Counting quickly the vectors with integer coordinates and with a given length

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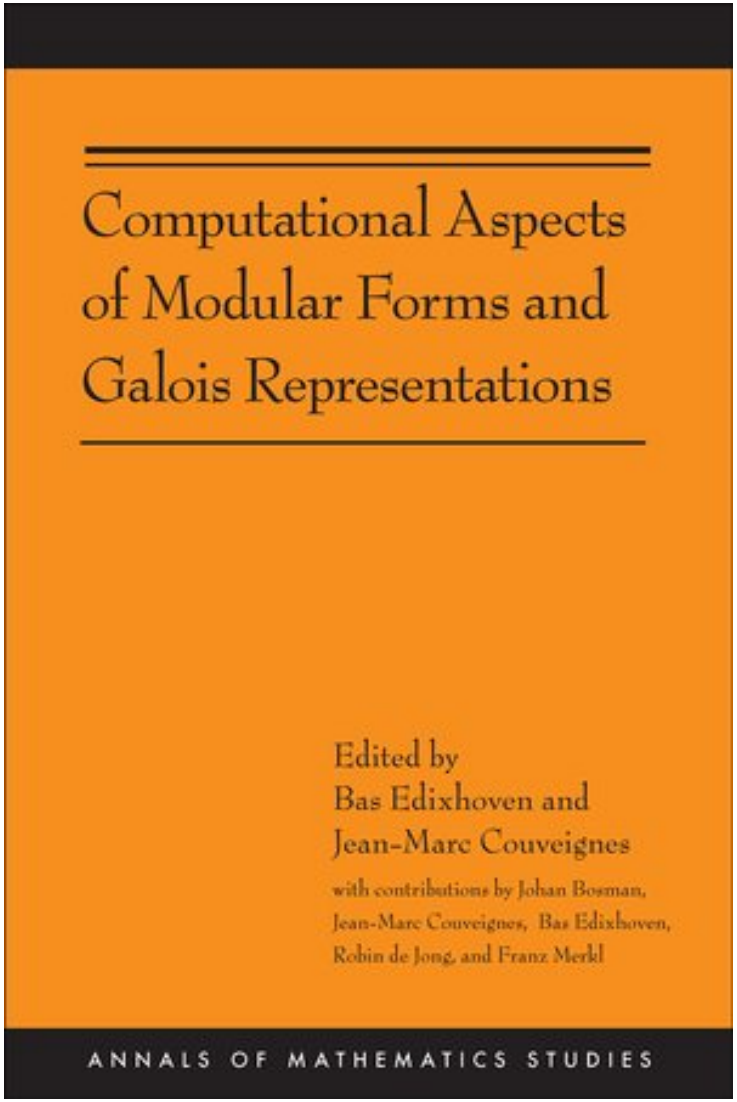
Joint work with Jean-Marc Couveignes, Robin de Jong,
Johan Bosman, Franz Merkl, Peter Bruin, Ila Varma

The commercial



The commercial, continued

Modular forms are tremendously important in various areas of mathematics, from number theory and algebraic geometry to combinatorics and lattices. Their Fourier coefficients, with Ramanujan's tau-function as a typical example, have deep arithmetic significance.

Prior to this book, the fastest known algorithms for computing these Fourier coefficients took exponential time, except in some special cases. The case of elliptic curves (Schoof's algorithm) was at the birth of elliptic curve cryptography around 1985.

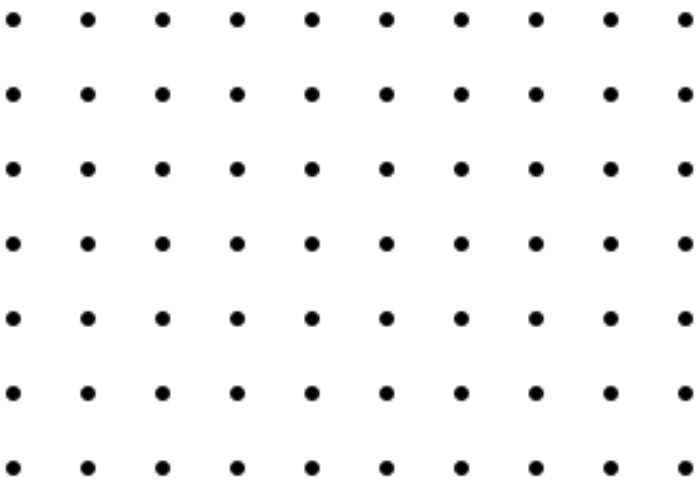
This book gives an algorithm for computing coefficients of modular forms of level one in polynomial time. For example, Ramanujan's tau of a prime number p can be computed in time bounded by a fixed power of the logarithm of p. . .

