a)(\mathrm{div}\, f)}{f(a_*\mathrm{div}\, f)} = \frac{f(a_*\mathrm{div}\, f)}{f(a_*\mathrm{div}\, f)} = 1.$$

To see "$=n$" occurs: Weil pairing.

2. Evidence for Pink's conjecture : $\beta_a(S)$ is special.

(in MHS-sense. / M-S-V. sense.)

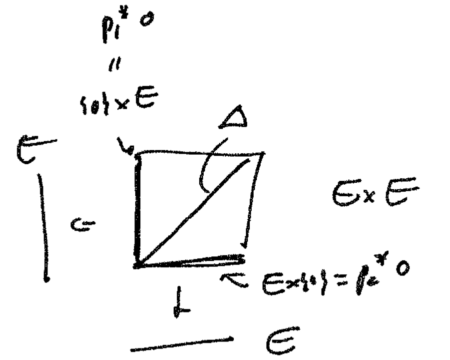
Mixed Shimura variety:

Let $\mathbb{E}$ be the Legendre family.
$\mathbb{E}$
$\downarrow$
Y

dim: 4 3

The M-S-V is: $\mathcal{P} \longrightarrow \mathbb{E}\times_Y \mathbb{E}$

Poincaré bi-extension.



$\mathcal{P} = \underline{\mathrm{Isom}}(\mathcal{O}, \mathcal{O}(\Delta - p_1^*0 - p_2^*0$

$\mathbb{C}$-analytic description:

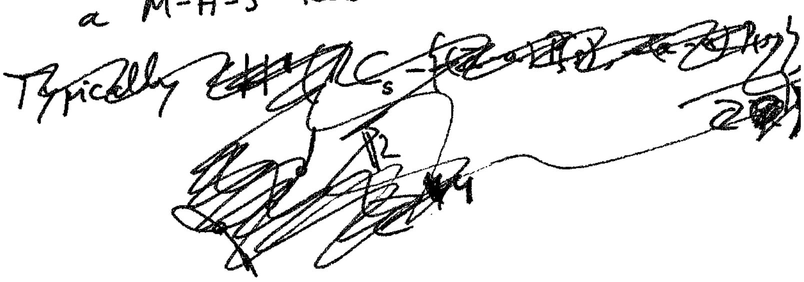
$(\mathbb{Z}^2 \times \mathbb{Z}^2) \rtimes \Gamma(2)$ ~~$SL_2(\mathbb{Z})$~~ $\circlearrowright$ $\mathbb{C}\times\mathbb{C}\times\mathbb{H}$ gives $\mathbb{E}\times_Y\mathbb{E}$,

st. rep. ↑ dual of st. repr. ↑

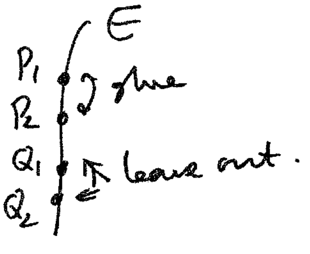
For $\mathcal{P}$: $\mathbb{C}\times\mathbb{C}\times\mathbb{C}\times\mathbb{H}$ $\circlearrowleft$ $\begin{pmatrix} 1 & x_1 & x_2 & z \\ 0 & a & b & y_1 \\ 0 & c & d & y_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ no z in the abelian case!

Mixed-H-S, M-T-groups.

$\beta_a \left(\begin{smallmatrix} \alpha J \\ \downarrow \\ S \end{smallmatrix} \right.$ gives $\mathbb{Z}_S \xrightarrow{1 \mapsto \beta_a} J$, a 1-motive. These have a M-H-S realisations at $s\in S$.



