EFFECTIVE RESULTS FOR POINTS ON CERTAIN SUBVARIETIES OF TORI

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SUBJECT OF OUR LECTURE

Let X be an algebraic subvariety of the N-dimensional torus $\mathbb{G}_m^N(\overline{\mathbb{Q}}) = (\overline{\mathbb{Q}}^*)^N$.

Let Γ be a free subgroup of $\mathbb{G}_m^N(\overline{\mathbb{Q}})$ of finite rank, and let $\overline{\Gamma}$ be the division group of Γ.

Laurent, Poonen, Ev., Rémond proved ineffective results which imply that the points of X which are in $\overline{\Gamma}$ or "close to $\overline{\Gamma}$" lie in finitely many "families."

We give some limited class of varieties X for which these families can be determined effectively.

HEIGHTS ON TORI

N-dimensional torus:
$\mathbb{G}_m^N(\overline{\mathbb{Q}}) = (\overline{\mathbb{Q}}^*)^N$ with multiplication
$(x_1, \ldots, x_N) \cdot (y_1, \ldots, y_N) = (x_1 y_1, \ldots, x_N y_N)$.

Absolute logarithmic height on $\overline{\mathbb{Q}}$:
$h(\alpha) := \frac{1}{d} \log \left(|a| \prod_{i=1}^{d} \max(1, |\alpha^{(i)}|) \right)$ for $\alpha \in \overline{\mathbb{Q}}$,
where $a(X - \alpha^{(1)}) \cdots (X - \alpha^{(d)})$ is the minimal polynomial in $\mathbb{Z}[X]$ of α.

Height on $\mathbb{G}_m^N(\overline{\mathbb{Q}})$:
$\widehat{h}(\mathbf{x}) := \sum_{i=1}^{N} h(x_i)$ for $\mathbf{x} = (x_1, \ldots, x_N) \in \mathbb{G}_m^N(\overline{\mathbb{Q}})$.

Metric on $\mathbb{G}_m^N(\overline{\mathbb{Q}})/\mathbb{G}_m^N(\overline{\mathbb{Q}})_{\text{tors}}$:
$d(\mathbf{x}, \mathbf{y}) := \widehat{h}(\mathbf{x} \cdot \mathbf{y}^{-1})$.

LANG'S CONJECTURE FOR TORI

Let X be an algebraic subvariety of $\mathbb{G}_m^N(\overline{\mathbb{Q}})$, i.e., the set of common zeros in $\mathbb{G}_m^N(\overline{\mathbb{Q}})$ of certain polynomials $f_1, \ldots, f_m \in \overline{\mathbb{Q}}[X_1, \ldots, X_N]$.

Let Γ be a free subgroup of $\mathbb{G}_m^N(\overline{\mathbb{Q}})$ of finite rank r, i.e., $\Gamma = \{\mathbf{v}_1^{w_1} \cdots \mathbf{v}_r^{w_r} : w_1, \ldots, w_r \in \mathbb{Z}\}$ where $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is a basis of Γ.

Let $\overline{\Gamma} = \{\mathbf{x} \in \mathbb{G}_m^N(\overline{\mathbb{Q}}) : \exists m \in \mathbb{Z}_{>0}$ with $\mathbf{x}^m \in \Gamma\}$ be the division group of Γ.

Theorem A. (Laurent, 1984)
$X \cap \overline{\Gamma}$ is contained in a finite union of translates of algebraic subgroups $\mathbf{u}_1 H_1 \cup \cdots \cup \mathbf{u}_t H_t$, where

$\mathbf{u}_i \in \overline{\Gamma}$, H_i *algebraic subgroup of* $\mathbb{G}_m^N(\overline{\mathbb{Q}})$,
$\mathbf{u}_i H_i \subseteq X$ *for* $i = 1, \ldots, t$.

