The Subspace Theorem and twisted heights

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Schmidt's Subspace Theorem

Theorem (Schmidt, 1972)

Let $L_0, \ldots, L_n$ be linearly independent linear forms in $n+1$ variables with real or complex algebraic coefficients and let $a_0, \ldots, a_n$ be reals with $a_0 + \cdots + a_n > 0$.
Then the set of solutions $\mathbf{x} = (x_0, \ldots, x_n) \in \mathbb{Z}^{n+1}$ of the system

$$(1) \quad |L_0(\mathbf{x})| \leqslant \|\mathbf{x}\|^{-a_0}, \ldots, |L_n(\mathbf{x})| \leqslant \|\mathbf{x}\|^{-a_n} \quad (\|\mathbf{x}\| = \max_{0 \leqslant i \leqslant n} |x_i|)$$

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is contained in a finite union of proper linear subspaces of $\mathbb{Q}^{n+1}$.

Consider the class of convex bodies parametrized by Q,

$$B(Q) := \{\mathbf{y} \in \mathbb{R}^{n+1} : \max_{0 \leqslant i \leqslant n} |L_i(\mathbf{y})| Q^{a_i} \leqslant 1\}.$$

If $\mathbf{x} \in \mathbb{Z}^{n+1}$ satisfies (1) then $\mathbf{x} \in B(Q)$ with $Q = \|\mathbf{x}\|$.
The Subspace Theorem follows from a study of the successive minima of the bodies $B(Q)$.

Notation

$\overline{\mathbb{Q}}$ is an algebraic closure of $\mathbb{Q}$.
$K \subset \overline{\mathbb{Q}}$ is an algebraic number field, M_K its set of places.

Choose normalized absolute values $|\cdot|_v$ ($v \in M_K$) on K such that

$|\cdot|_v$ extends $|\cdot|^{[K_v:\mathbb{R}]/[K:\mathbb{Q}]}$ if v is infinite,

$|\cdot|_v$ extends $|\cdot|_p^{[K_v:\mathbb{Q}_p]/[K:\mathbb{Q}]}$ if v is finite, $v|p$, p prime.

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For $\mathbf{x} = (x_0, \dots, x_n) \in \mathbb{P}^n(K)$ define

$$\|\mathbf{x}\|_v := \max(|x_0|_v, \dots, |x_n|_v) \quad (v \in M_K),$$
$$H(\mathbf{x}) := \prod_{v \in M_K} \|\mathbf{x}\|_v.$$

$H(\mathbf{x})$ is extended to $\mathbb{P}^n(\overline{\mathbb{Q}})$ in the usual way.

The height $H(F)$ of $F \in \overline{\mathbb{Q}}[X_0, \dots, X_n]$ is the height of its vector of coefficients.

The Subspace Theorem over number fields

Let S be a finite set of places of K.

For $v \in S$, let $L_{0v}, \ldots, L_{nv}$ be linearly independent linear forms in $X_0, \ldots, X_n$ with coefficients in K and let $c_{0v}, \ldots, c_{nv}$ be non-negative reals.

Consider the system of inequalities

$$(2) \qquad \frac{|L_{iv}(\mathbf{x})|_v}{\|\mathbf{x}\|_v} \leqslant H(\mathbf{x})^{-c_{iv}} \ (v \in S,\ i = 0, \ldots, n) \ \text{ in } \mathbf{x} \in \mathbb{P}^n(K).$$

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Theorem (Schmidt, Schlickewei, 1975–1977)

Assume that

$$\sum_{v \in S} \sum_{i=0}^{n} c_{iv} > n + 1.$$

Then the set of solutions of (2) *is contained in a finite union of proper linear subspaces of* $\mathbb{P}^n(K)$.

