II.3. Just present the definitions. I don't want to spend time on noetherian schemes/rings. Try to give examples with the def's.

Also: give the definition of $f\colon X \to Y$ locally of finite presentation and of finite presentation. These are important finiteness conditions.

See EGA I: 6.1.1, 6.1.3, 6.2.1 and 6.3.7. You cannot read this yet.

I can also look it up in Johan de Jong's Stacks project, in the Chapter "Morphisms of Schemes".

$f\colon X \to Y$ is locally of finite pres. if $\forall x \in X$ $\exists\, U \subset X$ ($x \in U$) affine open nbd, $\exists\, V \subset Y$ affine open nbd of fx, with $U \subset f^{-1}V$, s.t. $\mathcal{O}(V) \to \mathcal{O}(U)$ is of finite presentation. ($\mathcal{O}(U) = \mathcal{O}(V)[\text{fin. many variables}]/(\text{fin. many relations})$.)

$f\colon X \to Y$ is of finite presentation if $\forall y \in Y$ $\exists\, V \subset Y$ ~~open~~ aff. open nbd of y, $\exists$ finite affine open cover of $f^{-1}V$, $(U_i)_{i \in I}$, say, with $\forall i$: $\mathcal{O}(V) \to \mathcal{O}(U_i)$ of finite pres., and $\forall i,j$: $U_i \cap U_j$ is quasicompact (and then $U_i \cap U_j$ can be covered with finitely many open affines $U_{i,j,k}$ that are principal open both in U_i and in U_j). give the argument

$U \subset X \supset V$ ~~[illegible]~~ affine
$x \in U \cap V \ldots$

Consequence: $f^{-1}V$ can be described ~~by~~ as $\mathcal{O}(V)$-scheme in a finite way:

$$\prod_{i \in I} \mathcal{O}(U_i) \rightrightarrows \prod_{\substack{i,j \in I \\ k \in K_{i,j}}} \mathcal{O}(U_{i,j,k})$$

To get $f^{-1}V$: apply Spec; then glue. read Stacks, Lemma 19.14.1 ~~Also~~

$$f^{-1}V \leftarrow \coprod_{i \in I} U_i \leftleftarrows \coprod_{i,j,k} U_{i,j,k}$$

Before: explain that over a not nec. noeth. ring, finite presentation is important, both for modules as for algebras.

Example. $Y := \operatorname{Spec} k[x_1, x_2, \ldots]$, $y :=$ the max. ideal $(x_1, x_2, \ldots)$.
$X := Y - \{y\}$, $j\colon X \to Y$ the open immersion. Then j is locally of finite presentation. But X is not quasicompact. These kind of X can arise as intersection of 2 affines opens...

The fibered product.

give the def. ($\mathrm{Arr}(\mathcal{C}) \underset{t}{\overset{s}{\rightrightarrows}} \mathrm{Ob}(\mathcal{C})$, comp. id.)

Let $\mathcal{C}$ be a category.

Def. A diagram (comm.)

$$\begin{tikzcd} Z \arrow[r, "f'"] \arrow[d, "g'"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & S \end{tikzcd}$$

is called Cartesian (or: a fibered product) if: $\forall\, p: T\to X$, $q: T\to Y$ s.t. $fp = gq$ $\exists!\, r: T\to Z$ s.t. $p = g'r$ & $q = f'r$.

$$\begin{tikzcd} T \arrow[dr, dashed, "\exists! r"] \arrow[drr, bend left, "q"] \arrow[ddr, bend right, "p"'] & & \\ & Z \arrow[r, "f'"] \arrow[d, "g'"'] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"'] & S \end{tikzcd}$$

also: (Z, f', g') is called a fibered pr. of $X \xrightarrow{f} S$ and $Y \xrightarrow{g} S$. (Notation: $X \times_S Y \dots$)

Example For $\mathcal{C} = (\mathrm{Sets})$, $X \xrightarrow{f} S \xleftarrow{g} Y$ given, $Z := \{(x,y)\in X\times Y : f(x) = g(y)\}$, with its projections to X and Y is a fibered pr.

Note that for $s\in S$, $Z_s := (fp)^{-1}\{s\} = (gq)^{-1}\{s\} = X_s \times Y_s = X_s \times_{\{s\}} Y_s$. That explains the name "fibered product".

in (Sets): $Z(T) = X(T) \times_{S(T)} Y(T)$

Uniqueness: they're unique up to unique isomorphism.

Existence: in (Top): easy. in (C^∞-man): I think they do not exist; if $X \xrightarrow{f} S$ is a submersion then $\forall\, Y \xrightarrow{g} S$ it does exist.

in (LRS): I do not know.

But for $X \xrightarrow{f} S \xleftarrow{g} Y$ in (Sch) the fibered pr. in (LRS) exists. (but weird things happen.. think of $\otimes$ of fields..) (the underlying set is not nec. the fibered pr.). See Thm. II.3.3. + its proof.

