Oberwolfach, 2021/07/22, 20 minutes. !

Bas Edixhoven, joint work with Pierre Parent,

"stable & regular models for $X_{ns}(p)$, $X_{ns}^+(p)$."

Stable models: Documenta Math., 26 (2021).

Regular models: still being written up, we are not yet happy with all proofs/computations.

$X(p) = Y(p) \cup \{\text{cusps}\}$, smooth projective curve over $\mathbb{Q}$,

$\forall$ $\mathbb{Q}$-algebra A:

$$Y(p)(A) = \left\{ (E/A, \varphi) : \begin{array}{l} E \text{ ell. curve over } A \\ \varphi : (\mathbb{Z}/p\mathbb{Z})^2_A \xrightarrow{\sim} E[p] \end{array} \right\} \Big/ \cong$$

Note: $e_p(\varphi(1,0), \varphi(0,1)) \in A^\times$, order p,

so $X(p) \longrightarrow \mathrm{Spec}(\mathbb{Q}(\zeta_p)) \longrightarrow \mathrm{Spec}(\mathbb{Q})$; not a "nice" curve / $\mathbb{Q}$: geometrically reducible.

$GL_2(\mathbb{F}_p)$ acts on $X(p)$: $\varphi \mapsto \varphi \circ g$.

For $H \subset GL_2(\mathbb{F}_p)$ subgroup: $X(H) := X(p)/H$.

$$Y(H)(\mathbb{Q}) = \left(Y(p)(\overline{\mathbb{Q}})/H\right)^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}$$

$$= \left\{ \left(E/\mathbb{Q}, \underset{\text{up to } H}{\varphi : \mathbb{F}_p^2 \xrightarrow{\sim} E(\overline{\mathbb{Q}})[p]}\right) : \underset{\text{inv.}}{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\text{-}} \right\} \Big/ \cong$$

(or ~~twist~~ isom)

Let $\Gamma_{ns}(p) \subset GL_2(\mathbb{F}_p)$ be a cyclic subgr. of order p^2-1. $\quad (\mathbb{F}_{p^2}^\times)$.

$\Gamma_{ns}^+(p)$: its normalizer, order $2\cdot(p^2-1)$.

ns: non split (Cartan).

Then $X_{ns}(p) := X(p)/\Gamma_{ns}(p)$, $X_{ns}^+(p) := X(p)/\Gamma_{ns}^+(p)$.

Problem: find all $\mathbb{Q}$-points on $X_{ns}^+(p)$.

$p=13$: cursed curve, [BDMTV], quadratic Chabauty,

genus 3

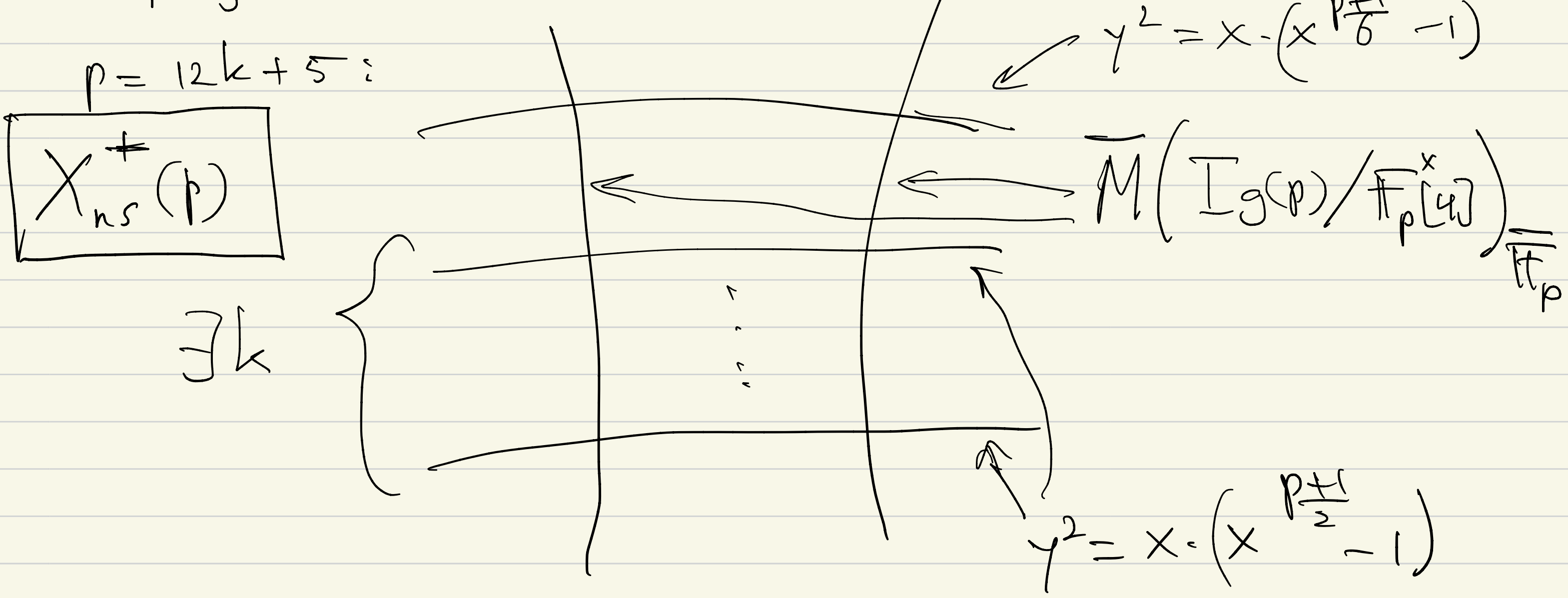
$p=17$?? etc... $\Leftarrow$ genus 6.

