Bas Edixhoven.

and Milan Lopuhaä's MSc thesis
and my preprint with D. Bertrand. 1.

On unlikely intersection conjectures, Leiden, 2019/10/07, 1 hour.

Reference: Richard Pink, A common generalization of the conjectures of André-Oort, Manin-Mumford and Mordell-Lang, preprint 2005, on the author's home page.
(and also: a preprint by me "On Pink's unlikely int. conj's", to appear on arxiv, soon.)

Manin-Mumford & Mordell-Lang are theorems (Raynaud, ...; Faltings, ...), and André-Oort is almost a theorem (Klingler-Ullmo-Yafaev, Pila-Tsimerman ...). $\mathcal{A}_g$: o.k.; general case under GRH, but expected to become indep.

Here are some statements.

Manin-Mumford: Let A be an ab. var. $/\mathbb{C}$, and $Z \subset A$ an irreducible closed algebr. subvar such that $Z \cap A_{tor} \subset Z$ is Zariski dense. Then $Z = x + B$ with $x \in A_{tor}$ and $B \subset A$ an abel. subvar.

Mordell-Lang. Let A be an ab. var. $/\mathbb{C}$, $\Lambda \subset A$ a fin. gen. subgroup, and $Z \subset A$ an irred. closed subv. s.t. $Z \cap \bigcup_{n \geq 1} \left(\frac{1}{n} \cdot \Lambda\right) \subset Z$ is Zariski dense. Then $Z = x + B$ with $x \in A$ and B an ab. subvar.
(Nice special case: C curve $/K$ (nr. field), genus ≥ 2, $J := \mathrm{jac}(C)$, $\Lambda := J(K)$.)

André-Oort. Let S be a Shim. var. $/\mathbb{C}$, $Z \subset S$ irred. closed subvar, s.t. $Z \cap \mathrm{Spec}$ (Spec = the set of "special" points in S) is Zar. dense in Z. Then Z is a "special" subvar. of S.

Nice case: $g \geq 1$, $S := \mathcal{A}_g$. Then special points are CM points, and special subvarieties are those coming from "sub Shimura data" (G, X) of $(\mathrm{GSp}_{2g,\mathbb{Q}}, \mathbb{H}_g^{\pm})$. More precisely: $\mathbb{H}_g^{\pm}$ is a $\mathrm{GSp}_{2g}(\mathbb{R})$-orbit in $\mathrm{Hom}_{\mathbb{R}\text{-alg.grps}}(\mathbb{C}^\times, \mathrm{GSp}_{2g}(\mathbb{R}))$, each connected sub-$\mathbb{Q}$-grp. $G \subset \mathrm{GSp}_{2g,\mathbb{Q}}$ gives

$\{h \in \mathbb{H}_g^{\pm} : h(\mathbb{C}^\times) \subset G(\mathbb{R})\}$, finite union of $G(\mathbb{R})^+$-orbits, and the images of such in $\mathcal{A}_g$ are the special subvarieties.

inspired by Bombieri, Masser, Zannier, Viada, Ratazzi, Rémond... He could also have cited Zilber, "exp. sums..." 2002, about "atypical intersections".

Pink generalised them to "unlikely intersection conjectures".

Conj. 5.1 Let A be a semi-ab. var. $/\mathbb{C}$ and Z an irred. closed subvar. of A, not contained in any proper alg. subgroup of A.

Then $Z \cap \bigcup_{\substack{B \subset A \text{ alg. subgr.} \\ \operatorname{codim}_A(B) > \dim Z}} B \subset Z$ is not Zariski-dense.

Conj. 5.2. Let A be a semi-ab. var. $/\mathbb{C}$, $\Lambda \subset A$ a fin. gen. subgroup, and $Z \subset A$ an irred. closed subvar., not contained in any translate of a proper alg. subgr. of A.

Then $Z \cap \bigcup_{\substack{n \geq 1,\ B \subset A \text{ alg. subg.} \\ \operatorname{codim}_A(B) > \dim Z}} (B + \frac{1}{n} \cdot \Lambda) \subset Z$ is not Zar. dense.

Thms. 5.3 and 5.5

Pink proves, in his §5: Conj. 5.1 $\Longleftrightarrow$ Conj. 5.2.
(I've read that proof in all detail, and agree with it.) (≈ 4 pages).

this is what the title of his preprint promises!

Conj. 1.3. Let S be a mixed Shim. var. $/\mathbb{C}$, and $Z \subseteq S$ a closed alg. subvar. that is not contained in a proper special sub-mixed-Shim variety of S.

Then $Z \cap \bigcup_{\substack{W \subset S \text{ special} \\ \operatorname{codim}_S(W) > \dim Z}} W \subset Z$ is not Zariski-dense.

Thm. 5.7

Pink states in his §5: Conj. 1.3 $\Longrightarrow$ Conj. 5.1, but there is an error in his argument. ($\approx 2/3$ page).

I claim: Conj. 1.3 $\Longrightarrow$ Conj. 5.2, and the proof is easy.
To see this, we consider the moduli space of:

$$\begin{array}{c} \mathbb{G}_m^n \hookrightarrow E \twoheadrightarrow A \\ \uparrow \\ \mathbb{Z}^m \end{array}$$

with A a princ. polarised ab. var. $/\mathbb{C}$, say of dim d.

