APPROXIMATION OF COMPLEX ALGEBRAIC NUMBERS BY ALGEBRAIC NUMBERS OF BOUNDED DEGREE

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APPROXIMATION BY RATIONALS

Let $\xi \in \mathbb{C}$.

Denote by $\kappa_1(\xi)$ the supremum of all $\kappa \in \mathbb{R}$ such that

$$|\xi - \frac{x}{y}| \leqslant (\max |x|, |y|)^{-\kappa} \text{ in coprime } x, y \in \mathbb{Z}$$

has infinitely many solutions.

So it has finitely many solutions if $\kappa > \kappa_1(\xi)$ and infinitely many solutions if $\kappa < \kappa_1(\xi)$.

Facts:

- $\kappa_1(\xi) = 0$ for $\xi \in \mathbb{C} \setminus \mathbb{R}$;
- $\kappa_1(\xi) \geqslant 2$ for $\xi \in \mathbb{R} \setminus \mathbb{Q}$ (Dirichlet, 1842);
- $\kappa_1(\xi) = 2$ for almost all $\xi \in \mathbb{R}$;
- $\kappa_1(\xi) = 2$ for $\xi \in \mathbb{R} \setminus \mathbb{Q}$ algebraic (Roth, 1955)

APPROXIMATION BY ALGEBRAIC NUMBERS OF HIGHER DEGREE

For an algebraic number α, denote by P_α its minimal polynomial in $\mathbb{Z}[X]$, i.e., $P_\alpha = \sum_{j=0}^{n} a_j X^j$ with $\gcd(a_0, \ldots, a_n) = 1$, and define its height

$$H(\alpha) = H(P_\alpha) := \max_i |a_i|.$$

Definition. For $\xi \in \mathbb{C}$, $n \in \mathbb{Z}_{\geqslant 1}$, denote by $\kappa_n(\xi)$ the supremum of all reals κ such that

$$|\xi - \alpha| \leqslant H(\alpha)^{-\kappa}$$

has infinitely many solutions in algebraic numbers $\alpha \in \mathbb{C}$ of degree at most n.

Remark. $\kappa_n(\xi) = w_n^*(\xi)+1$, where $w_n^*(\xi)$ was introduced by Koksma (1939) for a classification of transcendental numbers.

THE CASE $\xi \in \mathbb{R}$

Let $n \in \mathbb{Z}_{\geqslant 1}$.

Theorem (Sprindzhuk, 1966). *For almost all $\xi \in \mathbb{R}$ we have $\kappa_n(\xi) = n + 1$.*

Theorem (Schmidt, 1971). *Let ξ be a real algebraic number of degree $d \geqslant 2$.*
Then $\kappa_n(\xi) = \min(n + 1, d)$.

Real algebraic numbers of degree $d \geqslant n + 1$ are equally well approximable by algebraic numbers of degree at most n as almost all real numbers.

THE CASE $\xi \in \mathbb{C} \setminus \mathbb{R}$

Let $n \in \mathbb{Z}_{\geqslant 2}$.

Theorem (Sprindzhuk, 1966). *For almost all $\xi \in \mathbb{C}$ we have $\kappa_n(\xi) = (n+1)/2$.*

Lemma (Liouville's inequality). *If $\xi \in \mathbb{C} \setminus \mathbb{R}$ is algebraic of degree $d \leqslant n+1$ then $\kappa_n(\xi) = d/2$.*

Not considered so far: Computation of $\kappa_n(\xi)$ for $\xi \in \mathbb{C} \setminus \mathbb{R}$ algebraic of degree $d \geqslant n+2$.

Is it true that complex algebraic numbers ξ of degree $d \geqslant n+2$ are equally well approximable by algebraic numbers of degree at most n as almost all complex numbers, i.e., $\kappa_n(\xi) = (n+1)/2$?

A COUNTEREXAMPLE

Let $n \geqslant 2$ be an even integer, and η a positive real algebraic number of degree $d/2$ where d is even and $d \geqslant n+2$.

Let $\xi = \sqrt{-\eta}$. Then $\deg \xi = d \geqslant n+2$.

By Schmidt's Theorem in the real case, we have $\kappa_{n/2}(\eta) = (n/2)+1$.
Hence for every $\kappa < (n/2)+1$, there are infinitely many algebraic numbers β of degree at most $n/2$ such that $|\eta - \beta| \leqslant H(\beta)^{-\kappa}$.

Taking $\alpha = \sqrt{-\beta}$, we get infinitely many algebraic numbers α of degree at most n such that

$$|\xi - \alpha| \ll |\eta - \beta| \ll H(\beta)^{-\kappa} \ll H(\alpha)^{-\kappa}.$$

Hence $\kappa_n(\xi) \geqslant (n+2)/2$.

THE CASE $\xi \in \mathbb{C} \setminus \mathbb{R}$ ALGEBRAIC

Theorem 1. (Bugeaud, E.)
Let $n \in \mathbb{Z}_{\geqslant 2}$, $\xi \in \mathbb{C} \setminus \mathbb{R}$ algebraic of degree $d \geqslant n+2$.