For $X \xrightarrow{f} S \xleftarrow{g} Y$ ~ $B \xleftarrow{\varphi} A \xrightarrow{\psi} C$ in (Ring) we are looking for a co-Cartesian diagram

$$\begin{tikzcd} B\otimes_A C & C \arrow[l, "\varphi'"'] \\ B \arrow[u, "\psi'"] & A \arrow[l, "\varphi"] \arrow[u, "\psi"'] \end{tikzcd}$$

in (Ring) (a push-out diagram, also called amalgamated sum).

In (Ring), $\otimes$ products exist.

Proof. Write $B = A[\{x_i : i\in I\}]/(\{f_j : j\in J\})$, $C = A[\{y_k : k\in K\}]/(\{g_l : l\in L\})$.

Then $D := A[\{x_i : i\in I\}\cup\{y_k : k\in K\}]/(\{f_j : j\in J\}\cup\{g_l : l\in L\})$ does the job. Why: $\forall\, D' \in$ ~~Ring~~ disjoint A-alg: $\underbrace{\mathrm{Hom}_{A\text{-alg}}(D, D')}_{= \cdots\cdots} = \{\cdots : \cdots\}$

also: "resolve only one of the two".

Note: one can also define $M\otimes_A N$ for A-modules M & N: $M\times N \xrightarrow{b} M\otimes_A N$ is the univ. A-bil. map.

Then $B\otimes_A C$ as A-module is $B\otimes_A C \dots$

For $M = A^{(I)}$: $A^{(I)}\otimes_A N = N^{(I)}$ (from univ. prop.)

For $A^{(J)} \xrightarrow{r} A^{(I)} \xrightarrow{g} M$ (exact): $N^{(J)} \xrightarrow{r} N^{(I)} \xrightarrow{g} M\otimes_A N$ exact. ∎

Example. Let k be a field, $f \in k[x]$ a ~~separable~~ polynomial, and $k \to K$ a field extension s.t. in $K[x]$: $f = \prod_{i=1}^{d} (x-r_i)^{e_i}$, r_i dist., $e_i \geq 1$.

Then $K \otimes_k (k[x]/(f)) = K[x]/(f) = K[x]/((x-r_1)^{e_1} \cdots (x-r_d)^{e_d}) \overset{\text{Chinese rem. thm.}}{\cong} \prod_{i=1}^{d} K[x]/((x-r_i)^{e_i})$.

Special case: f irreducible, hence $k[x]/(f)$ a field.

$\overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ is not noetherian. (exercise: prove it).

We continue the proof of thm II 3.3.

If in $\mathcal{C}$ fibre products exist, then they are functorial:

for $X \to S$ and $Y_1 \to S$, $Y_2 \to S$ in $(\mathcal{C}/S)$ and $h: Y_1 \to Y_2$ over S,

we have $X \times_S Y_1 \xrightarrow{id_X \times h} X \times_S Y_2$, etc.

This functoriality helps for constructing them in (Sch).

Reduction to the case S affine. Let $S = \bigcup_{i \in I} S_i$, $S_i \subset S$ open affine.

Assume that all fibered pr. over affine schemes exist.

Let $X_i := f^{-1} S_i$, $Y_i := g^{-1} S_i$. Then we have the $X_i \times_{S_i} Y_i$ in (Sch).

For $i, j \in I$: $S_i \cap S_j \hookrightarrow S_i$, $S_i \cap S_j \hookrightarrow S_j$

$$\begin{array}{ccc} h_i^{-1}(S_i \cap S_j) & \hookrightarrow & X_i \times_{S_i} Y_i \\ \downarrow & & \downarrow h_i \\ S_i \cap S_j & \hookrightarrow & S_i \end{array}$$

$h_i^{-1}(S_i\cap S_j)$ = fibered pr. of $f_i^{-1} S_i \cap S_j$ and $g_i^{-1} S_i \cap S_j$ over $S_i \cap S_j$

this gives unique $h_i^{-1}(S_i \cap S_j) \xrightarrow{\sim} h_j^{-1}(S_i \cap S_j)$, glueing data.

Hence we get $X \times_S Y$.

Now S is affine, reduce to X is affine ... red. to Y is affine.

See also Stacks, §19.17 (and §19.16) And see EGA I (Springer) Ch. 0, §1.

Here is (I think) a very useful lemma:

in $\mathcal{C}$ suppose we have a comm. diagram

$$\begin{array}{ccccc} W & \to & Z & \to & Y \\ \downarrow & & \downarrow & \square & \downarrow \\ V & \to & X & \to & S \end{array}$$

Then $\begin{array}{ccc} W & \to & Y \\ \downarrow & & \downarrow \\ V & \to & S \end{array}$ is Cart. iff $\begin{array}{ccc} W & \to & Z \\ \downarrow & & \downarrow \\ V & \to & X \end{array}$ is.