Week 11, November 17.

We continue with Picard groups and divisors. ([H], II.6).

Def. Let $i: Y \to X$ be a closed immersion in (Sch). Then we have a short exact seq. in $\mathrm{QCoh}(\mathcal{O}_X)$: $0 \to I_Y \to \mathcal{O}_X \xrightarrow{i^\#} i_*\mathcal{O}_Y \to 0$.
Then i is called locally principal if I_Y is locally free of rank 1 as $\mathcal{O}_X$-module, that is, if I_Y is locally generated by a non-zero divisor.

Such closed immersions are very useful to produce elements in $\mathrm{Pic}(X)$.

Example Let $X := \mathbb{P}^n_{\mathbb{Z}}$, $n \geq 1$, $Y = V(x_0)$. Then $\mathcal{O}_X(-1) \xrightarrow[\sim]{x_0 \cdot} I_Y \subset \mathcal{O}_X$.
Not all I_Y for $Y \subset X$ closed of "codimension one" are locally free of rank one (unless X has very nice properties, as we see later), so one also introduces the concept of Weil divisors on schemes X that are noetherian, integral, separated and "regular in codimension 1":
$\forall x \in X$ s.t. $\dim \mathcal{O}_{X,x} = 1$, $\mathcal{O}_{X,x}$ is regular (i.e., m_x is generated by 1 element, equivalently: $\mathcal{O}_{X,x}$ is a unique fact. domain with 1 prime, say π, $k(X)^\times = \mathrm{Frac}(\mathcal{O}_{X,x})^\times \xleftarrow{\sim} \mathcal{O}_{X,x}^\times \times \mathbb{Z}$, $v_x : k(X)^\times \to \mathbb{Z}$
$u \cdot \pi^n \longleftarrow\!\shortmid (u, n)$ $\quad$ ($u \cdot \pi^n \mapsto n$.
valuation at x.

I do not want to spend time on Weil divisors.
Please read (without too much detail) that part of [H], II.6.
Here is the most important stuff.

A prime divisor on X is an integral closed subscheme Y of codimension 1, meaning the closure $\overline{\{x\}}$ of an $x \in X$ with $\dim(\mathcal{O}_{X,x}) = 1$. $P(X) :=$ the set of prime div. on X.
$\mathrm{Div}(X) := \mathbb{Z}^{(P(X))}$, free $\mathbb{Z}$-module with basis $P(X)$.
For $f \in k(X)^\times$ one defines $\mathrm{div}(f) := \sum_{Y \in P(X)} v_Y(f) \cdot Y$ (v_Y = val. at gen. pt. of Y).
(principal divisors)
Then one has the (Weil) divisor class group:

$$k(X)^\times \xrightarrow{\mathrm{div}} \mathrm{Div}(X) \twoheadrightarrow \mathrm{Cl}(X).$$

We have seen this in the Spring course in the case of X nonsingular curve, and X nonsingular quasi-projective connected variety.

Assume now that X is as before: noetherian, integral, separated and regular in codim. 1, but that moreover X is locally factorial: $\forall x \in X$: $\mathcal{O}_{X,x}$ is a u.f.d. (for example this is the case if $\forall x \in X$, $\mathcal{O}_{X,x}$ is regular: $\dim_{\kappa(x)}(m_x/m_x^2) = \dim(\mathcal{O}_{X,x})$) this is a rather deep result from comm. algebra.

Prop. Let $Y \in P(X)$. Then Y is locally principal.

Proof Let $x \in X$. If $x \notin Y$ then 1 is a generator of I_Y near x. Suppose $x \in Y$. Then η_Y in $\mathrm{Spec}(\mathcal{O}_{X,x})$, is a prime ideal of height 1: $\eta_Y \supsetneq (0)$ & no prime ideal in between. Let $0 \neq f \in \eta_Y$ and consider a fact. in $\mathcal{O}_{X,x}$: $f = u \cdot \prod_i p_i$ with the $p_i \in \mathcal{O}_{X,x}$ prime elements. Then $\exists i$ s.t. $p_i \in \eta_Y$, hence $(p_i) = \eta_Y$, and p_i generates I_Y near x. $\boxtimes$.