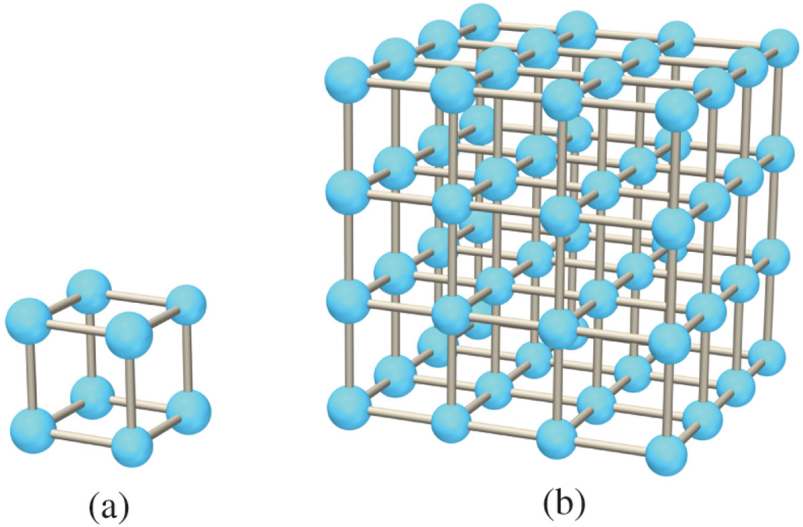
Back to mathematics: sums of squares

To illustrate the progress made in the book and Peter Bruin's PhD thesis (2010), we consider the problem of computing quickly, for d and n in $\mathbb{Z}$:

$$r_d(n) := \#\{x \in \mathbb{Z}^d : x_1^2 + \cdots + x_d^2 = n\}.$$

Geometric interpretation (Pythagoras): count the number of lattice points in $\mathbb{Z}^d$ at a given distance $\sqrt{n}$ from the origin.





Sums of squares: some examples

$r_2(3) = 0.$

$r_2(5) = 8$:

$$5 = (\pm 2)^2 + (\pm 1)^2$$
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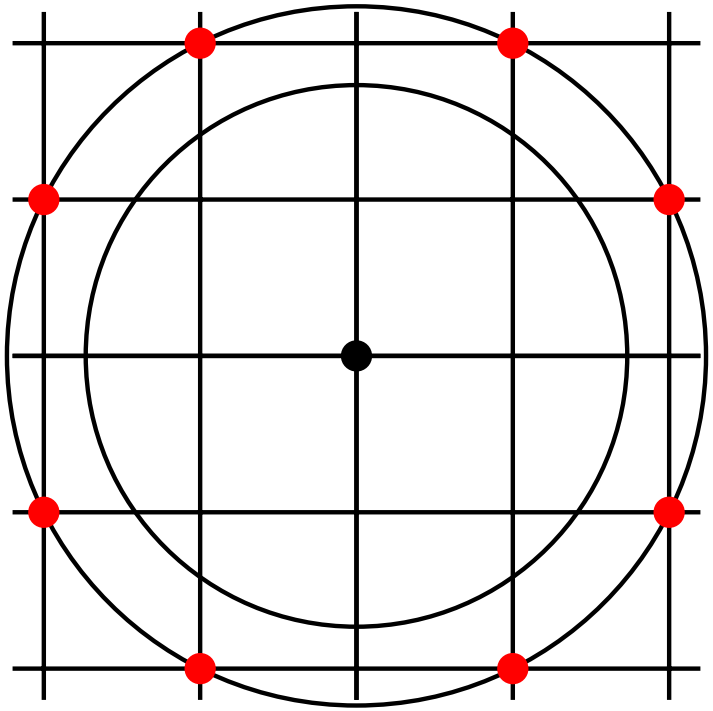


Dimension two: Diophantus

DIOPHANTI
ALEXANDRINI
ARITHMETICORVM
LIBRI SEX.
ET DE NVMERIS MVLTANGVLIS
LIBER VNVS.

