Week 6 (2011/10/13)

II.4. Separated morphisms. (See also Stacks, §19.21) (and/or EGA I.5)

Def. For $f: X \to Y$ in a cat. $\mathcal{C}$, s.t. $X \times_Y X$ exists,

the morphism $X \xrightarrow{(id,id)} X \times_Y X$ is called the diagonal morph. ass. with f,

notation: Δ_f. So:

$$\begin{tikzcd} X \arrow[r, "\Delta_f"] \arrow[rr, bend left, "id"] \arrow[dr, bend right, "id"'] & X \times_Y X \arrow[r, "pr_2"] \arrow[d, "pr_1"] & X \arrow[d, "f"] \\ & X \arrow[r, "f"] & Y. \end{tikzcd}$$

Note: $\forall\, T$ in $\mathcal{C}$: $\Delta_f: X(T) \longrightarrow X(T) \times_{Y(T)} X(T)$, $x \longmapsto (x, x)$.

Def. For $f: X \to Y$ in (Sch), f is separated if Δ_f is a closed immersion.

Prop 4.1. If $f: X \to Y$ with X, Y affine, then f is separated.

Proof.
$$\begin{tikzcd} \mathcal{O}(X) & \mathcal{O}(X) \otimes_{\mathcal{O}(Y)} \mathcal{O}(X) \arrow[l] & \mathcal{O}(X) \arrow[l] \arrow[ll, bend right, "id"'] \\ & \mathcal{O}(X) \arrow[u] \arrow[ul, bend left, "id"] & \mathcal{O}(Y) \arrow[l] \arrow[u] \end{tikzcd}$$
hence $\mathcal{O}(X \times_Y X) \longrightarrow \mathcal{O}(X)$ surjective. ∎

Cor. 4.2. Let $f: X \to Y$ in (Sch). Then f is separated iff $\Delta_f X \subset X \times_Y X$ is closed.

$\Rightarrow$: clear. ~~Write $X = \bigcup_{i \in I} U_i$ with $U_i \subset X$ open affine, $V_i \subset Y$. Then $U := \bigcup_{i\in I} (U_i \times_{V_i} U_i) \subset X \times_Y X$ is open, and $U \supset \Delta_f X$, bcs $\Delta_f U_i$ ... $U_i \times_{V_i} U_i = U_i \times_Y U_i$.~~

First, no assumptions. Let $V \subset Y$ be open affine, and $U \subset f^{-1}V$ open affine. Then $U \times_V U = U \times_Y U$ (use the def'n).

Also: $U \overset{j}{\hookrightarrow} X$ is a monomorphism hence (diagram chase in (Set))

$$\begin{tikzcd} X \arrow[d, "\Delta_f"'] \arrow[r, phantom, "\hookleftarrow^{j}"] & U \arrow[d, "\Delta_f"] \\ X \times_Y X \arrow[r, phantom, "\hookleftarrow_{j\times j}"] \arrow[ur, phantom, "\square"] & U \times_Y U \end{tikzcd}$$

hence $\Delta_f: U \to U \times_Y U$ is a closed immersion. Let $W \subset X \times_Y X$ be the union of all $U \times_Y U$. Then $X \xrightarrow{\Delta_f} W \hookrightarrow X \times_Y X$, closed imm., open imm. (Δ_f is an immersion (19.18.2)).

Now assume $\Delta_f X$ is closed in $X \times_Y X$.

Then $\Delta_f: X \to X \times_Y X$ is a closed imm: $\Delta_f X$ closed, Δ_f: homeo $X \to \Delta_f X$ + ind. top., $\Delta_f^{\#}$: surj on stalks.

Ex. II 4.3. Assume $f: X \to S$ separated, S affine, U, V open aff. in X. Then $U \cap V$ is affine.

Proof.
$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\square"] & U \times_S V \arrow[d, hook] \\ X \arrow[r, "\Delta_f"] & X \times_S X \end{tikzcd}$$
, + Ex. II.3.11. ∎

Let us prove Cor. 4.6 without valuative criterion for separatedness.

1. Closed imm. stable under base change:
$$\begin{tikzcd} X \arrow[d, "i"'] \arrow[dr, phantom, "\Box"] & X' \arrow[l, "f'"'] \arrow[d, "i'"] \\ Y & Y' \arrow[l, "f"] \end{tikzcd}$$
i closed imm. $\Downarrow$ i' too.

proof: Local question on Y'.