Proposition ([H], II. Cor. 6.16). Under the assumptions above, we have an exact sequence of $\mathbb{Z}$-modules:

$$\mathcal{O}(X)^\times \hookrightarrow k(X)^\times \xrightarrow{\text{div}} \mathrm{Div}(X) \longrightarrow \mathrm{Pic}(X) \longrightarrow 0.$$

$$P(X) \ni Y \longmapsto [I_Y^\vee] = [\mathcal{O}_X(Y)] = [\mathcal{L}(Y)]$$

$$D \longmapsto [\mathcal{O}_X(D)]$$

where $\forall\, \emptyset \neq U \subset X$ open: $(\mathcal{O}_X(D))(U) = \{ f \in k(X)^\times : \mathrm{div}(f|_U) + D|_U \geq 0 \} \cup \{0\}$.

So, if we write $D = \sum_{Y \in P(X)} D(Y) \cdot Y$, then for $x \in X$, $\mathcal{O}_X(D)$ is generated near x by $\prod_{Y \ni x} t_Y^{-D(Y)}$, where $t_Y \in \mathcal{O}_{X,x}$ is a gen. of $I_{Y,x}$.

The multiplication map $k(X) \times k(X) \to k(X)$ induces $I_Y \times \mathcal{O}_X(Y) \to \mathcal{O}_X$, giving an isom. $I_Y^\vee \xrightarrow{\sim} \mathcal{O}_X(Y)$.

Surjectivity of $\mathrm{Div}(X) \to \mathrm{Pic}(X)$: Let $\mathcal{L}$ be an inv. $\mathcal{O}_X$-module. Let $\eta \in X$ be the generic pt. ($\mathcal{O}_{X,\eta} = k(X)$), let $s \in \mathcal{L}_\eta$, $s \neq 0$. Then we get a divisor $\mathrm{div}_{\mathcal{L}}(s) = \sum_{Y \in P(X)} v_{\mathcal{L},Y}(s) \cdot Y$, and one checks that $+\mathrm{div}_{\mathcal{L}}(s) \longmapsto [\mathcal{L}]$. $\boxtimes$

Rem. There is also a notion of Cartier divisors. On integral schemes this also leads to a Cartier divisor class group that is naturally isomorphic to $\mathrm{Pic}(X)$. We skip it.

Examples. 1. Let A be a noetherian unique fact. domain, and $X = \mathrm{Spec}(A)$. Then X satisfies the conditions, and $\mathrm{Pic}(X) = 0$, because $\forall\, Y \in P(X)$, I_Y is generated by one element.

2. Let X satisfy the hypotheses, and let $n \in \mathbb{Z}_{\geq 1}$.

Then

$$\begin{tikzcd} \mathbb{P}^n_X \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\square"] & \mathbb{P}^n \arrow[d] \\ X \arrow[r] & \mathrm{Spec}\,\mathbb{Z} \end{tikzcd}$$

satisfies the hypotheses, and

$$\mathbb{Z} \times \mathrm{Pic}(X) = \mathrm{Pic}(\mathbb{P}^n) \times \mathrm{Pic}(X) \xrightarrow[p_1^* + p_2^*]{} \mathrm{Pic}(\mathbb{P}^n_X)$$

is an isomorphism. (See [H] II.6.4 + 6.5 & 6.6)

3. Let k be a field, E/k an elliptic curve

for example $V(-y^2z + x^3 + axz^2 + bz^3) \subset \mathbb{P}^2_k$ with $4a^3 + 27b^2 \neq 0$ if $6 \in k^\times$.

Then $E(k) \rightarrowtail \mathrm{Pic}(E) \overset{\deg}{\twoheadrightarrow} \mathbb{Z}$. to be explained.

$$P \longmapsto [\mathcal{O}_E(P - 0)]$$

recall: on a projective nonsing. connected curve, principal divisors have degree zero.

Week 12 November 24

Some examples of morphisms to $\mathbb{P}^n$

1. Let k be a field, $n \in \mathbb{Z}_{\geq 1}$. Then $GL_{n+1}(k)$ acts on $\mathbb{A}^{n+1}_k$, as follows: for $g \in GL_{n+1}(k)$ and for $k \to A$ any k-algebra, g acts on $\mathbb{A}^{n+1}_k(A) = A^{n+1}$ by $(g, P) \mapsto g \cdot P$ (matrix times columnvector).

This action commutes with the scalar multiplication $\mathbb{G}_{m,k} \circlearrowleft \mathbb{A}^{n+1}_k - \{0\}$, hence we get a faithful action $GL_{n+1}(k)/k^\times \circlearrowleft \mathbb{P}^n_k$.

Thm. $\mathrm{Aut}_k(\mathbb{P}^n_k) = GL_{n+1}(k)/k^\times$.