(i). *Suppose that n is odd. Then* $\kappa_n(\xi) = \frac{n+1}{2}$.

(ii). *Suppose that n is even. Then* $\kappa_n(\xi) \in \left\{\frac{n+1}{2}, \frac{n+2}{2}\right\}$.

Further, for every even $n \geqslant 2$, both cases may occur.

Proof. Schmidt's Subspace Theorem + elementary algebra.

THE CASE n EVEN

Let $n \in \mathbb{Z}_{\geqslant 2}$ even, $\xi \in \mathbb{C} \backslash \mathbb{R}$ algebraic of degree $d \geqslant n+2$.

Theorem 2 (Bugeaud, E.)
We have $\kappa_n(\xi) =$

$\frac{n+2}{2}$	$\deg \xi \geqslant n+2$, $\{1, \xi + \overline{\xi}, \xi \cdot \overline{\xi}\}$ $\mathbb{Q}$-*linearly dependent*
	$\deg \xi = n+2$, $[\mathbb{Q}(\xi) : \mathbb{Q}(\xi) \cap \mathbb{R}] = 2$
$\frac{n+1}{2}$	$\deg \xi > 2n-2$, $\{1, \xi + \overline{\xi}, \xi \cdot \overline{\xi}\}$ $\mathbb{Q}$-*linearly independent*
	$[\mathbb{Q}(\xi) : \mathbb{Q}(\xi) \cap \mathbb{R}] \geqslant 3$
??	*remaining cases*

We can determine $\kappa_n(\xi)$ in all cases, except n even, $n \geqslant 6$, $n+3 \leqslant \deg \xi \leqslant 2n-2$.

ANOTHER THEOREM

For $\xi \in \mathbb{C}$, $n \in \mathbb{Z}_{\geqslant 2}$, $\mu \in \mathbb{C}^*$ define the $\mathbb{Q}$-vector space

$V_n(\mu, \xi) := \{f \in \mathbb{Q}[X] : \deg f \leqslant n,\ \mu f(\xi) \in \mathbb{R}\}.$

Let

$$t_n(\xi) := \max\{\dim_{\mathbb{Q}} V_n(\mu, \xi) : \mu \in \mathbb{C}^*\}.$$

Theorem 3 (Bugeaud, E.)
Let $\xi \in \mathbb{C} \setminus \mathbb{R}$ algebraic of degree $d \geqslant n+2$. Then

$$\kappa_n(\xi) = \max\Big(\frac{n+1}{2},\ t_n(\xi)\Big).$$

Proof. Schmidt's Subspace Theorem. □

IDEA OF PROOF

Let $P_\alpha = \sum_{i=0}^{n} x_i X^i$ denote the minimal polynomial of α. Put $\mathbf{x} = (x_0, \ldots, x_n)$. Then $H(\alpha) = \max_i |x_i| =: \|\mathbf{x}\|$. Notice

$$|\xi - \alpha| \gg\ll \frac{|P_\alpha(\xi)|}{|P'_\alpha(\xi)|}.$$

Define linear forms

$$L_1(\mathbf{x}) = \mathrm{Re}\, P_\alpha(\xi),\ L_2(\mathbf{x}) = \mathrm{Im}\, P_\alpha(\xi),$$
$$M_1(\mathbf{x}) = \mathrm{Re}\, P'_\alpha(\xi),\ M_2(\mathbf{x}) = \mathrm{Im}\, P'_\alpha(\xi).$$

Use the Subspace Theorem and Minkowski's Theorem to decide for which u, v the following system has finitely or infinitely many solutions in $\mathbf{x} \in \mathbb{Z}^{n+1}$:

$$\begin{cases} |L_1(\mathbf{x})| \leqslant \|\mathbf{x}\|^u, & |L_2(\mathbf{x})| \leqslant \|\mathbf{x}\|^u, \\ |M_1(\mathbf{x})| \leqslant \|\mathbf{x}\|^v, & |M_2(\mathbf{x})| \leqslant \|\mathbf{x}\|^v. \end{cases}$$

PROPERTIES OF $t_n(\xi)$

Let $\xi \in \mathbb{C}\setminus\mathbb{R}$ algebraic, $\deg\xi \geqslant n+2$, $n \in \mathbb{Z}_{\geqslant 2}$.

Lemma 1. $t_n(\xi) \leqslant [(n+2)/2]$.

Lemma 2. *Suppose n is even. Then*

<table>
<tr><td rowspan="2">$t_n(\xi) = \frac{n+2}{2}$</td><td>$\deg\xi \geqslant n+2$,
$\{1, \xi+\overline{\xi}, \xi\cdot\overline{\xi}\}$ $\mathbb{Q}$-linearly dependent</td></tr>
<tr><td>$\deg\xi = n+2$, $[\mathbb{Q}(\xi) : \mathbb{Q}(\xi)\cap\mathbb{R}] = 2$</td></tr>
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<tr><td>??</td><td>remaining cases</td></tr>
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Proof of $t_n(\xi) \leqslant [(n+2)/2]$.