So, replace Y by an open affine, then Y'. So we may assume that Y and Y' are affine. But then X is affine (Ex. II. 3.11), and $\mathcal{O}(Y) \to \mathcal{O}(X)$ is surjective. Then X' is affine, $\mathcal{O}(Y') \to \mathcal{O}(X')$ is surj. $\boxtimes$

2. If $f: X \to Y$ (over S) and Y/S separated, then $X \xrightarrow{\Gamma_f} X \times_S Y$ is a closed imm.

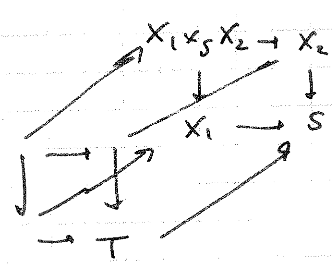
Proof:
$$\begin{tikzcd} X \arrow[r, "\Gamma_f"] \arrow[d, "f"'] \arrow[dr, phantom, "\Box"] & X \times_S Y \arrow[d, "f \times id"] \\ Y \arrow[r, "\Delta"'] & Y \times_S Y \end{tikzcd}$$

3. If $X \xrightarrow{f} Y \xrightarrow{g} Z$ (sep.) then $X \xrightarrow{gf} Z$ sep.

Proof:
$$\begin{tikzcd} X \arrow[r, "\Delta_f"] & X \times_Y X \arrow[r] \arrow[d] \arrow[dr, phantom, "\Box"] & X \times_Z X \arrow[d] \\ & Y \arrow[r, "\Delta_g"'] & Y \times_Z Y \end{tikzcd}$$

4. Separated morphisms are stable under base change: if
$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, phantom, "\Box"] & X_T \arrow[l] \arrow[d, "f_T"] \\ S & T \arrow[l] \end{tikzcd}$$
then: f sep. $\Rightarrow$ f_T separated.

Proof
$$\begin{tikzcd} X \arrow[d, "\Delta_f"'] \arrow[dr, phantom, "\Box"] & X_T \arrow[l] \arrow[d, "\Delta_{f_T}"] \\ X \times_S X \arrow[d] \arrow[dr, phantom, "\Box"] & X_T \times_T X_T \arrow[l] \arrow[d] \\ S & T \arrow[l] \end{tikzcd}$$
$\boxtimes$.



etc.

base change the Cart. square via $T \to S$.

5. $X_1 \xrightarrow{f_1} Y_1$, $X_2 \xrightarrow{f_2} Y_2$ (sep., over S) $\Rightarrow$ $f_1 \times f_2 : X_1 \times_S X_2 \to Y_1 \times_S Y_2$ sep.

Proof
$$\begin{tikzcd} & X_1 \times_S X_2 \arrow[dl] \arrow[dr, "id \times f_2"] & & Y_1 \times_S Y_2 \arrow[dr] & \\ X_2 \arrow[dr, "f_2"'] & \Diamond & X_1 \times_S Y_2 \arrow[ur, "f_1 \times id"] \arrow[dl] \arrow[dr] & \Diamond & Y_1 \\ & Y_2 & & X_1 \arrow[ur] & \end{tikzcd}$$

6. Let $f: X \to Y$, $g: Y \to Z$. If $g \circ f$ sep. then f is separated.

Proof.
$$\begin{tikzcd} X \arrow[r, "\Delta_f"] \arrow[rr, bend left, "\Delta_{gf}"] & X \times_Y X \arrow[r] \arrow[d] & X \times_Z X \arrow[d] \\ & Y \arrow[r, "\Delta_g"] & Y \times_Z Y \end{tikzcd}$$
immersion

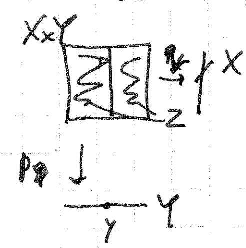
immersion; $X \times_Y X$ is a closed subset of an open of $X \times_Z X$.
Then $\Delta_f X$ is closed in $X \times_Y X$: $\boxtimes$

$A \subset B \subset C$
A & B induced top.
A closed in C
hence A closed in B.

Proper morphisms. II.4 & [Stacks, 22.39].

Fact. Let X in (Top). Then X is quasicompact iff $\forall$ Y in (Top): $p_Y: X \times Y \to Y$ is closed.

$\Longrightarrow$: Let $Z \subset X \times Y$ be closed, $y \in Y - p_Y Z$.



To show: $\exists$ open $U \subset Y$ with $y \in U$ and $U \cap p_Y Z = \emptyset$.

$\forall x \in X$ $\exists$ $\underset{x}{\underset{\Psi}{X_x}} \subset X$, $\underset{y}{\underset{\Psi}{Y_x}} \subset Y$ open s.t. $X_x \times Y_x \cap Z = \emptyset$.

$\exists$ finite I, $x_i \in X$ s.t. $\bigcup_{i \in I} X_{x_i} = X$. Then $\bigcap_{i \in I} Y_{x_i} =: U$. $\boxtimes$

$\Longleftarrow$: exercise (be creative, you must make a Y...) (compare 22.39.9)

Remark: I find that the definition of $f: X \to Y$ (in (Sch)) proper should have been: separated, finite presentation, univ. closed.
Be careful ~~...~~ when Y is not nec. noetherian. (many people make mistakes...).