Nunc primùm Græcè & Latinè editi, atque absolutissimis Commentariis illustrati.

AVCTORE CLAVDIO GASPARE BACHETO
MEZIRIACO SEBVSIANO, V. C.

LVTETIAE PARISIORVM,
Sumptibus SEBASTIANI CRAMOISY, via Iacobæa, sub Ciconiis.
M. DC. XXI.
CVM PRIVILEGIO REGIS.

Diophantus of Alexandria ($\approx$ 3rd century):

$$(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2.$$

Dimension two: Fermat

Pierre de Fermat (lawyer, Toulouse, 17th century), for $n \geq 1$: $r_2(n) \neq 0$ if and only if every prime factor of n that is 3 modulo 4, occurs an even number of times in the factorisation of n.

Dimensions 2 and 3: Legendre, Gauss

Adrien-Marie Legendre (1798) gave a formula for $r_2(2^a m^2)$.

Carl Friedrich Gauss (1801) gave a general formula for $r_2(n)$, and a formula for $r_3(n)$ that shows that the $r_d(n)$ for odd d are more complicated (involve class numbers).

For $n > 1$ squarefree, 1 or 2 mod 4, $r_3(n) = 12 \cdot h(\mathbb{Z}[\sqrt{-n}])$.

Higher even dimensions: Jacobi

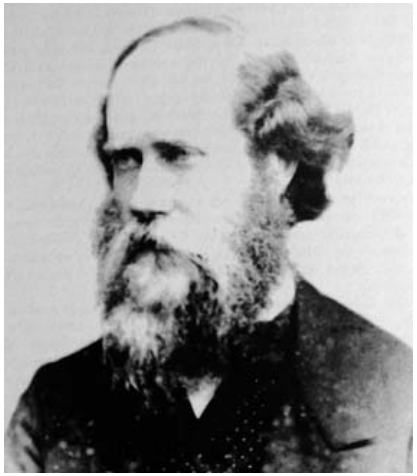
Carl Gustav Jacob Jacobi (1829) proved for $n \geq 1$:

$$r_2(n) = 4 \sum_{d|n} \chi(d), \quad \text{with} \quad \chi(d) = \begin{cases} 0 \text{ if } d \text{ is even,} \\ 1 \text{ if } d = 4r+1, \\ -1 \text{ if } d = 4r+3, \end{cases}$$

and:

$$r_4(n) = 8 \sum_{2 \nmid d | n} d + 16 \sum_{2 \nmid d | (n/2)} d.$$

Eisenstein, Smith



It follows from work of Jacobi, Ferdinand Eisenstein and Henry Smith that:

$$r_6(n) = 16 \sum_{d|n} \chi(n/d) d^2 - 4 \sum_{d|n} \chi(d) d^2,$$

$$r_8(n) = 16 \sum_{d|n} d^3 - 32 \sum_{d|(n/2)} d^3 + 256 \sum_{d|(n/4)} d^3.$$

Dimension 10: Liouville

For $d = 10$ Joseph Liouville (1865) found a formula in terms of the Gaussian integers $d = a + bi$ with a and b in $\mathbb{Z}$:

$$r_{10}(n) = \frac{4}{5}\sum_{d|n}\chi(d)d^4 + \frac{64}{5}\sum_{d|n}\chi(n/d)d^4 + \frac{8}{5}\sum_{d\in\mathbb{Z}[i],\,|d|^2=n} d^4.$$

Dimension 12: Glaisher, Ramanujan

James Whitbread Lee Glaisher, reinterpreted by Srinivasa Ramanujan in 1916, proved that:

$$r_{12}(n) = 8 \sum_{d|n} d^5 - 512 \sum_{d|(n/4)} d^5 + 16 a_n$$

where:

$$\sum_{n \geq 1} a_n q^n = q \prod_{m \geq 1} (1 - q^{2m})^{12} \quad \text{in } \mathbb{Z}[[q]].$$

Note: unlike for $d \leq 10$, this formula does *not* lead (directly) to computation of $r_{12}(n)$ in time polynomial in $\log n$, if n is given with its factorisation into primes.