Recall $\xi \in \mathbb{C} \setminus \mathbb{R}$, $\deg \xi \geqslant n+2$.
Choose $\mu \in \mathbb{C}^*$ such that $\dim_{\mathbb{Q}} V_n(\mu, \xi) = t_n(\xi)$.
Put $X \cdot V_n(\mu, \xi) := \{X \cdot f : f \in V_n(\mu, \xi)\}$.

Then $V_n(\mu, \xi) \cap X \cdot V_n(\mu, \xi) = (0)$.
Otherwise, there is non-zero $f \in V_n(\mu, \xi)$ such that also $X \cdot f \in V_n(\mu, \xi)$.
Then $\mu f(\xi) \in \mathbb{R}^*$ and $\mu \xi f(\xi) \in \mathbb{R}^*$, implying $\xi \in \mathbb{R}$, which is impossible.

Now

$$\begin{aligned} 2t_n(\xi) &= \dim_{\mathbb{Q}} V_n(\mu, \xi) + \dim_{\mathbb{Q}} X \cdot V_n(\mu, \xi) \\ &= \dim_{\mathbb{Q}} \Big(V_n(\mu, \xi) + X \cdot V_n(\mu, \xi) \Big) \\ &\leqslant \dim_{\mathbb{Q}} \{ f \in \mathbb{Q}[X] : \deg f \leqslant n+1 \} \\ &= n+2. \end{aligned}$$

□

APPROXIMATION BY ALGEBRAIC INTEGERS

Instead of approximation by algebraic numbers of degree at most n we consider approximation by algebraic integers of degree at most $n+1$.

Let $\xi \in \mathbb{C}$, $n \in \mathbb{Z}_{\geqslant 1}$.

Define $\lambda_n(\xi)$ to be the supremum of all $\lambda \in \mathbb{R}$ such that

$$0 < |\xi - \alpha| \leqslant H(\alpha)^{-\lambda}$$

has infinitely many solutions in algebraic integers α of degree $\leqslant n+1$.

Theorem (Bugeaud, Teulié).
Let $\xi \in \mathbb{R}$ be algebraic of degree $d \geqslant 2$ and $n \in \mathbb{Z}_{\geqslant 1}$.
Then $\lambda_n(\xi) = \min(n+1, d)$.

Theorem 4 (Bugeaud, E.)
Let $\xi \in \mathbb{C} \setminus \mathbb{R}$ be algebraic of degree $d \geqslant 2$ and $n \in \mathbb{Z}_{\geqslant 2}$.
Then

$$\lambda_n(\xi) = \begin{cases} \dfrac{d}{2} & \text{if } d \leqslant n+1, \\ \dfrac{n+1}{2} & \text{if } d \geqslant n+2,\ \kappa_n(\xi) = \dfrac{n+1}{2}, \\ \dfrac{n}{2} & \text{if } d \geqslant n+2,\ \kappa_n(\xi) = \dfrac{n+2}{2}. \end{cases}$$

NUMBER OF APPROXIMANTS

Let $n \in \mathbb{Z}_{\geqslant 2}$, $\kappa > 0$, and $\xi \in \mathbb{C}$ algebraic of degree $d > n$. Consider

$$(1) \qquad |\xi - \alpha| \leqslant H(\alpha)^{-\kappa}$$

in algebraic numbers α of degree at most n.

Theorem (E.)
Let $\delta > 0$. Suppose that

$$\kappa = \begin{cases} 2n + \delta & \text{if } \xi \in \mathbb{R}, \\ n + \delta & \text{if } \xi \in \mathbb{C} \setminus \mathbb{R}. \end{cases}$$

Then (1) has at most

$$2^{2n}(10n)^{20}(1 + \delta^{-4}) \log 4d \log\log 4d$$

solutions with $H(\alpha) \geqslant \max\left(2^{3n^2/\delta}, H(\xi)\right)$.

Proof. Quantitative Subspace Theorem.

VERY DIFFICULT OPEN PROBLEM:

let $\xi \in \mathbb{C}$ be algebraic of degree d.

Recall

$$\kappa_n(\xi) \begin{cases} = \min(n+1, d) & \text{if } \xi \in \mathbb{R}, \\ = \frac{d}{2} & \text{if } \xi \in \mathbb{C} \setminus \mathbb{R},\ d \leqslant n+1, \\ \in \{\frac{n+1}{2}, \frac{n+2}{2}\} & \text{if } \xi \in \mathbb{C} \setminus \mathbb{R},\ d \geqslant n+2. \end{cases}$$

The number of solutions of

$$|\xi - \alpha| \leqslant H(\alpha)^{-\kappa} \tag{1}$$

in algebraic numbers α of degree $\leqslant n$ is finite if $\kappa > \kappa_n(\xi)$.

Give an explicit upper bound for this number if

- $\kappa_n(\xi) < \kappa \leqslant 2n$ if $\xi \in \mathbb{R}$,
- $\kappa_n(\xi) < \kappa \leqslant n$ if $\xi \in \mathbb{C} \setminus \mathbb{R}$.