Proof. [H], II.7.1.1. Idea. Let $\varphi : \mathbb{P}^n_k \xrightarrow{\sim} \mathbb{P}^n_k$ be an isom. of k-schemes. Then φ is given by $(\varphi^*\mathcal{O}(1), \varphi^*x_0, \ldots, \varphi^*x_n)$. As $\mathrm{Pic}(\mathbb{P}^n_k) \xrightarrow[\sim]{\deg} \mathbb{Z}$, we see that $\varphi^*\mathcal{O}(1) \cong \mathcal{O}(1)$. Take an isom. $\psi : \varphi^*\mathcal{O}(1) \xrightarrow{\sim} \mathcal{O}(1)$. Then $\varphi \sim (\mathcal{O}(1), \psi(\varphi^*x_0), \ldots, \psi(\varphi^*x_n))$, and use that $(\mathcal{O}(1))(\mathbb{P}^n_k) = k[x_0, \ldots, x_n]_1 = k\cdot x_0 \oplus \cdots \oplus k \cdot x_n$. $\boxtimes$

Rem: $\mathrm{Aut}_k(\mathbb{A}^2_k)$ is more complicated.

2. On $\mathbb{P}^n$, $(\mathcal{O}(2), x_0^2, \dots, x_n^2)$ gives
$f\colon \mathbb{P}^n \longrightarrow \mathbb{P}^n$, $(x_0^2 : \dots : x_n^2)$, such that $\forall A$,
$\forall (a_0 : \dots : a_n) \in \mathbb{P}^n(A)$, $f\colon (a_0 : \dots : a_n) \mapsto (a_0^2 : \dots : a_n^2)$.
$\Big\}$ f is ~~locally~~ finite free of degree 2^n.

3. On $\mathbb{P}^1$, $(\mathcal{O}(2), x_0^2, x_0x_1, x_1^2)$: $\mathbb{P}^1 \xrightarrow{f} \mathbb{P}^2$, $(a_0 : a_1) \mapsto (a_0^2 : a_0a_1 : a_1^2)$
and f is a closed immersion with image $V(x_0x_2 - x_1^2)$
To prove it: $f^{-1}D_+(x_0) = D_+(x_0)$, $\mathbb{Z}\left[\frac{x_1}{x_0}\right] \twoheadleftarrow \mathbb{Z}\left[\frac{x_1}{x_0}, \frac{x_2}{x_0}\right]$, $\left(\frac{x_1}{x_0}\right)^2 \leftarrow\!\shortmid \frac{x_2}{x_0}$, $\frac{x_1}{x_0} \leftarrow\!\shortmid \frac{x_1}{x_0}$.
(etc.)
$\frac{a_1}{a_0} = (a_0 : a_1) \longmapsto (a_0^2 : a_0a_1 : a_1^2) = \left(\frac{a_1}{a_0}, \left(\frac{a_1}{a_0}\right)^2\right)$

4. On $\mathbb{P}^1$, $(\mathcal{O}(4), x_0^4, x_0^3x_1, x_0^2x_1^2, x_0x_1^3)$: $\mathbb{P}^1 \xrightarrow{f_1} \mathbb{P}^3$, $(a_0 : a_1) \mapsto (a_0^4 : a_0^3a_1 : a_0^2a_1^2 : a_0a_1^3)$
$(\mathcal{O}(4), x_0^3x_1, x_0^2x_1^2, x_0x_1^3, x_1^4)$: $\mathbb{P}^1 \xrightarrow{f_2} \mathbb{P}^3$, $(a_0 : a_1) \mapsto (a_0^3a_1 : a_0^2a_1^2 : a_0a_1^3 : a_1^4)$
both are closed immersions, but they do not differ by an automorphism of $\mathbb{P}^3$. (but they differ by an automorphism of $\mathbb{P}^1$).

5. $\mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^3$, $((a_0 : a_1), (b_0 : b_1)) \mapsto (a_0b_0 : a_0b_1 : a_1b_0 : a_1b_1)$
is a closed immersion.
It corresponds to: $(p_1^*\mathcal{O}(1) \otimes_{\mathcal{O}} p_2^*\mathcal{O}(1),\ p_1^*x_0 \otimes p_2^*x_0,\ p_1^*x_0 \otimes p_2^*x_1$, etc. $)$
So, in this case, "Yoneda" gives a somewhat simpler description than via line bundles. "local".

Remark. We see that to know generating spaces of sections of ~~a~~ invertible $\mathcal{O}$-modules is very good for us. Cohomology and duality is a good tool to be able to do so on more general spaces than just projective spaces and products of them. For example, think about curves of higher genus, to start with.
So, next time we go to cohomology, in [H], III.