# INEQUALITIES INVOLVING RESULTANTS

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For $f(X) = a_0X^r + a_1X^{r-1} + \dots + a_r$,
$g(X) = b_0X^t + b_1X^{t-1} + \dots + b_t \in \mathbb{Z}[X]$

define the <u>resultant</u>

$$R(f,g) = \begin{array}{|cccccccc|l}
a_0 & a_1 & \cdots & a_r & & & & & \\
 & a_0 & a_1 & \cdots & a_r & & & & \\
 & & \ddots & & & \ddots & & & t \\
 & & & a_0 & a_1 & \cdots & a_r & & \\
b_0 & b_1 & \cdots & b_t & & & & & \\
 & \ddots & & & \ddots & & & & r \\
 & & & b_0 & b_1 & \cdots & b_t & &
\end{array}$$

## PROPERTIES:

1) If $f(X) = a_0 \prod_{i=1}^{r}(X-\alpha_i)$, $g(X) = b_0 \prod_{j=1}^{t}(X-\beta_j)$
then
$R(f,g) = a_0^t\, g(\alpha_1)\cdots g(\alpha_r) = a_0^t b_0^r \prod_{i=1}^{r}\prod_{j=1}^{t}(\alpha_i-\beta_j)$

2) $|R(f,g)| \leq (r+1)^{t/2}\cdot(t+1)^{r/2}\, H(f)^t H(g)^r$,
where $H(f) = \max(|a_0|,\dots,|a_r|)$
$H(g) = \max(|b_0|,\dots,|b_t|)$

Fix $f \in \mathbb{Z}[X]$ of degree $r$.

Let $\varkappa > 0$.

Consider

(1) $0 < |R(f,g)| \leq H(g)^{r-\varkappa}$

in polynomials $g \in \mathbb{Z}[X]$ of degree $t$

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**Example:** $t = 1$, $g = g_0 X + g_1$. Then

$$R(f,g) = a_0(g_0\alpha_1 + g_1)\cdots(g_0\alpha_r + g_1) = F(g_0, g_1),$$

and (1) becomes

$$0 < |F(g_0,g_1)| \leq \{\max(|g_0|,|g_1|)\}^{r-\varkappa}$$

**THEOREM (Roth, 1955)**

If $f$ has only single zeros, $t=1$, $\varkappa > 2$, then (1) has only finitely many solutions.



Now let $t \geq 2$.

**THEOREM (Wirsing, Schmidt, Fujiwara, Ru & Wong)**

Let $f \in \mathbb{Z}[X]$ be a polynomial of degree $r$ with only single zeros. Let $\varkappa > 2t$. Then the inequality

(1) $0 < |R(f,g)| \leq H(g)^{r-\varkappa}$

in polynomials $g \in \mathbb{Z}[X]$ of degree $t$

has only finitely many solutions.

**HISTORY:**

Wirsing (1970) $\varkappa > 2t(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2t-1})$

Fujiwara (1971) $\varkappa > 2t$, $f$ irreducible

Schmidt (1971) $\varkappa > 2t$, $f$ no irreducible factors of degree $\leq t$

Ru & Wong (1991) general result

→ applied Schmidt's Subspace Thm.

## FACTS (Schmidt, 1971)

For every $f \in \mathbb{Z}[X]$ of degree $r$ with only single zeros, there is a $C$ such that

$$0 < |R(f,g)| \leq C \cdot H(g)^{r-t-1}$$

has infinitely many solutions $g \in \mathbb{Z}[X]$ of degree $t$

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There are infinitely many $r$ for which there exist $f \in \mathbb{Z}[X]$ of degree $r$ with only single zeros, and a constant $C$, such that

$$0 < |R(f,g)| \leq C \cdot H(g)^{r-2t}$$

has infinitely many solutions $g \in \mathbb{Z}[X]$ of degree $t$.

# A QUANTITATIVE RESULT

**THEOREM (E.)**

Let $f \in \mathbb{Z}[X]$ be a polynomial of degree $r$ with only single zeros.

Let $\varkappa = 2t + \delta$, $0 < \delta < 1$.

Then the number of polynomials $g \in \mathbb{Z}[X]$ of degree $t$ with

(a) $0 < |R(f,g)| \leq H(g)^{r-\varkappa}$

(b) $H(g) \geq (2^{12r^3} \cdot H(f)^{4r^2})^{1/\delta}$

c) $g$ irreducible, primitive

is at most

$(25t)^{2t+20} \cdot \underline{\delta^{-t-5}} \cdot r^t \log 4r \cdot \log\log 4r$.



