

Uitwerkingen werkcollege 3.

2.1.2 First we need to check that if we “add” two elements $x, y \in V$ with the usual operation \oplus , we actually get an element of V . This is indeed the case because if $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ satisfy $x_1 + x_2 + x_3 = 0$ and $y_1 + y_2 + y_3 = 0$, then the sum $z = x \oplus y$ equals (z_1, z_2, z_3) with $z_i = x_i + y_i$ for $1 \leq i \leq 3$, and we have

$$\begin{aligned} z_1 + z_2 + z_3 &= (x_1 + y_1) + (x_2 + y_2) + (x_3 + y_3) \\ &= (x_1 + x_2 + x_3) + (y_1 + y_2 + y_3) = 0 + 0 = 0, \end{aligned}$$

so also $z \in V$.

We also need to check that every scalar multiple of any element $x = (x_1, x_2, x_3) \in V$ is again contained in V . This is indeed the case, because for every $\lambda \in \mathbb{R}$ we have $\lambda \odot x = (\lambda x_1, \lambda x_2, \lambda x_3)$ and if $x_1 + x_2 + x_3 = 0$, then $\lambda x_1 + \lambda x_2 + \lambda x_3 = \lambda(x_1 + x_2 + x_3) = 0$, so $\lambda \odot x \in V$.

Now that we know that the operations \oplus and \odot are indeed well defined, we can check whether together with the zero $0_V = (0, 0, 0)$, they satisfy the eight axioms of Definition 2.1.

To check that for all $x, y \in V$ we have $x \oplus y = y \oplus x$, we note that we already knew this for all $x, y \in \mathbb{R}^3$, so it certainly holds for all $x, y \in V$, because V is a subset of \mathbb{R}^3 . Similarly, all axioms, except for the fourth, hold for all elements in \mathbb{R}^3 , so certainly for all elements in V .

It remains to check the fourth axiom. For $x = (x_1, x_2, x_3) \in V$ we define $x' = (-x_1, -x_2, -x_3)$, of which the sum of coordinates is $(-x_1) + (-x_2) + (-x_3) = -(x_1 + x_2 + x_3) = 0$, so we have $x' \in V$. Of course, we have $x \oplus x' = (x_1 + (-x_1), x_2 + (-x_2), x_3 + (-x_3)) = (0, 0, 0) = 0_V$, so also the fourth axiom is satisfied. It follows that V , together with the usual coordinate-wise addition and scalar multiplication and 0_V as zero vector is indeed a vector space.

2.1.3 In this case, V is not a vector space, because it is not closed under addition. To show this, it suffices to give two elements $x, y \in V$ with $x \oplus y \notin V$. For example, the elements $x = (1, 0, 0)$ and $y = (0, 1, 0)$ are contained in V , but $x \oplus y = (1, 1, 0)$ is not. [Alternatively, one can show that V is not closed under scalar multiplication: if $x \in V$, then $2x \notin V$. Or one notes that the suggested zero vector is not contained in V .]

2.1.4

a	b	$a \odot b$	$a \oplus b$	Remark
F	F	F	F	ordinary multiplication and addition on F
F	V	V	\times	
V	V	\times	V	

- 2.2.9 (3) Yes, all axioms can be checked.
 (4) Yes, all axioms can be checked.
 (5) No, the sum of two elements is not even in V , and the scalar multiples (by scalars other than 1) are not in V either, so we have no addition and scalar multiplication on V . So we can not even phrase any of the axioms, except for axiom (6).
 (6) Yes, all axioms can be checked.

(7) No, the sum of two elements is not even in V , and the scalar multiples (by scalars other than 1) are not in V either, so we have no addition and scalar multiplication on V . So we can not even phrase any of the axioms, except for axiom (6).

2.2.10 By Lemma 1.25, the set a^\perp is closed under addition and scalar multiplication: for every $x, y \in a^\perp$ and $\lambda \in \mathbb{R}$, we have $x \oplus y \in a^\perp$ and $\lambda \odot x \in a^\perp$. Therefore, we do indeed have a well defined addition and scalar multiplication on a^\perp . As in Exercise 2.1.2, we note that all axioms except axiom (4) state that some property has to hold for all vectors (or pairs or triples of vectors) in a^\perp and perhaps all scalars (or pairs of scalars). Given that we already know these properties hold for all (pairs or triples of) vectors in \mathbb{R}^n , they certainly hold for all (pairs or triples of) vectors in a^\perp . As for axiom (4), for every $x \in a^\perp$ we know that $x' = (-1) \cdot x$ is contained in a^\perp , and $x + x' = (1 + (-1))x = 0 \cdot x = 0$, so also axiom (4) is satisfied. This finishes the proof. Cf. Lemma 3.2.

2.3.2 For (3), we have

$$\lambda \odot 0_V = \lambda \odot (0_V \oplus 0_V) = (\lambda \odot 0_V) \oplus (\lambda \odot 0_V),$$

where the first equality follows from axiom (3) and the second from axiom (7). Proposition 2.16 applied to the case $0' = z = \lambda \odot 0_V$ then yields $\lambda \odot 0_V = 0_V$.

Statement (4) follows from statements (2) and (3) with $x = 0_V$ and $\lambda = -1$.