$H^1(C, \mathbb{Z}) =: V$



Weight filtration: ~~W_2~~

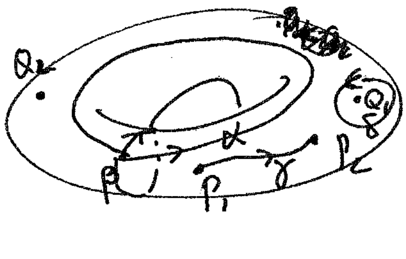
$W_0 =$ all

$W_{-1} = \mathbb{Z}\cdot\alpha + \mathbb{Z}\cdot\beta + \mathbb{Z}\cdot\delta$

$W_{-2} = \mathbb{Z}\cdot\delta$

$W_{-3} = 0.$

$\mathbb{Z}$-basis $H_1(C, \mathbb{Z}) = V^\vee$.



Graded pieces have a H.S.

exchange $P_1 \leftrightarrow Q_1$, $P_2 \leftrightarrow Q_2$: dual.

$$\begin{tikzcd} V \arrow[r, "\sim"] \arrow[loop below, "g"] & V^\vee \otimes \det \mathrm{gr}_1 V \arrow[loop below, "(g^\vee)^{-1}"] \end{tikzcd}$$

M-T of V: weight filtration gives upper triangular in blocks. $\mathrm{gr}_0 V \cong \mathbb{Q}(0)$, $\mathrm{gr}_1 V = H^1(\text{ell. curve})$ so $\det \mathrm{gr}_1 V \cong \mathbb{Q}(1) \cong \mathrm{gr}_2(V)$; $j^t = +j$.

see below.

$$\begin{pmatrix} \cdot & \cdot & 1 \\ \cdot & j & \cdot \\ 1 & \cdot & \cdot \end{pmatrix}\begin{pmatrix} 1 & x & z \\ \cdot & g & y \\ \cdot & \cdot & \det g \end{pmatrix} = \begin{pmatrix} 1 & x & z \\ \cdot & g & y \\ \cdot & \cdot & \det g \end{pmatrix}^{t,-1} \cdot \begin{pmatrix} & & 1 \\ & j & \\ 1 & & \end{pmatrix} \cdot \det g$$

On Lie algebra:

$g = 1 + \varepsilon \cdot a$

$$\begin{pmatrix} \cdot & \cdot & 1 \\ \cdot & j & \cdot \\ 1 & \cdot & \cdot \end{pmatrix}\cdot\begin{pmatrix} 0 & x & z \\ \cdot & a & y \\ \cdot & \cdot & \mathrm{tr}\, a \end{pmatrix} = -\begin{pmatrix} 0 & \cdot & \cdot \\ x^t & a^t & \cdot \\ z^t & y^t & \mathrm{tr}\, a \end{pmatrix}\cdot\begin{pmatrix} \cdot & \cdot & 1 \\ \cdot & j & \cdot \\ 1 & \cdot & \cdot \end{pmatrix} + \mathrm{tr}\, a \cdot \begin{pmatrix} & & 1 \\ & j & \\ 1 & & \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & \mathrm{tr}\, a \\ 0 & ja & jy \\ 0 & x & z \end{pmatrix} \qquad\qquad \begin{pmatrix} 0 & 0 & \mathrm{tr}\, a \\ 0 & (\mathrm{tr}\, a - a^t) j & -x^t \\ 0 & -y^t j & -z \end{pmatrix}$$

Conclusion: $\begin{cases} x = -y^t j \\ z = 0 \end{cases}$

So, on $\beta_a(S)$ the generic MT group is of dimension $\underbrace{2}_{g} + \underbrace{2}_{y} = 4$. At special points on $\beta_a(S)$ ~~this~~ it is $\underbrace{2}_{g}$.

And if we take another section of $J \to S$, for example given by the divisor $\mathrm{tr}_{Q*}\left(a^*(P - (-P)) - a_*(P - (-P))\right)$ for some $Q \in E(S)$, then the gen. M-T will be of dim. $5 = 2 + 2 + 1$, 1 for $z \neq 0$.

