APPROXIMATION OF ALGEBRAIC NUMBERS BY ALGEBRAIC NUMBERS

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DIRICHLET'S THEOREM

Rational numbers are represented as x/y, where x, y are integers such that $\gcd(x, y) = 1$, $y > 0$.

Theorem (Dirichlet, 1842)
Let ξ be an irrational real number. Then there are infinitely many rational numbers x/y such that

$$|\xi - (x/y)| \leqslant y^{-2}.$$

ROTH'S THEOREM

A number ξ is called algebraic if there exists a non-zero polynomial P with integer coefficients such that $P(\xi) = 0$.

Theorem (Roth, 1955) *Let ξ be a real, irrational, algebraic number. Let $\kappa > 2$. Then there are only finitely many rational numbers x/y such that*

$$|\xi - (x/y)| \leqslant y^{-\kappa}.$$

This result is a culmination of earlier work of Thue (1909), Siegel (1921), Dyson, Gel'fond (1949).

METRIC DIOPHANTINE APPROXIMATION

We recall a very special (and easy) case of a result of Khintchine (1924):

Theorem.
Let $\kappa > 2$. Then the set of real numbers ξ such that

$|\xi - (x/y)| \leqslant y^{-\kappa}$ *for infinitely many* $x/y \in \mathbb{Q}$,

has Lebesgue measure 0.

APPROXIMATION BY RATIONALS

Denote by $\kappa_1(\xi)$ the supremum of all $\kappa \in \mathbb{R}$ such that

(*) $$|\xi - \frac{x}{y}| \leqslant y^{-\kappa}$$

has infinitely many solutions in rational numbers x/y.

So (*) has infinitely many solutions if $\kappa < \kappa_1(\xi)$ and only finitely many solutions if $\kappa > \kappa_1(\xi)$.

Facts:

- $\kappa_1(\xi) = 2$ for almost all real numbers ξ (Dirichlet, Khintchine)
- $\kappa_1(\xi) = 2$ for real, irrational algebraic numbers ξ (Dirichlet, Roth)

So the κ_1-value of a real irrational algebraic number ξ is equal to that of almost all real numbers.

ALGEBRAIC NUMBERS

For every algebraic number α (in $\mathbb{C}$) there is a unique polynomial P of minimal degree such that $P(\alpha) = 0$ and P has integer coefficients with gcd 1 and positive leading coefficient.

P is called the *minimal polynomial* of α.

The *degree* $\deg \alpha$ of α is the degree of P.

The *height* $H(\alpha)$ of α is the maximum of the absolute values of the coefficients of P.

EXAMPLES:

$\alpha = x/y$ ($x, y \in \mathbb{Z}$, gcd$(x, y) = 1$, $y > 0$) has minimal polynomial $yX - x$, degree 1 and height max$(|x|, y)$.

$\alpha = \frac{1}{2}\sqrt{2} + \sqrt{3}$ has minimal polynomial $4X^4 - 28X^2 + 25$, degree 4 and height 28.

APPROXIMATION BY ALGEBRAIC NUMBERS OF HIGHER DEGREE

Definition. For a complex number ξ and a positive integer n, denote by $\kappa_n(\xi)$ the supremum of all reals κ such that

$$|\xi - \alpha| \leqslant H(\alpha)^{-\kappa}$$

has infinitely many solutions in algebraic numbers α of degree at most n.

APPROXIMATION TO REAL ξ

Let n be a positive integer.

Theorem (Sprindzhuk, 1966).
For almost all real numbers ξ we have $\kappa_n(\xi) = n+1$.

Theorem (W.M. Schmidt, 1971).
Let ξ be a real algebraic number of degree $d \geqslant 2$. Then $\kappa_n(\xi) = \min(n+1, d)$.

Real algebraic numbers of degree $d > n$ have the same κ_n-value as almost all real numbers.

APPROXIMATION TO COMPLEX ξ

Let n be an integer $\geqslant 2$.

Theorem (Sprindzhuk, 1966). *For almost all $\xi \in \mathbb{C}$ we have $\kappa_n(\xi) = (n+1)/2$.*

Lemma. *If ξ is a complex, non-real algebraic number of degree $d \leqslant n$ then $\kappa_n(\xi) = d/2$.*

Not considered so far: Determination of $\kappa_n(\xi)$ for complex, non-real algebraic numbers of degree $d > n$.