A RESULT FOR CYLINDERS

Define the cylinder with radius ε around $\overline{\Gamma}$ by

$$\overline{\Gamma}_\varepsilon := \{\mathbf{x} = \mathbf{y}\cdot\mathbf{z} : \mathbf{y} \in \overline{\Gamma},\ \mathbf{z} \in \mathbb{G}_m^N(\overline{\mathbb{Q}}),\ \widehat{h}(\mathbf{z}) < \varepsilon\}.$$

Theorem B. (Poonen, 1998)
There is $\varepsilon_0 = \varepsilon_0(N, \deg X) > 0$ (effectively computable) such that $X \cap \overline{\Gamma}_{\varepsilon_0}$ is contained in a finite union of translates of algebraic subgroups $\mathbf{u}_1 H_1 \cup \cdots \cup \mathbf{u}_t H_t$ of $\mathbb{G}_m^N(\overline{\mathbb{Q}})$, where

$$\mathbf{u}_i \in \overline{\Gamma}_{\varepsilon_0},\ \mathbf{u}_i H_i \subseteq X \ \textit{for}\ i = 1, \ldots, t.$$

Let X^{exc} denote the union of all translates $\mathbf{u}H$ such that $\mathbf{u} \in X$ and H is an alg. subgroup of $\mathbb{G}_m^N(\overline{\mathbb{Q}})$ of dimension > 0 with $\mathbf{u}H \subseteq X$.

Put $X^0 := X \setminus X^{\mathrm{exc}}$.

Corollary. *The set $X^0 \cap \overline{\Gamma}_{\varepsilon_0}$ is finite.*

A RESULT FOR CONES

Define the truncated cone around $\overline{\Gamma}$ by

$$\mathcal{C}(\overline{\Gamma},\varepsilon) := \Big\{ \mathbf{x} = \mathbf{y}\cdot\mathbf{z} : \ \mathbf{y} \in \overline{\Gamma},\, \mathbf{z} \in \mathbb{G}_m^N(\overline{\mathbb{Q}}), \quad \widehat{h}(\mathbf{z}) < \varepsilon(1+\widehat{h}(\mathbf{x})) \Big\}.$$

Theorem C. (Ev.; Rémond, 2002)
There is $\varepsilon_1 = \varepsilon_1(N, X, \Gamma) > 0$ such that the set $X^0 \cap \mathcal{C}(\overline{\Gamma}, \varepsilon_1)$ is finite.

Remarks.

1) There are varieties X and groups Γ such that for any $\varepsilon > 0$, the intersection $X \cap \mathcal{C}(\overline{\Gamma}, \varepsilon)$ is not contained in a finite union of translates $\mathbf{u}_i H_i \subset X$.

2) Rémond proved that the dependence of ε_1 on X, Γ is necessary.

3) In general, from the proof ε_1 can not be computed effectively.

EFFECTIVITY

For some limited class of varieties X, we obtained effective versions of Theorems A,B,C.

In these versions we give:

• explicit expressions for ε_0, ε_1;

• explicit upper bounds, in terms of a set of defining equations for X and a basis for Γ, for the heights $\widehat{h}(\mathbf{u}_i)$ and degrees $[\mathbb{Q}(\mathbf{u}_i):\mathbb{Q}]$ for the translates $\mathbf{u}_1 H_1, \ldots, \mathbf{u}_t H_t \subset X$ occurring in Thms. A,B or for the heights $\widehat{h}(\mathbf{x})$ and degrees $[\mathbb{Q}(\mathbf{x}):\mathbb{Q}]$ of the solutions $\mathbf{x} \in X^0 \cap \mathcal{C}(\overline{\Gamma}, \varepsilon_1)$ in Thm. C.

These data suffice to determine effectively in principle the translates $\mathbf{u}_i H_i$ in Thms. A,B or the solutions $\mathbf{x}$ in Thm. C.

CURVES IN $\mathbb{G}_m^2(\overline{\mathbb{Q}})$

Let $N = 2$, let $f \in \overline{\mathbb{Q}}[X, Y]$ be a non-zero, irreducible polynomial not of the shape $aX^m - bY^n$ or $aX^mY^n - b$, and

$$X = \Big\{\mathbf{x} = (x, y) \in \mathbb{G}_m^2(\overline{\mathbb{Q}}) : \; f(x, y) = 0\Big\}$$

(i.e., X is not a translate of an algebraic subgroup).