Such complexes $\overset{-1}{\mathbb{Z}^m} \longrightarrow \overset{0}{E}$ are called 1-motives $/\mathbb{C}$.

it looks big but I try to make it small. ← 3×3 block matrices, that's as small as it gets.

This moduli space is described as $P(\mathbb{Z})^+\backslash D$, with:

← note $\mathrm{Ext}^1(A_\tau, \mathbb{G}_m) = A_\tau^\vee$.

$$\underset{\tau}{\mathbb{H}_d^+} \times \underset{A_\tau^{\vee,n}}{M_{n,d}(\mathbb{C})} \times \underbrace{M_{d,m}(\mathbb{C}) \times M_{n,m}(\mathbb{C})}_{E_\tau^m} \xrightarrow{\sim} D$$

$$(\tau, u, v, w) \longmapsto \left(\begin{pmatrix} u & w \\ \tau & v \\ 1_d & 0 \\ 0 & 1_m \end{pmatrix} \cdot \mathbb{C}^{d+m} \subset \mathbb{C}^{n+2d+m} \right) \in Gr_{d+m}(\mathbb{C}^{n+2d+m}).$$

and, ∀ ring R:

$$P(R) = \left\{ \begin{pmatrix} \mu(g)\cdot 1_n & x & z \\ 0 & g & y \\ 0 & 0 & 1_m \end{pmatrix} : \begin{array}{l} (g,\mu(g)) \in GSp_{2g}(R) \\ x \in M_{n,2d}(R) \\ y \in M_{2d,m}(R) \\ z \in M_{n,m}(R) \end{array} \right\}$$

$\cap$

$GL_{n+2d+m}(R)$ $\quad \cup$

$$U(R) = \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : z \in M_{n,m}(R) \right\}.$$

$$\begin{array}{ccccc} D & \supset & D_\tau \circlearrowleft & P^u(\mathbb{Z}) = \left\{ \begin{pmatrix} 1_n & x & z \\ 0 & 1_{2d} & y \\ 0 & 0 & 1_m \end{pmatrix} : x \in M_{n,2d}(\mathbb{Z}) \text{ etc.} \right\} \\ \downarrow & & \downarrow \ \ni \tilde{\tau} = (\tau,0,0,0) & \\ \mathbb{H}_d^+ & \ni & \{\tau\} & \end{array}$$

Then: $\quad P^u(\mathbb{R})\cdot U(\mathbb{C}) \xrightarrow{\sim} D_\tau, \quad p \longmapsto p\cdot\tilde{\tau}$

$\quad\circlearrowleft \qquad\qquad\qquad\qquad \circlearrowleft$

$\quad P^u(\mathbb{Z}) \qquad\qquad\qquad P^u(\mathbb{Z}).$

skip??

$$P^u(\mathbb{Z})\backslash D_\tau \longrightarrow (A_\tau^m \times (A_\tau^\vee)^n)$$

$\uparrow$ $(\mathbb{C}^\times)^{nm}$-bundle, product of pullbacks of Poincaré torsor on $A_\tau \times A_\tau^\vee$ via all nm projections.

Fibre over a point in $(A_\tau^\vee)^n$: group str., ext. of A_τ^m by $\mathbb{G}_m^{n\cdot m}$

———————— A_τ^m : ———————— $(A_\tau^\vee)^n$ by $\mathbb{G}_m^{nm}$.

Crucial point: intersection of such a fibre with a special subvariety is a translate of ~~[illegible]~~ a sub-semi-ab. variety by a point, but not necessarily by a torsion point.

To prove that, I gave a description of all connected alg. subgroups of $P_{\mathbb{Q}}$ whose image in $GSp_{2d,\mathbb{Q}}$ contains the central $\mathbb{G}_{m,\mathbb{Q}}$.

Here is a recipe for obtaining all such Q with image G.
Via the given splitting of $P^u_{\mathbb{Q}} \hookrightarrow P \twoheadrightarrow GSp_{2g}$, G acts on P^u by conjugation.

Take V_{-1} and V_{-2} G-invariant sub $\mathbb{Q}$-v.spaces of $M_{n,2d}(\mathbb{Q}) \oplus M_{2d,m}(\mathbb{Q})$ and $M_{n,m}(\mathbb{Q})$, such that $\forall (x_1,y_1),(x_2,y_2) \in V_{-1}$, $x_1y_2 - x_2y_1 \in V_{-2}$.
Then $V := V_{-2} \oplus V_{-1} \subset \mathrm{Lie}(P)_{\mathbb{Q}}$ is a sub-Lie-algebra.
Take $p \in P^u(\mathbb{Q})$.
Take

$$Q := p\cdot\left(\exp(V) \rtimes G\right) p^{-1}$$

So, $\forall \mathbb{Q}$-alg. R,

$$Q(R) = p\cdot\left\{\begin{pmatrix} \nu(g) & x & z \\ 0 & g & y \\ 0 & 0 & 1_m \end{pmatrix} : \begin{array}{l} (x,y)\in R\otimes V_{-1} \\ z\in R\otimes V_{-2} \\ g\in G(R) \end{array}\right\}\cdot p^{-1}$$