A refinement of the Subspace Theorem

Theorem (Vojta, Schmidt, Faltings-Wüstholz, 1989–1994)

Assume again that

$$\sum_{v \in S} \sum_{i=0}^{n} c_{iv} > n + 1.$$

Then there is a proper linear subspace T of $\mathbb{P}^n(K)$ such that the system

$$\frac{|L_{iv}(\mathbf{x})|_v}{\|\mathbf{x}\|_v} \leqslant H(\mathbf{x})^{-c_{iv}} \ (v \in S,\ i = 0, \ldots, n) \quad \text{in } \mathbf{x} \in \mathbb{P}^n(K)$$

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has only finitely many solutions outside T.

Moreover, T can be determined effectively, and T belongs to a finite collection independent of the c_{iv}.

Twisted heights

For $\mathbf{x} \in \mathbb{P}^n(K)$, $Q \geqslant 1$, define

$$H_Q(\mathbf{x}) = \prod_{v \in S} \max_{0 \leqslant i \leqslant n} \left(|L_{iv}(\mathbf{x})|_v Q^{c_{iv}} \right) \cdot \prod_{v \in M_K \setminus S} \|\mathbf{x}\|_v.$$

Lemma

Let $\mathbf{x} \in \mathbb{P}^n(K)$ and $Q := H(\mathbf{x})$. If $\mathbf{x}$ is a solution to

$$\frac{|L_{iv}(\mathbf{x})|_v}{\|\mathbf{x}\|_v} \leqslant H(\mathbf{x})^{-c_{iv}} \ (v \in S, \ i = 0, \ldots, n)$$

then $H_Q(\mathbf{x}) \leqslant Q$.

The Parametric Subspace Theorem

Theorem (Schlickewei, E., 2002)

Suppose that $\sum_{v \in S} \sum_{i=0}^{n} c_{iv} > n+1$. *Then there are* $Q_0 > 1$, *and proper linear subspaces* $T_1, \ldots, T_t$ *of* $\mathbb{P}^n(K)$, *such that the following holds:*
for every $Q \geqslant Q_0$, *there is* $T_i \in \{T_1, \ldots, T_t\}$ *such that*

$$\{\mathbf{x} \in \mathbb{P}^n(K) : H_Q(\mathbf{x}) \leqslant Q\} \subset T_i.$$

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Proof of the Subspace Theorem.

Pick a solution $\mathbf{x} \in \mathbb{P}^n(K)$ with $H(\mathbf{x}) \geqslant Q_0$ of

$$\frac{|L_{iv}(\mathbf{x})|_v}{\|\mathbf{x}\|_v} \leqslant H(\mathbf{x})^{-c_{iv}} \quad (v \in S,\ i = 0, \ldots, n).$$

Then with $Q = H(\mathbf{x})$ we have $H_Q(\mathbf{x}) \leqslant Q$, hence $\mathbf{x} \in \bigcup_i T_i$. □

The successive minima of a twisted height

Define $\lambda_i(Q)$ to be the minimum of all λ such that

$$\{\mathbf{x} \in \mathbb{P}^n(K) : \; H_Q(\mathbf{x}) \leqslant \lambda\}$$

contains i linearly independent points.

Then

$$0 < \lambda_1(Q) \leqslant \cdots \leqslant \lambda_{n+1}(Q) < \infty.$$

We are interested in the behaviour of the $\lambda_i(Q)$ as $Q \to \infty$.

Weights

Let $\mathbf{c} = (c_{iv} :\ v \in S,\ i = 0, \ldots, n)$ be a tuple of non-negative reals.

For a linear subspace T of $\mathbb{P}^n(K)$ we define

$$w_v(T) := \max_A \sum_{i \in A} c_{iv} \ \ (v \in S), \quad w(T) := \sum_{v \in S} w_v(T)$$

where for $v \in S$ the maximum is taken over all subsets A of $\{0, \ldots, n\}$ such that the restrictions $L_{iv}|_T$ $(i \in A)$ are linearly independent. Let

$$P(T) := \big(\dim T + 1,\ w(T)\big) \in \mathbb{R}^2.$$

A filtration

Denote by $C(\mathbf{c})$ the convex hull of the points
$P(T) = \big(\dim T + 1,\, w(T)\big)$, for all linear subspaces T of $\mathbb{P}^n(K)$.

