LINEAR EQUATIONS WITH UNKNOWNS FROM A MULTIPLICATIVE GROUP WHOSE SOLUTIONS LIE IN A FEW SUBSPACES

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K is a field of characteristic 0.
K^* is the multiplicative group of K.

Γ is a subgroup of K^* of finite rank, i.e., there is a free subgroup Γ_0 of Γ of finite rank such that for every $x \in \Gamma$ $\exists m \in \mathbb{N}$ with $x^m \in \Gamma_0$.

Define $\operatorname{rank} \Gamma := \operatorname{rank} \Gamma_0$.

We consider equations

(1) $a_1 x_1 + \cdots + a_n x_n = 1 \quad$ in $x_1, \ldots, x_n \in \Gamma$

with $a_1, \ldots, a_n \in K^*$.

Theorem A. (Schlickewei, Schmidt, E., 2002) *Let* $\operatorname{rank} \Gamma = r$. *Then the number of non-degenerate solutions of* (1), *i.e., with*

$$\sum_{i \in I} a_i x_i \neq 0 \quad \textit{for each subset } I \textit{ of } \{1, \ldots, n\},$$

is at most $e^{(6n)^{4n}(r+1)}$.

Two tuples $(a_1, \ldots, a_n)$, $(b_1, \ldots, b_n) \in (K^*)^n$ are called Γ-*equivalent* if

$$\frac{b_1}{a_1} \in \Gamma, \ldots, \frac{b_n}{a_n} \in \Gamma .$$

Two equations

(1) $\quad a_1 x_1 + \cdots + a_n x_n = 1 \quad$ in $x_1, \ldots, x_n \in \Gamma$

with Γ-equivalent tuples of coefficients have the same number of non-degenerate solutions.

Aim: Obtain more precise results for the set of solutions of (1), valid for "almost all" equivalence classes of $(a_1, \ldots, a_n)$.

Theorem B. (Győry, Stewart, Tijdeman, E., 1988)
Let Γ *be a given subgroup of* K^* *of finite rank. Then for all pairs* $(a, b) \in (K^*)^2$ *with the exception of finitely many* Γ*-equivalence classes, the equation*

$$ax + by = 1 \quad \textit{in } x, y \in \Gamma$$

has at most **two** *solutions.*

Fact: The bound 2 is best possible.

Problem: How to generalize this to equations in $n \geqslant 3$ unknowns?

Let $n \geqslant 3$.

Fact: (Győry, Stewart, Tijdeman, E., 1988) *For every h, there exist a multiplicative subgroup Γ of $\mathbb{Q}^*$ of finite rank, and infinitely many Γ-equivalence classes of tuples $(a_1, \ldots, a_n) \in (\mathbb{Q}^*)^n$, such that the equation*

$$a_1x_1 + \cdots + a_nx_n = 1 \quad \textit{in } x_1, \ldots, x_n \in \Gamma$$

has at least h non-degenerate solutions.

Let Γ be a given subgroup of K^* of finite rank.

Theorem C. (Győry, E., 1989)
For all tuples $(a_1, \ldots, a_n) \in (K^*)^n$ with the exception of finitely many Γ-equivalence classes, the set of solutions of

$$a_1x_1 + \cdots + a_nx_n = 1 \quad \textit{in } x_1, \ldots, x_n \in \Gamma$$

is contained in the union of not more than $2^{(n+1)!}$ proper linear subspaces of K^n.

Improvements:
E. (1993): $(n!)^{2n+2}$;
E. (2000) 2^{n^2} (unpublished)

Theorem. (E.) *For all tuples* $(a_1, \ldots, a_n) \in (K^*)^n$ *with the exception of finitely many* Γ-*equivalence classes, the set of* **non-degenerate** *solutions of*

(1) $\quad a_1x_1 + \cdots + a_nx_n = 1 \quad$ *in* $x_1, \ldots, x_n \in \Gamma$

is contained in the union of not more than

$$2^n$$

proper linear subspaces of K^n.

Remark. The degenerate solutions lie in at most 2^n subspaces $\sum_{i \in I} a_i x_i = 0$ ($I \subseteq \{1, \ldots, n\}$).

Our main tool.

Assume w.l.o.g. that K is algebraically closed. Let Γ be a subgroup of K^* of finite rank.