Put

$$\begin{aligned} h(f) &:= \max\Big(1, \text{heights of the coeff. of } f\Big), \\ \delta &:= \deg_X f + \deg_Y f. \end{aligned}$$

Let Γ be a free subgroup of $\mathbb{G}_m^2(\overline{\mathbb{Q}})$ with basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$. Put

$$\begin{aligned} h(\Gamma) &:= \max\Big(1, \widehat{h}(\mathbf{v}_1), \ldots, \widehat{h}(\mathbf{v}_r)\Big), \\ d &:= [\mathbb{Q}(\Gamma) : \mathbb{Q}], \\ C(\Gamma) &:= \exp\Big(50(r+1)d \cdot h(\Gamma)\Big), \\ K &:= \mathbb{Q}(\Gamma, \text{ coeff. of } f). \end{aligned}$$

EFFECTIVE RESULTS FOR CURVES

Theorem 1. (BEGP)
Let $\varepsilon_0 := \left(2^{48}\delta(\log\delta)^5\right)^{-1}$.
Then for every $\mathbf{x} \in X \cap \overline{\Gamma}_{\varepsilon_0}$ *we have*

$$\begin{aligned} \widehat{h}(\mathbf{x}) &\leqslant C(\Gamma)^{\delta^2} \cdot h(f), \\ [K(\mathbf{x}) : K] &\leqslant 2^{50}\delta^2(\log\delta)^6. \end{aligned}$$

Theorem 2. (BEGP)
Let $\varepsilon_1 := \left(2^{48}\delta(\log\delta)^5 \cdot C(\Gamma)^{\delta^2} h(f)\right)^{-1}$.
Then for every $\mathbf{x} \in X \cap \mathcal{C}(\overline{\Gamma}, \varepsilon_1)$ *we have*

$$\begin{aligned} \widehat{h}(\mathbf{x}) &\leqslant C(\Gamma)^{\delta^2} \cdot h(f), \\ [K(\mathbf{x}) : K] &\leqslant 2^{50}\delta^2(\log\delta)^6. \end{aligned}$$

THE ESTIMATE FOR $\widehat{h}(\mathbf{x})$

Step 1. *Estimate* $\widehat{h}(\mathbf{x})$ *for* $\mathbf{x} \in X \cap \Gamma$.

Use an idea of Bombieri and Gubler ("Heights in Diophantine Geometry") and lower bounds for linear forms in (ordinary and p-adic) logarithms to obtain

$$\widehat{h}(\mathbf{x}) \leqslant C_1(\Gamma)^{\delta^2} h(f).$$

Step 2. *Reduce Thms. 1, 2 to Step 1.*

For instance, in the case of Thm. 2, let $\mathbf{x} \in X \cap \mathcal{C}(\overline{\Gamma}, \varepsilon_1)$. Then $\mathbf{x} = \mathbf{y} \cdot \mathbf{z}$ with

$$\mathbf{y} \in \overline{\Gamma}, \quad \widehat{h}(\mathbf{z}) < \varepsilon_1(1 + \widehat{h}(\mathbf{x})).$$

Choose $\mathbf{y}' \in \Gamma$ with minimal distance to $\mathbf{y}$. This gives $\mathbf{x} = \mathbf{y}' \cdot \mathbf{z}'$ with

$$\mathbf{y}' \in \Gamma, \quad \widehat{h}(\mathbf{z}') \leqslant C_2(\Gamma) + \varepsilon_1 \widehat{h}(\mathbf{x}).$$

Apply Step 1 to $\mathbf{y}' \in \mathbf{z}'^{-1} X \cap \Gamma$. This gives $\widehat{h}(\mathbf{y}') \leqslant C_1(\Gamma)^{\delta^2}(h(f) + \delta h(\mathbf{z}'))$ and thus

$$\widehat{h}(\mathbf{x}) \leqslant \widehat{h}(\mathbf{y}') + \widehat{h}(\mathbf{z}') \leqslant \cdots + \theta \widehat{h}(\mathbf{x}) \text{ with } \theta < 1.$$

THE ESTIMATE FOR $[K(\mathbf{x}):K]$ (I)

We use the following explicit Bogomolov type result.