Need to know how $X_{ns}^{(+)}(p)$ extends as a curve $/\mathbb{Z}$.

Over $\mathbb{Z}[1/p]$: smooth projective, via moduli interpr.

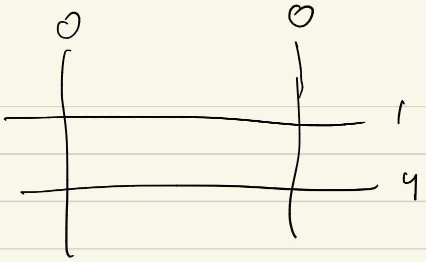
Over $\mathbb{Z}_p$: bad reduction, want regular projective model

Let $\pi := p^{1/\frac{p^2-1}{2}}$, over $\mathbb{Z}_p^{unr}[\pi]$: stable model, i.e., projective curve with "nodal" reduction mod π:



(the numbers are genera)

For $p=17$:



genus 6, toric rank 1.

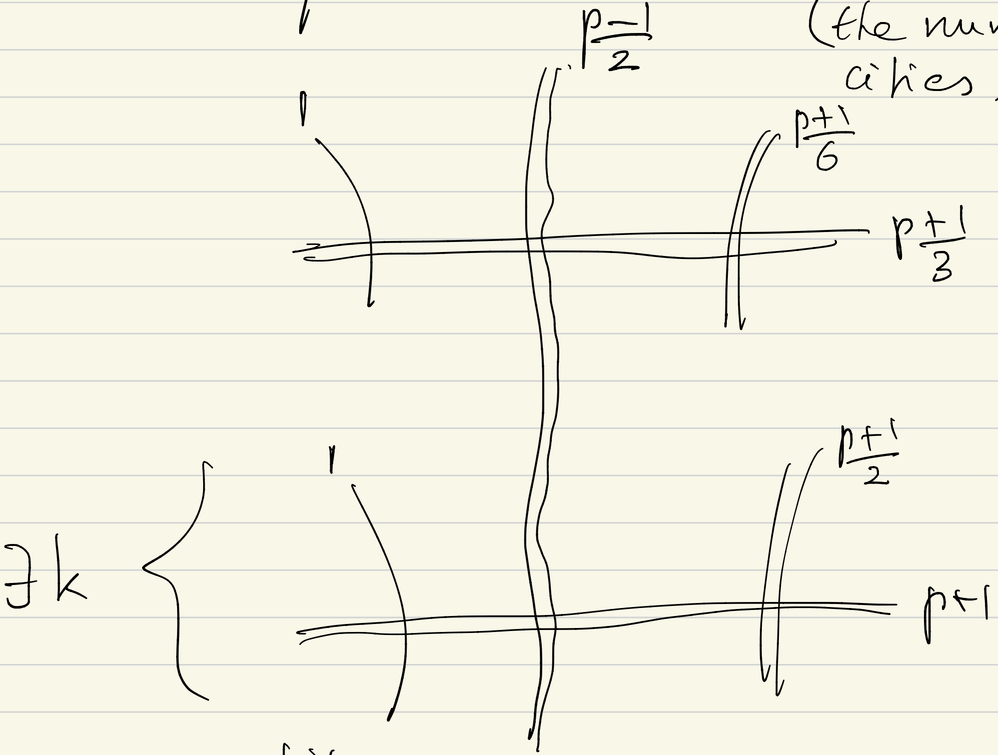
$\mathrm{Gal}\left(\mathbb{Z}_p^{unr}[\pi]/\mathbb{Z}_p\right) = \mu_{\frac{p^2-1}{2}}(\mathbb{F}_{p^2})$ acts on this stable model, with given action on the base.

Take quotient, resolve the singularities in it.

That gives a regular model over $\mathbb{Z}_p^{unr}$.

Picture for $\boxed{p=12k+5, \quad X_{ns}^{+}(p):}$

(the numbers are multiplicities, all irred. comp. have genus 0)



Curious, but a bit expected, fact: all irr. comp. with mult. >1 can (successively) be contracted, hence the minimal regular model has reduced fibre $/\mathbb{F}_p$.