Lecture 15. Serre duality on curves.

We follow [H] III §7 except that we use a finite morphism to $\mathbb{P}^1$ instead of an embedding in $\mathbb{P}^n$. See also Lipman, Astérisque 117. Of course, the approach is due to Grothendieck. The idea is to generalise it ~~to~~ from invertible $\mathcal{O}$-modules to coherent $\mathcal{O}$-modules.

We begin with $\mathbb{P}^1_k$. (So, let k be a field.) For V a k-vect.sp., $V^\vee := \mathrm{Hom}(V,k)$.

Thm. Let $Y = \mathbb{P}^1_k$. The functor $\mathrm{Coh}(\mathcal{O}_Y) \longrightarrow k\text{-vect.sp.}$, $\mathcal{F} \longmapsto H^1(Y,\mathcal{F})^\vee$ is represented by $\mathcal{O}(-2)$.

Proof. First we give a morphism of functors $\mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{F},\mathcal{O}(-2)) \to H^1(Y,\mathcal{F})^\vee$.

For $\varphi : \mathcal{F} \to \mathcal{O}(-2)$, we have $H^1(Y,\mathcal{F}) \xrightarrow{H^1(Y,\varphi)} H^1(Y,\mathcal{O}(-2)) = k$.

This is functorial in $\mathcal{F}$. Let us now show that it is an isomorphism.

For $\mathcal{F} = \mathcal{O}(n)$: $\mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{O}(n),\mathcal{O}(-2)) = \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{O},\mathcal{O}(-2-n)) =$

$= k[x_0,x_1]_{-2-n}$, $\quad H^1(Y,\mathcal{O}(n)) = \left(\frac{1}{x_0x_1}k[x_0^{-1},x_1^{-1}]\right)_n$

basis $x_0^{i_0}x_1^{i_1}$, $i_0+i_1 = -2-n$, $i_0 \geq 0$, $i_1 \geq 0$ $\qquad = \frac{1}{x_0x_1}\cdot \underbrace{k[x_0^{-1},x_1^{-1}]_{n+2}}_{\text{basis } x_0^{j_0}\cdot x_1^{j_1},\ j_0+j_1=n+2,\ j_0\leq 0,\ j_1\leq 0.}$

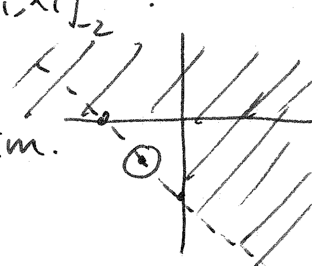
The map $\mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{O}(n),\mathcal{O}(-2)) \times H^1(Y,\mathcal{O}(n)) \to H^1(Y,\mathcal{O}(-2))$

is given by $\left(x_0^{i_0}x_1^{i_1},\ x_0^{j_0}x_1^{j_1}\cdot\frac{1}{x_0x_1}\right) \longmapsto \left[x_0^{i_0+j_0}x_1^{i_1+j_1}\cdot\frac{1}{x_0x_1}\right] \in H^1(Y,\mathcal{O}(-2))$

$= 0 \Leftrightarrow (i_0+j_0,\ i_1+j_1) \neq (0,0)$

Recall Čech complex for $\mathcal{O}(-2)$:

$$k[x_0,x_1,x_0^{-1}]_{-2} \oplus k[x_0,x_1,x_1^{-1}]_{-2} \longrightarrow k[x_0,x_0^{-1},x_1,x_1^{-1}]_{-2} :$$



So, for all $\mathcal{O}(n)$, the map

$\mathrm{Hom}(\mathcal{O}(n),\mathcal{O}(-2)) \to H^1(Y,\mathcal{O}(n))^\vee$ is an isomorphism.

Let now $\mathcal{F}$ be in $\mathrm{Coh}(\mathcal{O}_Y)$. We take a presentation $\mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{F} \to 0$, with $\mathcal{E}_1$ and $\mathcal{E}_0$ finite direct sums of $\mathcal{O}_Y(n_i)$'s. Then we have:

$$\begin{array}{ccccccc}
0 \to & \mathrm{Hom}(\mathcal{F},\mathcal{O}(-2)) & \to & \mathrm{Hom}(\mathcal{E}_0,\mathcal{O}(-2)) & \to & \mathrm{Hom}(\mathcal{E}_1,\mathcal{O}(-2)) \\
 & \downarrow & & \downarrow\wr & & \downarrow\wr \\
0 \to & H^1(Y,\mathcal{F})^\vee & \longrightarrow & H^1(Y,\mathcal{E}_0)^\vee & \longrightarrow & H^1(Y,\mathcal{E}_1)^\vee
\end{array}$$

$\boxtimes$

Let now $X \to \operatorname{Spec}(k)$ be smooth of rel. dim. 1, projective and geom. irreducible. For example: $X = V(F) \subset \mathbb{P}^2_k$, $F \in k[x_0, x_1, x_2]_d$, s.t. $V\left(F, \frac{\partial F}{\partial x_0}, \frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}\right)$ is empty in $\mathbb{A}^3_k - \{0\}$.

