On consequences of Pink's Unlikely intersection conjecture for mixed Shimura Varieties.

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This talk concerns conjecture 1.3 in Pink's 2005 preprint "A common generalization of the conjectures of André–Oort, Manin–Mumford and Mordell–Lang!".

Here is that conjecture.

Conj. 1.3: If X is a Hodge generic irreducible closed subvariety of a mixed Shimura variety S, then the union

$$\bigcup_{\substack{S' \subset S \\ \text{Shimura subv. of codim } > \dim X}} (S' \cap X) \quad \text{is not Zariski dense in } X.$$

In the same preprint, there is:

Conj. 6.1: if $B \to C$ is a family of complex semi-abelian varieties, and $X \subset B$ an irreducible closed subvariety, not contained in any proper closed subgroup scheme of $B \to C$, then

$$\bigcup_{\substack{c \in C \\ B' \subset B_c \text{ subgrp. of codim } > \dim X}} (B' \cap X) \quad \text{is not Zariski dense in } X.$$

And there is the theorem:

<u>Thm. 6.3</u>: Conj. 1.3 $\Longrightarrow$ Conj. 6.1.

But Daniel Bertrand found (in 2011) counterexamples against Conj. 6.1. See our joint article from 2020 (J.E.P.).

We explain: these counterexamples are not against Conj. 1.3, in fact, they give evidence for Conj. 1.3.

The simplest counterexample is this:

$E := \mathbb{C}/\mathbb{Z}[i]$, elliptic curve with $\mathrm{End}(E) = \mathbb{Z}[i]$;

$C := E$, viewed as $\mathrm{Ext}^1(E, \mathbb{G}_m)$;

$B \to C$: the universal extension:

$$\begin{tikzcd} \mathbb{G}_m \times E \arrow[r, hook] \arrow[dr] & B \arrow[r, two heads] \arrow[d, shift right] & E \times E \arrow[dl, "pr_2"] \\ & E \arrow[u, shift right, "r"'] \arrow[ur, bend right, "{(i, id)}"'] & \end{tikzcd}$$

$X := r(E)$, r is called a "Ribet section".

Then $\forall\, c \in E$ torsion, $r(c) \in B_c$ is torsion.

In fact, B is a mixed Shimura variety and X is a special subvariety, of a kind that Pink had missed: it is not true that for $B' \subset B$ special, the fibres $B'_\varepsilon \subset B_\varepsilon$ over $\varepsilon \in E$ are translates of subgroups by torsion points. (In the example: take ε not torsion, then $r(\varepsilon)$ is not torsion.)

The same "torsion" appears in Pink's proof that Conj. 1.3 $\Rightarrow$ (Conj. 5.1 and Conj 5.2).

Now the good news: with a little modification, that proof can be completed.

(5.1) Let E be a complex semiabelian variety, and $X \subset E$ an irreducible closed subvar. that is not contained in any proper closed subgroup of E; then $\bigcup (E' \cap X)$ is not Zariski dense in X.

$E' \subset E$ algebr. subgr. of codim. $> \dim X$

("unlikely" generalisation of Manin-Mumford)

("unlikely Mordell–Lang generalisation")

Conj. 5.2 Let E be a complex semi-ab. var., $\Gamma \subset E$ a fin. gen. subgroup, $X \subset E$ a closed irred. sub-variety that is not contained in any translate of a proper algebr. subgr. of E.

Then $\bigcup_{\substack{n \in \mathbb{N}_{\geq 1} \text{ s.t. } n\cdot\gamma \in \Gamma \\ E' \subset E \text{ algebr. subgr. codim} > \dim X}} \big((\gamma + E') \cap X\big)$ is not Zar. dense in X.

Pink Thms 5.3 and 5.8: Conj. 5.1 $\Longleftrightarrow$ Conj 5.2.

I have read the proof of $5.2 \Rightarrow 5.1$ detail and have found it o.k.

Pink Thm. 5.7. Conj. 1.3 $\Longrightarrow$ Conj. 5.1

Here I do have a problem with the proof given: "translate by torsion point...".

But: it is in fact quite easy to see that

$$\text{Conj. } 1.3 \Longrightarrow \text{Conj. } 5.2.$$

Theorem: Conj. 1.3. $\Rightarrow$ Conj. 5.2 .

Proof. Let $\mathbb{G}_m^n \hookrightarrow E \twoheadrightarrow A$, $\Gamma \subset E$ and $X \subset E$ be as in Conj. 5.2.

(Strategy: include this in a family that is a mixed Shimura variety, then apply Conj. 1.3.)

