Root separation of polynomials

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Let $f = a_0X^n + \cdots + a_n \in \mathbb{Z}[X]$ (with $n \geqslant 2$) be a polynomial with n distinct roots in $\mathbb{C}$, say $\alpha_1, \ldots, \alpha_n$. Define the minimal root distance of f by $\mathrm{sep}(f) := \min_{i<j} |\alpha_i - \alpha_j|$ and define the height of f by $H(f) := \max_i |a_i|$. According to an elementary inequality of Mahler, we have

$$\mathrm{sep}(f) \geqslant c(n)H(f)^{1-n}$$

where $c(n)$ is an effectively computable number, depending on n only. In 2006, Schönhage proved that in terms of $H(f)$ this is best possible if $n \geqslant 3$, i.e., for $n = 2, 3$ there are $c_1 > 0$ and polynomials $f \in \mathbb{Z}[X]$ of degree n and of arbitrarily height, such that $\mathrm{sep}(f) \leqslant c_1 H(f)^{1-n}$. Schönhage's proof uses continued fractions. On the other hand, for polynomials $f \in \mathbb{Z}[X]$ of degree $n \geqslant 4$ one has $\mathrm{sep}(f) \geqslant c_2(n)H(f)^{1-n}(\log 2H(f))^{1/(10n-6)}$, with $c_2(n)$ effectively computable in terms of n. Here, the proof is based on Baker's method.

We would like to discuss generalizations for the minimal m-cluster distance $\mathrm{sep}_m(f) := \min_I \prod_{\{i,j\}\subset I} |\alpha_i - \alpha_j|$, where the minimum is taken over all subsets I of $\{1, \ldots, n\}$ of cardinality m, and for p-adic analogues where we take distances with respect to the p-adic absolute value instead of the ordinary absolute value.