Lemma (Faltings-Wüstholz)

There exists a unique filtration

$$\emptyset = T_0 \subsetneq T_1 \subsetneq \cdots \subsetneq T_t = \mathbb{P}^n(\overline{\mathbb{Q}})$$

of linear subspaces of $\mathbb{P}^n(K)$, *such that the lower boundary of* $C(\mathbf{c})$ *is a concave polygon with vertices* $P(T_0), \ldots, P(T_t)$.

An asymptotic result for the successive minima

Let $\emptyset = T_0 \subsetneq T_1 \subsetneq \cdots \subsetneq T_t = \mathbb{P}^n(K)$ be the filtration of Faltings-Wüstholz. Put

$$d_i := \dim T_i + 1 \quad (i = 0, \ldots, t),$$

$$s_i := \frac{w(T_i) - w(T_{i-1})}{\dim T_i - \dim T_{i-1}} \quad (i = 1, \ldots, t)$$

(the slope of the line segment from $P(T_{i-1})$ to $P(T_i)$).

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Theorem (Schmidt, 1993, Faltings-Wüstholz, 1994; Ferretti, E.)

For every $\delta > 0$ there is $Q_1 = Q_1(\delta) > 1$ such that for $Q \geqslant Q_1$, $i = 1, \dots, t$,

$$Q^{s_i - \delta} \leqslant \lambda_{d_{i-1}+1}(Q) \leqslant \cdots \leqslant \lambda_{d_i}(Q) \leqslant Q^{s_i + \delta};$$

$$H_Q(\mathbf{x}) \geqslant Q^{s_i - \delta} \text{ for } \mathbf{x} \in \mathbb{P}^n(K) \setminus T_{i-1}.$$

About the spaces T_i

For a linear form L define $H(L)$ to be the height of its coefficient vector.
For a linear subspace T of $\mathbb{P}^n(K)$, define
$H(T) := H(\mathbf{a}_0 \wedge \cdots \wedge \mathbf{a}_m)$, where $\mathbf{a}_0, \ldots, \mathbf{a}_m$ is any basis of T.

Lemma

Suppose that $H(L_{iv}) \leqslant H$ for $v \in S$, $i = 0, \ldots, n$. Then

$$H(T_i) \leqslant (\sqrt{n}H)^{4^n} \quad (i = 1, \ldots, t).$$

Thus, the spaces T_i belong to a finite, effectively computable collection independent of the c_{iv}.

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Thus, the spaces T_i belong to a finite, effectively computable collection independent of the c_{iv}.

Open problem: To give more precise information about the spaces T_i.

An example

Let $a_0, \ldots, a_n \in K^*$ and suppose that

$$\{L_{0v}, \ldots, L_{nv}\} \subset \left\{X_0, \ldots, X_n, \sum_{i=0}^{n} a_i X_i\right\} \text{ for } v \in S.$$

Then

$$T_i = \left\{\mathbf{x} \in \mathbb{P}^n(K) : \sum_{k \in A_{ij}} a_k x_k = 0 \ \ (j = 0, \ldots, n - d_i)\right\}$$

where $d_i = \dim T_i + 1$, and $A_{i0}, \ldots, A_{i,n-d_i}$ are non-empty, pairwise disjoint subsets of $\{0, \ldots, n\}$.

Back to systems of inequalities

Consider again

$$(2) \qquad \frac{|L_{iv}(\mathbf{x})|_v}{\|\mathbf{x}\|_v} \leqslant H(\mathbf{x})^{-c_{iv}} \ (v \in S,\ i = 0, \ldots, n) \quad \text{in } \mathbf{x} \in \mathbb{P}^n(K).$$

Assume again $c_{iv} \geqslant 0$, $\sum_{v \in S} \sum_{i=0}^{n} c_{iv} > n+1$.

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Corollary

Let m be the smallest index such that $s_m > 1$. Then (2) has only finitely many solutions outside T_{m-1}.

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Corollary

Let m be the smallest index such that $s_m > 1$. Then (2) has only finitely many solutions outside T_{m-1}.

Proof.