Choose $\varphi : \mathbb{Z}^m \twoheadrightarrow \Gamma$. That gives us the 1-motive

$$\begin{array}{ccccc} \mathbb{G}_m^n & \hookrightarrow & E & \twoheadrightarrow & A \\ & & \uparrow \varphi & & \\ & & \mathbb{Z}^m & & \end{array}$$

For the mixed Shim. variety: take the moduli space of 1-motives of this kind, with A principally polarised, and with some level structure that gives a fine moduli space:

$$\begin{array}{c} T \circlearrowleft \mathbb{G}_m^{nm}\text{-torsor} \\ \downarrow \\ A^m \times B^n \\ \downarrow \\ \mathcal{A}_{d,1,3} \end{array}$$

$\mathbb{G}_m^{nm}$-torsor: pullback via all nm projections of Poincaré torsor

$$\begin{array}{c} P^\times \\ \downarrow \\ A \times_{\mathcal{A}_{d,1,3}} B \end{array}$$

Mixed Shimura datum:

(P, D) set of Fil^0 on $M_{\mathbb{C}}$

$$\underbrace{\mathbb{Z}^n}_{W_{-2}} \subset \mathbb{Z}^{n+2d} \subset \underbrace{\mathbb{Z}^{n+2d+m}}_{= M}$$

$W_{-1} = \mathbb{Z}^{n+2d}$

that give a MHS of type $\{(-1,-1), (-1,0), (0,-1), (0,0)\}$, s.t. standard sympl. form is pol'n.

$$P = \left\{ \begin{pmatrix} \mu(g)\cdot 1_n & x & z \\ & g & y \\ & & 1_m \end{pmatrix} \;\middle|\; \begin{array}{l} (g, \mu(g)) \text{ in } GSp_{2d} \\ x \in M_{n,2d} \\ y \in M_{2d,m} \\ z \in M_{n,m} \end{array} \right\}$$

<u>Note</u>: P^u (unipotent radical) is a (generalised) Heisenberg group.

([It is nicer to work with this matrix group than with a group defined by a 2-cocycle....])

Then carefully study subgroups of $P_{\mathbb{Q}}$, using a bit of Lie-algebra theory for those of $P^u_{\mathbb{Q}}$.

That gives: **1.** $\forall$ special subvar. $T' \subset T$, every irr. comp. of a fibre over $b \in B^n$ is a <u>translate</u> of an irred. subgr. of E_b^m;

2. "up to $P^u(\mathbb{Q})$" the special subvarieties of T are obtained by:

- take a special subv. $S \subset \mathcal{A}_{d,1,3}$,
- take a sub-ab. scheme $C \subset A^m \times_S B^n$ and a sub torus $\mathbb{G} \subset \mathbb{G}_m^{nm}$ on S s.t. the $(\mathbb{G}_m^{nm}/\mathbb{G})$-torsor $\mathbb{G}\backslash(T|_C)$ on C is trivial,
- let T' be the inverse image in $T|_C$ of the canonical trivialisation of $\begin{matrix}\mathbb{G}\backslash T|_C \\ \downarrow \\ C\end{matrix}$.

With this available, the proof is quite direct.

The
$$\begin{tikzcd} \mathbb{G}_m^n \arrow[r, hook] & E \arrow[r, two heads] & A \\ & \mathbb{Z}^m \arrow[u, "\varphi"] & \end{tikzcd}$$
as in Conj. 5.2 gives
$$\begin{tikzcd} T \ni t \arrow[d] \\ A^m \times_{\mathcal{A}_{d,1,3}} B^n \ni (a,b) \arrow[d] \\ \mathcal{A}_{d,1,3} \ni s. \end{tikzcd}$$

Over T we then have the universal family:

$$\begin{tikzcd} \mathbb{G}_m^n \arrow[r, hook] & E^{univ} \arrow[r, two heads] & A^{univ} \\ & \mathbb{Z}^m \arrow[u, "\varphi^{univ}"] & \end{tikzcd}$$

and the fibre $/t$ is the one given.

Let X^{sp} be the special closure of $X \subset E_t$. As $X \not\subset$ any translate of a proper irred. algebr. subgroup, $(X^{sp})_t = E_t$, and therefore $X^{sp} = E^{univ}|_{T'}$, with $T' \subset T$ the special closure of t.

Now let $E' \subset E$ be an irred. alg. subgroup, $n \in \mathbb{N}_{\geq 1}$, $\gamma \in \Gamma_0$.

Then E' extends uniquely to an $E^{univ\prime} \subset E^{univ}|_{T'}$, and γ to a γ', and so we see:

$\gamma + E'$ = fibre at t of a special subv. of $E^{univ}|_{T'}$

codim in E = codim in $E^{univ}|_{T'}$. $\boxtimes$.