Cor. 4.8: proofs are like we did for "separated".

II Exercise 4.4.
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "\text{proper } gf"'] & Y \arrow[d, "g \text{ separated}"] \\ & S \end{tikzcd}$$
then f is univ closed.
(in top: $f: X \to Y$, q.compact, Hausdorff $\Rightarrow fX \subset Y$ closed)

Proof.
$$\begin{tikzcd} & & X \arrow[r] & S \\ X \arrow[r, "\Gamma_f = (id, f)"] \arrow[d, "f"'] & X \times_S Y \arrow[r] \arrow[d] & Y \\ Y \arrow[r, "\Delta_g"] & Y \times_S Y \end{tikzcd}$$
closed imm.
proper.
$\boxtimes$

II Exercise 4.5. Let $k = \bar{k}$, X/k proper, integral, $f \in \mathcal{O}(X)$. Then $f \in k$.

Proof.
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "\text{proper}"'] & \mathbb{A}^1_k \arrow[r, hook] \arrow[d, "g"] & \mathbb{P}^1_k \arrow[dl, "\text{separated}"] \\ & \operatorname{Spec} k \end{tikzcd}$$
So fX is a closed pt. May assume it is 0.
Then $\forall$ $U \subset X$ open affine $\neq \emptyset$, $\mathcal{O}(U) \xleftarrow{f^\#} k[x]$
$\ker f^\# = (x)$. $\boxtimes$

II Exercise 4.8: same kind of proofs as we did for "separated."

edix/talks/2010_04_12.pdf

Proj(S) for S a graded ring. II.2 in Hartshorne, 20.7 in [Stacks].

What are graded rings? Why are we interested in them??

$\mathbb{Z}$-grading $\leftrightarrow$ $\mathbb{G}_m$-action $\longrightarrow$ projective spaces: $(\mathbb{A}^{n+1}_{\mathbb{Z}} - 0(\operatorname{Spec}\mathbb{Z}))/\mathbb{G}_m$

$$\mathbb{Z}[x_0,\ldots,x_n] = \bigoplus_{d\geq 0} \underbrace{\mathbb{Z}[x_0,\ldots,x_n]_d}_{\text{finite rank free } \mathbb{Z}\text{-modules.}}$$

(Can you find a better one?)

$\mathbb{Z}$-grading $\leftrightarrow$ $\mathbb{G}_m$-action. Reference: "hidden" in SGA 3, part 1.

Let A be a ring, M an A-module. ~~…~~

$\mathbb{G}_{m,A} : (A\text{-alg}) \longrightarrow (\mathrm{Grp})$, $B \longmapsto B^\times$, repr. by $A[x,x^{-1}]$.

Def. An action of $\mathbb{G}_{m,A}$ on M is: $\forall A\to B \xrightarrow{\varphi} C$ an action of $B^\times$ on the B-module $B\otimes_A M \circlearrowleft \alpha(u)$, $u\in B^\times$,

functorial in B: $\downarrow \varphi\otimes \mathrm{id}_M =: \varphi_M$, $C\otimes_A M$

$\forall u\in B^\times$, $\forall m\in B\otimes_A M$:

$$\varphi_M((\alpha u)m) = (\alpha(\varphi u))(\varphi_M m)$$

$$\begin{array}{ccc} B^\times \times B\otimes_A M & \longrightarrow & B\otimes_A M \\ \downarrow \varphi\times\varphi_M & \circlearrowleft & \downarrow \varphi_M \\ C^\times \times C\otimes_A M & \longrightarrow & C\otimes_A M. \end{array}$$

This looks like something very complicated.

But it's not! $\exists$ universal $(A\to B, \overset{B^\times}{\overset{\in}{u}})$, namely $(A\to A[x,x^{-1}], x)$.

So to give, $\forall B, \forall u\in B^\times$ an ~~auto~~ endomorphism αu of $B\otimes_A M$, functorial in B, is to give just the one $A[x,x^{-1}]$ ~~autom.~~ endom. αx of $A[x,x^{-1}]\otimes_A M$.

$A[x,x^{-1}] = \bigoplus_{n\in\mathbb{Z}} A\cdot x^n$, as A-modules. So $A[x,x^{-1}]\otimes_A M = \bigoplus_{n\in\mathbb{Z}} x^n\cdot M = \bigoplus_{n\in\mathbb{Z}} M$.