View $(K^*)^n$ as an algebraic group with coordinatewise multiplication $(x_1, \ldots, x_n) * (y_1, \ldots, y_n) = (x_1 y_1, \ldots, x_n y_n)$.

Let X be an algebraic subvariety of $(K^*)^n$.

Call a point $\mathbf{x} \in X$ *degenerate* if there is a one-dimensional algebraic subgroup H of $(K^*)^n$ with $\mathbf{x} * H \subset X$, and non-degenerate otherwise.

Theorem D. (Laurent, 1980's)
X has at most finitely many non-degenerate points with coordinates in Γ.

Remark. A one-dimensional algebraic subgroup H of $(K^*)^n$ can be expressed as

$$H = \{(\lambda^{c_1}, \ldots, \lambda^{c_n}) : \lambda \in K^*\}$$

where $c_1, \ldots, c_n$ are integers with gcd 1.

Hence $\mathbf{x} = (x_1, \ldots, x_n)$ is a degenerate point of X if and only if there are integers $c_1, \ldots, c_n$ with gcd 1 such that

$$(\lambda^{c_1} x_1, \ldots, \lambda^{c_n} x_n) \in X \quad \text{for every } \lambda \in K^*.$$

A reduction.

Consider tuples $(a_1, \ldots, a_n) \in (K^*)^n$ such that

(1) $\quad a_1x_1 + \cdots + a_nx_n = 1 \quad$ in $x_1, \ldots, x_n \in \Gamma$

has non-degenerate solutions.

Every Γ-equivalence class of such tuples contains a **normalized** tuple, i.e., a tuple $(a_1, \ldots, a_n)$ such that $(1, 1, \ldots, 1)$ is a non-degenerate solution of (1).

Hence it suffices to show:

Theorem.
For all but finitely many normalized tuples $\mathbf{a} \in (K^*)^n$, the set of non-degenerate solutions of (1) *is contained in the union of not more than 2^n proper linear subspaces of K^n.*

For every $\mathbf{a} = (a_1, \ldots, a_n) \in (K^*)^n$ the equation

(1) $\quad a_1 x_1 + \cdots + a_n x_n = 1 \quad$ in $x_1, \ldots, x_n \in \Gamma$

has at most $A := e^{(6n)^{4n}(r+1)}$ non-degenerate solutions where $r = \text{rank } \Gamma$.

Given a normalized tuple $\mathbf{a}$, we can order the non-degenerate solutions of (1) in a sequence

$$(1, \ldots, 1), (x_{21}, \ldots, x_{2n}), \ldots, (x_{A1}, \ldots, x_{An}),$$

where we have copied some of the solutions if the number of non-degenerate solutions is smaller than A.

Thus we get

$$\text{rank} \begin{pmatrix} 1 & \cdots & 1 & 1 \\ x_{21} & \cdots & x_{2n} & 1 \\ \vdots & & \vdots & \vdots \\ x_{A1} & \cdots & x_{An} & 1 \end{pmatrix} \leqslant n\,.$$

This defines an algebraic subvariety X of $(K^*)^{n(A-1)}$ which is independent of $\mathbf{a}$.

Each normalized tuple of coefficients $\mathbf{a} = (a_1, \ldots, a_n)$ gives rise to a point $(x_{21}, \ldots, x_{An}) \in X$ with coordinates in Γ.

$\mathbf{a} \in$ **CLASS I**
if $(x_{21}, \ldots, x_{An})$ is a non-degenerate point of X.

$\mathbf{a} \in$ **CLASS II**
if $(x_{21}, \ldots, x_{An})$ is a degenerate point of X.

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We will prove:

CLASS I is finite.

If $\mathbf{a}$ is in CLASS II, then the non-degenerate solutions of

$$a_1 x_1 + \cdots + a_n x_n = 1 \quad \text{in } x_1, \ldots, x_n \in \Gamma$$

lie in not more than 2^n subspaces of K^n.

CLASS I.

$\mathbf{a} = (a_1, \ldots, a_n)$ is such that $(x_{21}, \ldots, x_{An})$ is a non-degenerate point with coordinates in Γ of

$$X: \text{ rank } \begin{pmatrix} 1 & \cdots & 1 & 1 \\ x_{21} & \cdots & x_{2n} & 1 \\ \vdots & & \vdots & \vdots \\ x_{A1} & \cdots & x_{An} & 1 \end{pmatrix} \leqslant n\,.$$

By Laurent's Theorem, $(x_{21}, \ldots, x_{An})$ belongs to a finite set independent of $\mathbf{a}$.