$r_d(n)$ for all even d

Negative. Ila Varma (masters thesis, Leiden, June 2010): there is no even $d > 10$ for which there is an "elementary" formula for $r_d(n)$.

Positive (book and Peter Bruin's PhD thesis). For every even d one can compute $r_d(n)$ in time polynomial in $\log n$, if $n \in \mathbb{N}$ is given with its factorisation into primes.

Note: for $n = pq$ with p and q distinct odd primes:

$$r_4(n) = 8(1 + p + q + n).$$

Conclusion. From an algorithmic perspective this classical problem is now solved for *all* even d. The question for *formulas* has a negative answer, but for *computing* that negative answer does not matter and we now have a *positive* answer.

Explanation: generating series

It is more than time to explain what is going on behind all these formulas. Generating series:

$$\theta_d := \sum_{x \in \mathbb{Z}^d} q^{x_1^2 + \cdots + x_d^2} = \sum_{n \geq 0} r_d(n) q^n \quad \text{in } \mathbb{Z}[[q]].$$

Let $\theta := \theta_1$ (Jacobi theta function at $z = 0$). Then:

$$\theta^d = \left(\sum_{x_1 \in \mathbb{Z}} q^{x_1^2} \right) \cdots \left(\sum_{x_d \in \mathbb{Z}} q^{x_d^2} \right) = \sum_{x \in \mathbb{Z}^d} q^{x_1^2 + \cdots + x_d^2} = \theta_d.$$

Compute θ^d in $\mathbb{Z}[[q]]/(q^{n+1})$: gives $r_d(n)$, but takes time at least linear in nd.

Theta functions are modular forms

Key idea: $q\colon \mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\} \to \mathbb{C}, \quad z \mapsto e^{2\pi i z}$.

Then $\theta_d\colon \mathbb{H} \to \mathbb{C}$, and for $z \in \mathbb{H}$: $\theta_d(z+1) = \theta_d(z)$, and Jacobi proved (Poisson summation formula):

$$\theta_d(-1/4z) = (2z/i)^{d/2}\theta_d(z).$$

This implies: θ_d is in the $\mathbb{C}$-vector space $M_{d/2}(\Gamma_1(4))$ of modular forms of weight $d/2$ on the subgroup $\Gamma_1(4)$ of $\mathrm{SL}_2(\mathbb{Z})$. Assume from now on that d is even. Then $k = d/2$ is in $\mathbb{Z}$.

The $M_k(\Gamma_1(4))$ are finite dimensional. This causes many identities.

For $0 \leq k \leq 4$, $M_k(\Gamma_1(4))$ is generated by Eisenstein series, hence the formulas for $r_d(n)$ for $d \leq 8$. For $d = 10$: also a Hecke character. Ila Varma: for $d > 10$, θ_d is not linear combination of Eisenstein and Hecke.

Complex analytic geometry

Let E^{k-2} be the quotient of $\mathbb{C}^{k-2} \times \mathbb{H}$ by an action of $\mathbb{Z}^{2(k-2)} \rtimes \Gamma$:

$$(x, z) \mapsto \left(x + n_1 + n_2 z, \frac{az + b}{cz + d} \right)$$

For $f \in M_k(\Gamma)$, $f\, dx_1 \cdots dx_{k-2} dz$ is a $\mathbb{Z}^{2(k-2)} \rtimes \Gamma$-invariant and closed holomorphic $(k-1)$-form. $M_k(\Gamma)$ is a subspace of the de Rham cohomology of E^{k-2}.

Via de Rham's comparison theorem, $M_k(\Gamma)$ is a piece of $H^{k-1}(E^{k-2}, \mathbb{C})$, Betti cohomology, defined topologically: combinatorics of a cell decomposition of E^{k-2}.

The coefficients $a_n(f)$ modular forms $f = \sum_{n \geq 0} a_n(f) q^n$ are closely related to Hecke operators T_n coming from the $\mathrm{GL}_2(\mathbb{Q})^+$-action on $\mathbb{H}$.