Choose $\delta > 0$ with $s_m - \delta > 1$. Let $\mathbf{x}$ be a solution of (2) outside T_{m-1} with $H(\mathbf{x}) > Q_1(\delta)$. Put $Q := H(\mathbf{x})$.
Then $H_Q(\mathbf{x}) \leqslant Q$ since $\mathbf{x}$ satisfies (2) and $H_Q(\mathbf{x}) \geqslant Q^{s_m - \delta}$ since $\mathbf{x} \notin T_{m-1}$. Contradiction. It follows that $H(\mathbf{x}) \leqslant Q_1(\delta)$. □

What about the solutions in T_{m-1}?

Consider

$$(2) \qquad \frac{|L_{iv}(\mathbf{x})|_v}{\|\mathbf{x}\|_v} \leqslant H(\mathbf{x})^{-c_{iv}} \ (v \in S,\ i = 0, \dots, n) \quad \text{in } \mathbf{x} \in \mathbb{P}^n(K),$$

and keep our above assumptions. Let m be as above and assume $m \geqslant 2$, $\dim T_{m-1} > 0$. Put

$$r_v := \operatorname{rank}\{L_{iv}|_{T_{m-1}} : c_{iv} > 0\}.$$

Theorem (E.)

(i) Assume there is $v \in S$ with $r_v = \dim T_{m-1} + 1$. Then (2) has only finitely many solutions in T_{m-1}.

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Theorem (E.)

(i) Assume there is $v \in S$ with $r_v = \dim T_{m-1} + 1$. Then (2) has only finitely many solutions in T_{m-1}.

(ii) Assume that $r_v \leqslant \dim T_{m-1}$ for all $v \in S$ and $s_{m-1} < 1$. Then the solutions of (2) in T_{m-1} are Zariski dense in T_{m-1}.

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Open problem: What if $r_v \leqslant \dim T_{m-1}$ for all $v \in S$ and $s_{m-1} = 1$?

An extension of the Subspace Theorem: Assumptions

Let $X \subset \mathbb{P}^R$ be a projective subvariety defined over K, of dimension n, degree $\deg X$ and exponential height $H(X) := H(\text{Chow form of } X)$.

We consider systems of inequalities

$$\frac{|f_{iv}(\mathbf{x})|_v^{1/\deg f_{iv}}}{\|\mathbf{x}\|_v} \leqslant H(\mathbf{x})^{-c_{iv}} \;(v \in S,\, i = 0, \ldots, n) \;\text{ in } \mathbf{x} \in X(K),$$

with homogeneous $f_{iv} \in K[X_0, \ldots, X_R]$ and $c_{iv} \geqslant 0$.

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with homogeneous $f_{iv} \in K[X_0, \dots, X_R]$ and $c_{iv} \geqslant 0$.

We assume that

$$\{\mathbf{x} \in X(\overline{\mathbb{Q}}) : \ f_{0v}(\mathbf{x}) = \cdots = f_{nv}(\mathbf{x}) = 0\} = \emptyset \ \text{ for } v \in S,$$

$$\sum_{v \in S} \sum_{i=0}^{n} c_{iv} = n + 1 + \delta \text{ with } \delta > 0.$$

An extension of the Subspace Theorem: Statement

Theorem (Ferretti, E., 2008; Corvaja, Zannier 2004 for $X = \mathbb{P}^n$)

The set of solutions of the system

$$\frac{|f_{iv}(\mathbf{x})|_v^{1/\deg f_{iv}}}{\|\mathbf{x}\|_v} \leqslant H(\mathbf{x})^{-c_{iv}} \ (v \in S,\ i = 0, \ldots, n) \quad \text{in } \mathbf{x} \in X(K)$$

is not Zariski dense in X.