Theorem. (Pontreau)
Let $X \subset \mathbb{G}_m^2(\overline{\mathbb{Q}})$ be a curve given by $f(x,y) = 0$, where $f \in \overline{\mathbb{Q}}[X,Y]$ is an irreducible polynomial not of the form $aX^m - bY^n$ or $aX^mY^n - b$. Let $\delta := \deg_X f + \deg_Y f$. Put

$$A := 2^{47}\delta(\log\delta)^5, \quad B := 2^{50}\delta^2(\log\delta)^6.$$

Then there are at most B points $\mathbf{x} \in X$ with $\widehat{h}(\mathbf{x}) \leqslant A^{-1}$.

There are similar such results for arbitrary varieties $X \subset \mathbb{G}_m^N(\overline{\mathbb{Q}})$, due to Zhang, Zagier, Bombieri& Zannier, Schmidt, David, Philippon, Amoroso, Viada.

THE ESTIMATE FOR $[K(\mathbf{x}):K]$ **(II)**

For instance, let $\mathbf{x} \in X \cap \mathcal{C}(\overline{\Gamma}, \varepsilon_1)$. Then $\mathbf{x} = \mathbf{y} \cdot \mathbf{z}$ with $\mathbf{y} \in \overline{\Gamma}$, $\widehat{h}(\mathbf{z}) < \varepsilon_1(1 + \widehat{h}(\mathbf{x}))$.

Note that $[K(\mathbf{x}) : K]$ equals the number of distinct points among $\mathbf{x}_\sigma := \sigma(\mathbf{x}) \cdot \mathbf{x}^{-1}$ where $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/K)$.

On the one hand, $\sigma(\mathbf{x}) \in X$, hence
$\mathbf{x}_\sigma \in \mathbf{x}^{-1} X$.

On the other hand, since $\sigma(\mathbf{y})\mathbf{y}^{-1}$ is a torsion point, and by our upper bound for $\widehat{h}(\mathbf{x})$,

$$\widehat{h}(\mathbf{x}_\sigma) = \widehat{h}(\sigma(\mathbf{z})\mathbf{z}^{-1}) \leqslant 2\widehat{h}(\mathbf{z}) < A^{-1}.$$

So the number of distinct points $\mathbf{x}_\sigma$, and hence $[K(\mathbf{x}) : K]$, is at most B.

HIGHER DIMENSIONAL VARIETIES

We obtained effective versions of Theorems A,B,C for varieties $X \subset \mathbb{G}_m^N(\overline{\mathbb{Q}})$ given by equations

$$
\begin{aligned}
&f_1(\mathbf{x}) = 0, \ldots, f_m(\mathbf{x}) = 0, \text{ with} \\
&f_1, \ldots, f_m \in \overline{\mathbb{Q}}[X_1, \ldots, X_N], \\
&f_1, \ldots, f_m \text{ binomials or trinomials.}
\end{aligned}
$$

Example: Grassmann type varieties:

$$
N = \binom{n}{k} \quad (n \geqslant 4,\ 2 \leqslant k \leqslant n-2),
$$

$$
X = \Big\{ \mathbf{x} \in \mathbb{G}_m^N(\overline{\mathbb{Q}}) : \exists\, \mathbf{y}_1, \ldots, \mathbf{y}_k \in \overline{\mathbb{Q}}^n \text{ such that } \mathbf{x} = \mathbf{y}_1 \wedge \cdots \wedge \mathbf{y}_k \Big\}.
$$

Main ingredients: lower bounds for linear forms in logarithms; explicit Bogomolov type results.

A LEMMA

In addition, we needed the following lemma which seems to be non-trivial.

Lemma. (BEGP)
Let $\varepsilon > 0$*, let* Γ *a free subgroup of* $\mathbb{G}_m^N(\overline{\mathbb{Q}})$ *of finite rank, let* H *be a positive dimensional algebraic subgroup of* $\mathbb{G}_m^N(\overline{\mathbb{Q}})$*, and let* $\mathbf{u} \in \mathbb{G}_m^N(\overline{\mathbb{Q}})$*.*
If $\mathbf{u}H \cap \overline{\Gamma}_\varepsilon \neq \emptyset$*, there exists* $\mathbf{u}' \in \mathbf{u}H \cap \overline{\Gamma}_\varepsilon$*, with both* $\widehat{h}(\mathbf{u}')$*,* $[\mathbb{Q}(\mathbf{u}') : \mathbb{Q}]$ *bounded above by effectively computable numbers depending on* ε*,* Γ*,* $\mathbf{u}$*,* H*.*