__Proposition.__ There is a finite morphism $f \colon X \to \mathbb{P}^1_k$, over k.

__Proof__ We can embed X into $\mathbb{P}^n_k$ (some n, sufficiently large if k is finite) s.t. $V(x_0, x_1) \cap X = \emptyset$. Then let $L = V(x_0, x_1)$,

$$\begin{tikzcd} \pi\colon \mathbb{P}^n - L \arrow[r] & \mathbb{P}^1_k \\ X \arrow[u, "i"] \arrow[ur, "f"'] & \end{tikzcd} \quad : (a_0 : \cdots : a_n) \mapsto (a_0 : a_1)$$

Then f is affine: $f^{-1} D_+(x_0)$ is closed in $\pi^{-1} D_+(x_0) = D_+(x_0)$ is affine, and similarly for $f^{-1} D_+(x_1)$.

Also: $f_* \mathcal{O}_X$ is a coherent $\mathcal{O}_{\mathbb{P}^1_k}$-module because f is projective:

$$\begin{tikzcd} & \mathbb{P}^n_k \times_k \mathbb{P}^1_k \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathbb{P}^n_k \arrow[d] \\ X \arrow[ur] \arrow[urr, bend left] \arrow[r, "f"'] & \mathbb{P}^1_k \arrow[r] & \operatorname{Spec}(k). \end{tikzcd} \quad \boxtimes$$

__Proposition.__ ([H], Exercise II.5.17(e)) Let $f \colon X \to Y$ be an affine morphism of schemes. Then $f_* \colon \mathrm{QCoh}(\mathcal{O}_X) \longrightarrow f_*\mathcal{O}_X\text{-mod}$ induces an equivalence with the category of $f_*\mathcal{O}_X$-modules that are quasi-coherent as $\mathcal{O}_Y$-modules. An inverse is given by:

$\mathcal{M} \mapsto \tilde{\mathcal{M}}$, where $\tilde{\mathcal{M}}|_{f^{-1}U} = \widetilde{\mathcal{M}(U)}$ where we view $\mathcal{M}(U)$ as
($U \subset Y$ open affine) $\mathcal{O}_X(f^{-1}U)$-module

__Proof__ If Y, and hence X, are affine, say $\operatorname{Spec} A$ and $\operatorname{Spec} B$, $A \to B$, then f_* is the functor that sends a B-module M to the underlying A-module ${}_AM$, plus its structure of B-module. So that is an equivalence. To finish: compatibility with localisation. $\boxtimes$

Let X/k, $f : X \to Y$ over k be as before. ($Y = \mathbb{P}^1_k$).

<u>Thm.</u> The functor $\mathrm{Coh}(\mathcal{O}_X) \longrightarrow k\text{-v.sp}$, $\mathcal{F} \mapsto H^1(X, \mathcal{F})^\vee$, is representable.

<u>Proof</u> We have the $f_*\mathcal{O}_X$-module $\mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{O}(-2))$ on Y. Let ω_X denote the coherent $\mathcal{O}_X$-module corresponding to it.

Consider:

$$\begin{array}{ccc}
\mathrm{Coh}(\mathcal{O}_X) & \xrightarrow{\mathcal{H}om_{\mathcal{O}_X}(-,\omega_X)} & \mathrm{Coh}(\mathcal{O}_X) \\
{\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\
\mathrm{Coh}(\mathcal{O}_Y) & \xrightarrow[\mathcal{H}om_{\mathcal{O}_Y}(-,\mathcal{O}(-2))]{} & \mathrm{Coh}(\mathcal{O}_Y) \\
{\scriptstyle H^1(Y,\cdot)}\downarrow & & \downarrow{\scriptstyle H^0(Y,\cdot)} \\
k\text{-v.sp.} & \xrightarrow[\mathrm{Hom}_k(\cdot,k)]{} & k\text{-v.sp.}
\end{array}$$

We claim it is commutative.

<u>Lower square:</u>

$$\begin{array}{ccccc}
\mathcal{F} & \longmapsto & & & \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{O}(-2)) \\
\rotatebox{90}{$\mapsto$} & & & & \rotatebox{90}{$\mapsto$} \\
H^1(Y,\mathcal{F}) & \longmapsto & H^1(Y,\mathcal{F})^\vee & \xleftarrow{\sim} & \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{F}, \mathcal{O}(-2))
\end{array}$$

(the isom. above.