To give αx is to give a morphism of A-modules: $M = x^0\cdot M \longrightarrow \bigoplus_{n\in\mathbb{Z}} x^n\cdot M$, $m\longmapsto \sum_{i\in\mathbb{Z}} x^i\cdot(\alpha_i m)$,

where $\alpha_i \in \operatorname{End}_A M$, "locally finite"

$\forall m\in M$: $\alpha_i m = 0$ for almost all i.

Now let $(\alpha_i)_{i\in\mathbb{Z}}$ be such. When is it an action? $\forall B$ $B^\times \circlearrowleft$ group action.

$\forall (B,u)$; $\forall b\in B$, $\forall m\in M$: $(\alpha u)\cdot b\otimes m = \sum_{i\in\mathbb{Z}} (u^i b)\otimes(\alpha_i m)$

This is an action by $B^\times$ on $B\otimes_A M$ iff $\forall u_1,u_2\in B^\times$; $\forall b\in B$, $\forall m\in M$

$$(\alpha(u_1u_2))\cdot b\otimes m = \sum_i (u_1^i u_2^i b)\otimes(\alpha_i m)$$

$$\| $$

$$(\alpha u_1)\cdot((\alpha u_2)\cdot b\otimes m) = \sum_{j,k} (u_1^j u_2^k b)\otimes(\alpha_j\alpha_k m)$$

Take $B = A[u_1,u_2,u_1^{-1},u_2^{-1}]$: $(\alpha_i)_{i\in\mathbb{Z}}$ is an action iff: $\forall j,k$: $\alpha_j\alpha_k = \begin{cases} 0 & j\neq k \\ \alpha_j & j=k \end{cases}$

In other words: an action of $\mathbb{G}_{m,A}$ on M is a set of orthogonal idempotents in $\mathrm{End}_A(M)$, or, in yet other words, a $\mathbb{Z}$-grading of M: $M = \bigoplus_{i\in\mathbb{Z}} M_i$ $\quad (M_i = \alpha_i M)$

This decomposition is the one in <u>eigenspaces</u> for the $\mathbb{G}_{m,A}$-action!

Namely: $(\alpha u)\cdot(b\otimes(\alpha_j m)) = \sum_i (u^i b)\otimes(\alpha_i\alpha_j m) = (u^j b)\otimes(\alpha_j m)$

$= u^j\cdot(b\otimes\alpha_j m)$.

<u>Consequence</u>: now you know how to grade $\mathrm{Hom}_A(M,N)$, $M\otimes_A N$, etc. if M & N are graded! And you understand why one speaks of "weights". <u>Exercise</u>: prove that $\mathrm{Hom}(\mathbb{G}_{m,A}, \mathbb{G}_{m,A}) =$

$= \mathrm{Hom}_{(\mathrm{Top})}(\mathrm{Spec}\,A, (\mathbb{Z}, \text{discrete}))$.

$= (\mathbb{Z}_{\mathrm{Spec}A})(\mathrm{Spec}(A))$, global sections of const. sheaf $\mathbb{Z}$ on $\mathrm{Spec}A$.

Let $A \to S$ in (Ring). What is a $\mathbb{G}_{m,A}$-action on S:

well: it is an action on the A-<u>module</u> S such that $\forall B, \forall u\in B^\times$

αu is a <u>B-algebra automorphism</u> of $B\otimes_A S$.

Hence: it is a $\mathbb{Z}$-grading of S as A-algebra:

$S = \bigoplus_{i\in\mathbb{Z}} S_i$, and $\begin{cases} S_i\cdot S_j \subset S_{i+j}. \\ 1\in S_0. \end{cases}$ $\left(\alpha u : \begin{array}{l} s_i \mapsto u^i s_i \\ s_j \mapsto u^j s_j \\ \text{so } s_i s_j \mapsto u^{i+j} s_i s_j \end{array} \right)$.

$1 \mapsto u^0\cdot 1$.

We say S is <u>positively graded</u> if $\begin{cases} \forall i<0: S_i = \{0\}. \\ S_0 \end{cases}$

For $f\in S_d$, $\quad S \to S_f$, $\forall B$: $B\otimes_A(S_f) = (B\otimes_A S)_f$,

$$\begin{array}{cccc} \forall u\in B^\times & f & B\otimes_A S & \longrightarrow (B\otimes_A S)_f \\ & \updownarrow & \alpha u \downarrow\wr & \exists!\downarrow\wr \\ & u^d\cdot f & B\otimes_A S & \longrightarrow (B\otimes_A S)_f = (B\otimes_A S)_{u^d\cdot f} \end{array}$$

So we get a unique $\mathbb{Z}$-grading on S_f, such that $\frac{1}{f}\in (S_f)_{-d}$.

(Another way to see this: $S_f = S[x]/(x\cdot f - 1)$ homogeneous of degree 0 if $x\in (S[x])_{-d}$)

<u>Now</u> we are ready for the Proj-construction.