$j^t = +j$. Let $H = H^1(E, \mathbb{Z})$. Then we have the princ. polarisation $\psi : H \to H^\vee$ and $\alpha : \bar{a} - a : H \to H$ both antisymmetric:

$$\begin{tikzcd} \psi^\vee : H^{\vee\vee} \arrow[r] \arrow[d, equal] & H^\vee \\ H & \end{tikzcd} \qquad \psi^\vee = -\psi, \qquad \begin{tikzcd} H^\vee \arrow[r, "\alpha^\vee"] \arrow[d, "\psi^{-1}"] & H^\vee \\ H \arrow[r, "-\alpha"] \arrow[u, "\psi", shift left] & H \arrow[u, "\psi^{-1}"'] \end{tikzcd}$$

Then: $j = \psi \circ \alpha$,

$j^\vee = (\psi \circ \alpha)^\vee = \alpha^\vee \psi^\vee = \psi(-\alpha)\psi^{-1}(-\psi)$

$= -(-\psi\alpha) = \psi\alpha = j.$

Why is there no trouble in the ~~ab~~ relative abelian case??

There the mixed Shimura group or say generic M-T group ~~of the~~ is first $\mathbb{Q}^{2g} \rtimes GSp_{2g}$ for the univ. fam. $\mathcal{A}$ over $A_{g,1}$.

Let S be a special subv. of $\mathcal{A}$ and $\overline{S}$ its image in $A_{g,1}$. Then S is a special subv. of $\mathcal{A}_{\overline{S}}$, that surjects to $\overline{S}$. The generic M-T group G of S is a subgroup of $\mathbb{Q}^{2g} \rtimes \overline{G}$, where G is the generic M-T on $\overline{S}$.

We have the exact sequence $G \cap \mathbb{Q}^{2g} \hookrightarrow G \twoheadrightarrow \overline{G}$.

As $\overline{G}$ is reductive over $\mathbb{Q}$, $H^2(\overline{G}, G \cap \mathbb{Q}^{2g}) = 0$ (the cat. of repr. of $\overline{G}$ is semi-simple), hence this extension is split.

As $H^1(\overline{G}, G \cap \mathbb{Q}^{2g}) = 0$, there is a unique splitting up to conjugation by elmts of $G \cap \mathbb{Q}^{2g}$.

So we conclude that S is a special subvariety given by the subgroup $(G \cap \mathbb{Q}^{2g}) \rtimes \overline{G}$, and that all special subv. of $\mathcal{A}_{\overline{S}}$ that surject to $\overline{S}$ are given by sub-$\overline{G}$-reps of $\mathbb{Q}^{2g}$. So, in the relative abelian case, only subgroupschemes matter for special subvarieties.

Here is an example that shows that we have used that $\overline{G}$ is reductive. We have $\mathbb{G}_a \hookrightarrow \left\{ \begin{pmatrix} 1 & a & c \\ & 1 & b \\ & & 1 \end{pmatrix} \right\} \twoheadrightarrow \mathbb{G}_a^2$, a non-split extension.

$$\begin{tikzcd}
\mathbb{G}_a \arrow[r, hook] \arrow[d, hook] & \left\{ \begin{pmatrix} 1 & a & c \\ & 1 & b \\ & & 1 \end{pmatrix} \right\} \arrow[r, two heads] \arrow[d, hook] & \mathbb{G}_a^2 \arrow[d, hook] \\
\mathbb{G}_a^4 \arrow[r, hook] & \mathbb{G}_a^4 \rtimes GL_4 \arrow[r, two heads] \arrow[d, hook] & GL_4 \\
& GL_5 &
\end{tikzcd}$$

$$\begin{pmatrix} 1 & a & c \\ & 1 & b \\ & & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & b & & & \\ & 1 & & & \\ & & 1 & a & c \\ & & & 1 & b \\ & & & & 1 \end{pmatrix}$$

$$\left(\begin{array}{c|c} 4\times 4 & \\ \hline & 1\times 1 \end{array} \right)$$