Algebraic geometry

To get further, $\mathrm{SL}_2(\mathbb{Z})$ does not suffice, we need *Galois symmetry*. In fact, E^{k-2} is an algebraic variety, defined over $\mathbb{Q}$.

Grothendieck: for $m \in \mathbb{Z}_{>0}$, $H^{k-1}(E^{k-2}, \mathbb{Z}/m\mathbb{Z})$ is defined algebraically, as étale cohomology, hence $\mathrm{Aut}(\mathbb{C})$ acts on it.

Simple example. Let $a_1, \ldots, a_n$ be the roots of a polynomial f in $\mathbb{Q}[x]$: then each σ in $\mathrm{Aut}(\mathbb{C})$ permutes the a_i.

$$H^1(\mathbb{C} - \{a_1, \ldots, a_n\}, \mathbb{Z}/m\mathbb{Z}) = \frac{\mathbb{C}[x, ((x-a_1)\cdots(x-a_n))^{-1}]^\times}{m\text{th powers}}$$
$$= (\mathbb{Z}/m\mathbb{Z})^{\{a_1,\ldots,a_n\}}$$

$$\sigma \in \mathrm{Aut}(\mathbb{C}) \colon (x-a_1)^{e_1}\cdots(x-a_n)^{e_n} \mapsto (x-\sigma(a_1))^{e_1}\cdots(x-\sigma(a_n))^{e_n},$$
$$(e_1, \ldots, e_n) \mapsto (e_{\sigma^{-1}(a_1)}, \ldots, e_{\sigma^{-1}(a_n)}).$$

The book and two theses

The *book* explains, in about 400 pages, how the Galois action on $H^{k-1}(E^{k-2}, \mathbb{Z}/m\mathbb{Z})$ can be used to compute coefficients a_n of modular forms in time polynomial in $\log n$, if the factorisation of n into primes is given.

Johan Bosman (thesis, 2008): did real computations.

Recently (2013): computations extended by Nicolas Mascot, Xinjian Zheng, Tian Peng, with some important ideas contributed by Maarten Derickx.

Peter Bruin's PhD thesis (2010): generalises the theory from $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ to more general congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$.
Javanpeykar (2013): arbitrary congruence subgroups.

An example by Johan Bosman

The polynomial:

$$\begin{aligned} f = x^{24} &- 2x^{23} + 115x^{22} + 23x^{21} + 1909x^{20} + 22218x^{19} \\ &+ 9223x^{18} + 121141x^{17} + 1837654x^{16} - 800032x^{15} \\ &+ 9856374x^{14} + 52362168x^{13} - 32040725x^{12} \\ &+ 279370098x^{11} + 1464085056x^{10} + 1129229689x^{9} \\ &+ 3299556862x^{8} + 14586202192x^{7} + 29414918270x^{6} \\ &+ 45332850431x^{5} - 6437110763x^{4} - 111429920358x^{3} \\ &- 12449542097x^{2} + 93960798341x - 31890957224 \end{aligned}$$

has Galois group $\mathrm{PGL}_2(\mathbb{Z}/23\mathbb{Z})$, and (reduced) discriminant 23^{43}; it comes from étale cohomology of degree 21 of a variety of complex dimension 21.

The commercial, end

. . . The computation of the Galois representations uses their realization, following Shimura and Deligne, in the torsion subgroup of Jacobian varieties of modular curves.

The main challenge is then to perform the necessary computations in time polynomial in the dimension of these highly nonlinear algebraic varieties.

Exact computations involving systems of polynomial equations in many variables take exponential time.

This is avoided by numerical approximations with a precision that suffices to derive exact results from them.

Bounds for the required precision–in other words, bounds for the height of the rational numbers that describe the Galois representation to be computed–are obtained from Arakelov theory. . .

The end

Terima kasih!

Questions?

Nederlandse Organisatie voor Wetenschappelijk Onderzoek

With: Jean-Marc Couveignes (Bordeaux), Robin de Jong, Franz Merkl (München), Johan Bosman, Peter Bruin, Ila Varma.