We can determine $\mathbf{a}$ uniquely from $(x_{21}, \ldots, x_{An})$ by solving

$$\begin{aligned} a_1 + \cdots + a_n &= 1 \\ a_1 x_{i1} + \cdots + a_n x_{in} &= 1 \quad (i = 2, \ldots, A). \end{aligned}$$

Hence CLASS I is finite.

CLASS II.

$\mathbf{a} = (a_1, \ldots, a_n)$ is such that $\mathbf{x} = (x_{21}, \ldots, x_{An})$ is a degenerate point of

$$X: \ \text{rank} \begin{pmatrix} 1 & \cdots & 1 & 1 \\ x_{21} & \cdots & x_{2n} & 1 \\ \vdots & & \vdots & \vdots \\ x_{A1} & \cdots & x_{An} & 1 \end{pmatrix} \leqslant n\,.$$

Then there are integers $c_{21}, \ldots, c_{An}$ with $\gcd(c_{21}, \ldots, c_{An}) = 1$ such that

$$\text{rank} \begin{pmatrix} 1 & \cdots & 1 & 1 \\ \lambda^{c_{21}} x_{21} & \cdots & \lambda^{c_{2n}} x_{2n} & 1 \\ \vdots & & \vdots & \vdots \\ \lambda^{c_{A1}} x_{A1} & \cdots & \lambda^{c_{An}} x_{An} & 1 \end{pmatrix} \leqslant n$$

for every $\lambda \in K^*$.

Substitute $\lambda = -1$. Then we get

$$\text{rank} \begin{pmatrix} 1 & \cdots & 1 & 1 \\ \pm x_{21} & \cdots & \pm x_{2n} & 1 \\ \vdots & & \vdots & \vdots \\ \pm x_{A1} & \cdots & \pm x_{An} & 1 \end{pmatrix} \leqslant n\,.$$

Not all signs are $+$ since not all c_{ij} are even.

Hence there are $b_1, \ldots, b_n, b_0 \in K$, not all 0 such that

$$\begin{aligned} b_1 + \cdots + b_n &= b_0 \\ \pm b_1 x_{i1} \pm \cdots \pm b_n x_{in} &= b_0 \ (i = 2, \ldots, A). \end{aligned}$$

Conclusion:

Recall that $(1,\ldots,1)$, $(x_{i1},\ldots,x_{in})$ $(i=2,\ldots,A)$ contain all non-degenerate solutions of

(1) $\quad a_1x_1+\cdots+a_nx_n=1 \quad$ in $x_1,\ldots,x_n\in\Gamma$.

So for each non-degenerate solution of (1) there are n signs $\pm$ such that

$$\pm b_1x_1\pm\cdots\pm b_nx_n=b_0\,.$$

Thus, if $\mathbf{a}\in$ CLASS II, then the non-degenerate solutions of (1) lie in at most 2^n proper linear subspaces of K^n.

QED

A speculation.

For every tuple $\mathbf{a} = (a_1, \ldots, a_n) \in (K^*)^n$ with the exception of finitely many Γ-equivalence classes, the following holds:

1) If $\mathbf{a}$ is Γ-equivalent to $(b, b, \ldots, b)$ for some $b \in K^*$, then the set of solutions of

$$(1) \quad a_1x_1 + \cdots + a_nx_n = 1 \quad \text{in } x_1, \ldots, x_n \in \Gamma$$

is contained in the union of not more than n proper linear subspaces of K^n.

2) If $\mathbf{a}$ is not Γ-equivalent to $(b, b, \ldots, b)$ for any $b \in K^*$, then the set of solutions of (1) is contained in the union of not more than 2 proper linear subspaces of K^n.

Remark. If $\mathbf{u} = (u_1, \ldots, u_n)$ is a solution of

$$bx_1 + bx_2 + \cdots + bx_n = 1$$

then so are the points $\mathbf{u}_\sigma = (u_{\sigma(1)}, \ldots, u_{\sigma(n)})$ for each permutation σ of $1, 2, \ldots, n$.

For "generic" $\mathbf{u}$, precisely n proper linear subspaces of K^n are needed to cover the set of all points $\mathbf{u}_\sigma$.