<u>Upper square</u>. Restrict to $U = \mathrm{Spec}\, A \subset Y$, $\mathrm{Spec}\, B = f^{-1}U \subset X$, M a B-module, N an A-module. Then: $\mathrm{Hom}_B(M, \mathrm{Hom}_A(B,N)) = \mathrm{Hom}_A({}_AM, N)$.
($N \mapsto \mathrm{Hom}_A(B,N)$ right adjoint to $M \mapsto {}_AM$).

<u>**Note:**</u> $H^0(Y,\cdot) \circ f_* \circ \mathcal{H}om_{\mathcal{O}_X}(\cdot, \omega_X) = \mathrm{Hom}_{\mathcal{O}_X}(\cdot, \omega_X)$, by definition.

$H^1(Y,\cdot) \circ f_* = H^1(X, \cdot)$: consider Čech complex for $Y = D_+(x_0) \cup D_+(x_1)$, and f affine. $\boxtimes$

<u>Remark</u> For $\mathcal{F}$ a loc. free $\mathcal{O}_X$-module of finite rank:

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_X) = \big(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega_X)\big)(X) = (\mathcal{F}^\vee \otimes_{\mathcal{O}_X} \omega_X)(X) = H^0(X, \mathcal{F}^\vee \otimes_{\mathcal{O}_X} \omega_X).$$

<u>Thm.</u> $\omega_X = \Omega^1_{X/k}$. For a proof, see Lipman, Hartshorne, or my syllabus "variétés abéliennes". No time for details now.

<u>Ext and cohomology</u> For exercise 7(b) of the assignment.

Let $(X, \mathcal{O})$ be a ringed space.

Then $\forall$ $\mathcal{O}$-module $\mathcal{F}$: $\quad \mathrm{Hom}_{\mathcal{O}\text{-mod}}(\mathcal{O}, \mathcal{F}) \longrightarrow \mathcal{F}(X)$ is an isom. of $\mathcal{O}(X)$-modules.

$$\varphi \longmapsto \left(\begin{array}{c} \mathcal{O}(X) \xrightarrow{\varphi(X)} \mathcal{F}(X) \\ 1 \longmapsto (\varphi X)(1) \end{array} \right)$$

Therefore, the derived functors are the same.

That is: $\forall i \in \mathbb{Z}_{\geq 0}$: $\quad \mathrm{Ext}^i_{\mathcal{O}\text{-mod}}(\mathcal{O}, \mathcal{F}) = H^i(X, \mathcal{F})$.

These are by definition the derived functors of Hom; can be computed using a projective res. of $\mathcal{O}$, or an inj. res. of $\mathcal{F}$.

<u>Ext^1 and extensions.</u> Let $\mathcal{A}$ be an abelian category, with sufficiently many injectives. Then $\forall$ A in $\mathcal{A}$: $\mathrm{Hom}(A, \cdot)$ is left-exact, and its derived functors are denoted $\mathrm{Ext}^i(A, \cdot)$.

For A and B in $\mathcal{A}$, an extension of A by B is a short exact sequence: $\quad 0 \to B \to E \to A \to 0$, and one has the notion of isomorphism:

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r, "i"] \arrow[d, "\mathrm{id}_B"'] \arrow[dr, phantom, "\circlearrowleft"] & E \arrow[r, "p"] \arrow[d, "\wr"] \arrow[dr, phantom, "\circlearrowleft"] & A \arrow[r] \arrow[d, "\mathrm{id}_A"] & 0 \\ 0 \arrow[r] & B \arrow[r, "i'"] & E' \arrow[r, "p'"] & A \arrow[r] & 0. \end{tikzcd}$$

<u>Def.</u> $\quad \mathrm{Ext}(A, B) := \{ 0 \to B \to E \to A \to 0 \} / \cong$.

<u>Thm.</u> $\quad \mathrm{Ext}(A, B) \longrightarrow \mathrm{Ext}^1(A, B)$ is a bijection.

$$(0 \to B \to E \to A \to 0) \underset{\mathrm{Hom}(-,B)}{\longmapsto} 0 \to \mathrm{Hom}(A, B) \to \mathrm{Hom}(E, B) \to \underset{\substack{\Downarrow \\ \mathrm{id}_B}}{\mathrm{Hom}(B, B)} \to \underset{\substack{\Downarrow \\ \text{image of } \mathrm{id}_B}}{\mathrm{Ext}^1(A, B)}$$

$$\mathrm{id}_B \longmapsto \text{image of } \mathrm{id}_B$$