Reasonable question: Do complex, non-real algebraic numbers ξ of degree $> n$ have the same κ_n-value as almost all complex numbers, i.e., $\kappa_n(\xi) = (n+1)/2$?

A COUNTEREXAMPLE

Let n, d be even integers with $d > n \geqslant 2$, and η a positive real algebraic number of degree $d/2$.

Let $\xi := \sqrt{-\eta}$. Then $\deg \xi = d$.

By Schmidt's Theorem we have
$\kappa_{n/2}(\eta) = (n/2) + 1$.

Hence for every $\kappa < (n/2) + 1$, there are infinitely many algebraic numbers β of degree at most $n/2$ such that $|\eta - \beta| \leqslant H(\beta)^{-\kappa}$.

Taking $\alpha = \sqrt{-\beta}$, we get infinitely many algebraic numbers α of degree at most n such that for some constant $A > 0$,

$$|\xi - \alpha| \leqslant A \cdot |\eta - \beta| \leqslant A \cdot H(\beta)^{-\kappa} = A \cdot H(\alpha)^{-\kappa}.$$

Hence $\kappa_n(\xi) \geqslant (n + 2)/2$.

APPROXIMATION TO COMPLEX ALGEBRAIC ξ

Theorem 1. (Bugeaud, E.)
Let n be an integer $\geqslant 2$ and ξ a complex, non-real algebraic number of degree $> n$.

(i). *Suppose that n or $\deg\xi$ is odd. Then*

$$\kappa_n(\xi) = \frac{n+1}{2}.$$

(ii). *Suppose that both n and $\deg\xi$ are even. Then*

$$\kappa_n(\xi) \in \left\{\frac{n+1}{2}, \frac{n+2}{2}\right\}.$$

Further, for every even n, d with $d > n \geqslant 2$ there are ξ of degree d with $\kappa_n(\xi) = (n+1)/2$ and ξ of degree d with $\kappa_n(\xi) = (n+2)/2$.

THE CASE n AND $\deg \xi$ EVEN

Theorem 2 (Bugeaud, E.)
Let n be an even integer $\geqslant 2$ and ξ a complex, non-real algebraic number of even degree $\geqslant 2n$.

Then $\kappa_n(\xi) = (n+2)/2 \iff$
$1, \xi + \overline{\xi}, \xi \cdot \overline{\xi}$ are linearly dependent over $\mathbb{Q}$.

The description of the set of ξ with $\kappa_n(\xi) = (n+2)/2$ and $n < \deg \xi < 2n$ is more complicated, and is not completely known.

ANOTHER THEOREM

For complex numbers ξ, μ, and for integers $n \geqslant 2$, denote by $V_n(\mu, \xi)$ the set of polynomials $f(X)$ with coefficients in $\mathbb{Q}$ such that

$$\deg f \leqslant n, \quad \mu f(\xi) \in \mathbb{R}.$$

This is a vector space over $\mathbb{Q}$.

Denote by $t_n(\xi)$ the maximum of the dimensions of the spaces $V_n(\mu, \xi)$, taken over all $\mu \in \mathbb{C} \setminus \{0\}$.

Theorem 3 (Bugeaud, E.)
Let ξ be a complex, non-real algebraic number of degree $> n$. Then

$$\kappa_n(\xi) = \max\left(\frac{n+1}{2}, t_n(\xi)\right).$$

MAIN TOOL: SCHMIDT'S SUBSPACE THEOREM

Let $n \geqslant 2$, $\delta > 0$ and let

$$L_i = \alpha_{i1}X_1 + \cdots + \alpha_{in}X_n \ (i = 1, \ldots, n)$$

be linear forms with algebraic coefficients α_{ij} in $\mathbb{C}$.

Theorem (W.M. Schmidt, 1972). *Suppose that the linear forms $L_1, \ldots, L_n$ are linearly independent. Then the set of solutions*

$$\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{Z}^n$$

of the inequality

$$|L_1(\mathbf{x}) \cdots L_n(\mathbf{x})| \leqslant \left(\max_i |x_i|\right)^{-\delta}$$

is contained in some union $T_1 \cup \cdots \cup T_t$ of proper linear subspaces of $\mathbb{Q}^n$.