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$$\frac{|f_{iv}(\mathbf{x})|_v^{1/\deg f_{iv}}}{\|\mathbf{x}\|_v} \leqslant H(\mathbf{x})^{-c_{iv}} \ (v \in S,\ i = 0, \ldots, n) \quad \text{in } \mathbf{x} \in X(K)$$

is not Zariski dense in X. More precisely, the set of solutions is contained in $X \bigcap \left(\bigcup_{i=1}^{t} \{G_i = 0\} \right)$, where $G_1, \ldots, G_t \in K[X_0, \ldots, X_R]$ are homogeneous polynomials not vanishing identically on X, of degree at most

$$8(n+2)^3 \Delta^{n+1} (\deg X)(1 + \delta^{-1})$$

where $\Delta := \mathrm{lcm}\{\deg f_{iv} :\ v \in S,\ i = 0, \ldots, n\}$.

A further refinement

Theorem

There is an effectively computable, proper projective subvariety $Y \subsetneq X$, defined over K, such that

$$\frac{|f_{iv}(\mathbf{x})|_v^{1/\deg f_{iv}}}{\|\mathbf{x}\|_v} \leqslant H(\mathbf{x})^{-c_{iv}} \ (v \in S,\ i = 0, \ldots, n) \quad \text{in } \mathbf{x} \in X(K),$$

has only finitely many solutions outside Y.

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Theorem

There is an effectively computable, proper projective subvariety $Y \subsetneq X$*, defined over* K*, such that*

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has only finitely many solutions outside Y*. We may choose* $Y = X \cap \{G = 0\}$*, where* $G \in K[X_0, \ldots, X_R]$ *is homogeneous with*

$$\deg G \leqslant 8(n+2)^3 \deg X \Delta^{n+1}(1 + \delta^{-1}), \qquad H(G) \leqslant c_1 H^{c_2},$$

where $\Delta := \mathrm{lcm}\{\deg f_{iv} :\ v \in S,\ i = 0, \ldots, n\}$,
$H := \max\{H(X);\ H(f_{iv}) :\ v \in S,\ i = 0, \ldots, n\}$,
c_1, c_2 *effectively computable in terms of* $R, n, \deg X, \Delta, \delta$.

Idea of the proof

Construct an embedding $\Psi : X \hookrightarrow \mathbb{P}^M : \mathbf{x} \mapsto (g_0(\mathbf{x}), \ldots, g_M(\mathbf{x}))$ with homogenenous polynomials $g_0, \ldots, g_M$ of equal degree such that if $\mathbf{x} \in X(K)$ is a solution to

$$\frac{|f_{iv}(\mathbf{x})|_v^{1/\deg f_{iv}}}{\|\mathbf{x}\|_v} \leqslant H(\mathbf{x})^{-c_{iv}} \ (v \in S,\ i = 0, \ldots, n),$$

then $\mathbf{y} = \Psi(\mathbf{x})$ satisfies

$$\frac{|L_{iv}(\mathbf{y})|_v}{\|\mathbf{y}\|_v} \ll H(\mathbf{y})^{-d_{iv}} \quad (v \in S,\ i = 0, \ldots, M), \quad \mathbf{y} \in \mathbb{P}^M(K)$$

for certain linear forms L_{iv} and reals d_{iv} with $d_{iv} \geqslant 0$ and $\sum_{v \in S} \sum_{i=0}^{M} d_{iv} > M + 1$.

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for certain linear forms L_{iv} and reals d_{iv} with $d_{iv} \geqslant 0$ and $\sum_{v \in S} \sum_{i=0}^{M} d_{iv} > M + 1$.

Ferretti, E.: Such an embedding exists with $\deg g_i = [8(n+2)^3 \deg X \Delta^{n+1} \delta^{-1}]$.

Open problems

1. The system of inequalities

$$\frac{|f_{iv}(\mathbf{x})|_v^{1/\deg f_{iv}}}{\|\mathbf{x}\|_v} \leqslant H(\mathbf{x})^{-c_{iv}} \ (v \in S,\ i = 0, \dots, n)$$

has only finitely many solutions in $(X \setminus Y)(K)$, where Y can be chosen from a finite collection depending on
$\delta = \sum_{v \in S} \sum_{i=0}^{n} c_{iv} - (n+1)$.

Is this dependence on δ necessary?

2. More precise information on Y.