# REMARKS

1) In 1996 I proved a similar result (with a slightly better bound) for $\varkappa > 2t(1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2t-1})$ using Wirsing's method

2) A result similar to Theorem 1 (with upper bound depending on $t$ like $c^{t^2}$) can be derived from work of **Hirata-Kohno** (1998)

# SKETCH OF PROOF OF THM.

Go through proof of Subspace Thm. and show that all occurring subspaces are one-dimensional.

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Take $g = g_0 X^t + g_1 X^{t-1} + \dots + g_t \in \mathbb{Z}[X]$ satisfying (a), (b), (c) of the Theorem, i.e.

(a) $0 < |R(f,g)| \leq H(g)^{r-\varkappa}$ with $\varkappa = 2t + \delta$

(b) $H(g) \geq \left(2^{12r^3} \cdot H(f)^{4r^2}\right)^{1/\delta}$

(c) $g$ irreducible, primitive.

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To $g$ we associate a convex body

$$C(g) = \left\{ \begin{array}{l} \underline{x} = (x_0, \dots, x_t) \in \mathbb{R}^{t+1} \\ |x_0 \alpha_i^t + x_1 \alpha_i^{t-1} + \dots + x_t| \leq |g(\alpha_i)| \\ \qquad\qquad (i = 1, \dots, r) \end{array} \right\}$$

Note that $\underline{g} = (g_0, g_1, \dots, g_t) \in C(g)$.



Let $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_{t+1}$ be the successive minima of

$$C(g) = \left\{ \underline{x} \in \mathbb{R}^{t+1} : \left| \sum_{i=0}^{t} x_i \alpha_i^{t-i} \right| \leq |g(\alpha_i)| \quad (i = 1, \dots, r) \right\}$$

So

$$\lambda_k = \min \left\{ \lambda > 0 : \begin{array}{l} \lambda C(g) \cap \mathbb{Z}^{t+1} \text{ contains} \\ k \text{ \underline{linearly} independent points} \end{array} \right\}$$

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Choose linearly independent $\underline{h}_1, \underline{h}_2, \dots, \underline{h}_{t+1} \in \mathbb{Z}^{t+1}$ such that

$$\underline{h}_1, \dots, \underline{h}_k \in \lambda_k \cdot C(g) \qquad (k = 1, \dots, t+1)$$

Put $V_k = V_k(g) = \text{span}\{\underline{h}_1, \dots, \underline{h}_k\}$.

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**FACTS:** If $\lambda_k \geq 1$, $\lambda_{k+1} > \lambda_k$ then

$$\underline{g} = (g_0, g_1, \dots, g_t) \in \lambda_k \cdot C(g)$$

$$\underline{g} \in V_k(g).$$

The proof of the Subspace Theorem runs as follows:

(i) $\exists k$ with $\lambda_k \geq 1$ and $\lambda_{k+1}/\lambda_k$ large
$\Rightarrow \underline{g} = (g_0, g_1, \dots, g_t) \in V_k(g)$.

(ii). $V_k(g)$ belongs to a finite collection of $k$-dimensional spaces independent of $g$.

**LEMMA:** In our situation we may take $k=1$, more precisely

$$\lambda_1(g) = 1, \quad \lambda_2(g) \geq H(g)^{3\delta/4t}, \quad \delta = \varkappa - 2t$$

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So $g$ lies in the union of finitely many one-dimensional subspaces

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In the proof of the lemma it is crucial that $g$ is irreducible.



What about the number of reducible polynomials $g \in \mathbb{Z}[X]$ with

(1) $0 < |R(f,g)| \leq H(g)^{r-\varkappa}$ ?

**THEOREM (E.)** Let $f \in \mathbb{Z}[X]$ be a polynomial of degree $r$ with only single zeros. Let $t = 2$, $\varkappa > 4$.
Let $N$ be the number of primitive, reducible $g \in \mathbb{Z}[X]$ of degree 2 with (1).
Then for every zero $\alpha$ of $f$, and every pair $(x,y) \in \mathbb{Z}^2$ with $\frac{x}{y} \neq \alpha$, we have

$$\left|\alpha - \frac{x}{y}\right| \geq C^{\mathrm{eff}}(f) \cdot N^{-\varkappa/2} \cdot \{\max(|x|,|y|)\}^{-\varkappa}.$$

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Similar observation by Schmidt in 1988: explicit upper bounds for the number of solutions of certain inequalities imply strong effective results for other inequalities.