Results and open problems related to Schmidt's Subspace Theorem

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http://pub.math.leidenuniv.nl/∼evertsejh/lectures.shtml

Roth's Theorem

Define

$$H(\xi) = \max(|p|, |q|), \text{ where } \xi = p/q,\ p, q \in \mathbb{Z},\ \gcd(p, q) = 1.$$

Theorem (Roth, 1955)

Let α be a real algebraic number and $\delta > 0$. Then the inequality

(1) $$|\alpha - \xi| \leq H(\xi)^{-2-\delta} \text{ in } \xi \in \mathbb{Q}$$

has only finitely many solutions.

Roth's proof, and later proofs of his Theorem, are ineffective, i.e., they do not give a method to determine the solutions.

A semi-effective result

The minimal polynomial of an algebraic number α is the irreducible polynomial $F \in \mathbb{Z}[X]$ with coprime coefficients such that $F(\alpha) = 0$.

We define the height $H(\alpha) := \max|\text{coeff. of } F|$.

Theorem (Bombieri, van der Poorten, 1987)

Let $\delta > 0$, $K = \mathbb{Q}(\alpha)$, $[K : \mathbb{Q}] = d$. Then for the solutions $\xi \in \mathbb{Q}$ of

$$|\alpha - \xi| \leq H(\xi)^{-2-\delta}$$

we have $H(\xi) \leq \max\left(B^{\mathrm{ineff}}(\delta, K),\ H(\alpha)^{c^{\mathrm{eff}}(\delta,d)}\right)$.

Here c^{eff}, B^{ineff} are constants, effectively, resp. not effectively computable from the method of proof, depending on the parameters between the parentheses.

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Equivalent formulation, 'Roth's theorem with moving targets'

Let K be a number field of degree d and $\delta > 0$. Then there are only finitely many pairs $(\xi, \alpha) \in \mathbb{Q} \times K$ such that

$$|\alpha - \xi| \leq H(\xi)^{-2-\delta}, \quad H(\xi) > H(\alpha)^{c^{\text{eff}}(\delta,d)}.$$

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Similar results follow from work of Vojta (1995), Corvaja (1997), McQuillan ($c^{\mathrm{eff}}(\delta, d) = O(d(1 + \delta^{-2}))$, published ?).

Schmidt's Subspace Theorem

Let $\overline{\mathbb{Q}}$ denote the field of algebraic numbers in $\mathbb{C}$ and let

$$L_i(\mathbf{X}) = \alpha_{i1}X_1 + \cdots + \alpha_{in}X_n \quad (i = 1, \ldots, n)$$

be linearly independent linear forms with coefficients $\alpha_{ij} \in \overline{\mathbb{Q}}$.

For $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{Z}^n$, put $\|\mathbf{x}\| := \max_i |x_i|$.

Theorem (W.M. Schmidt, 1972)

Let $\delta > 0$. Then the set of solutions of

(2) $$|L_1(\mathbf{x}) \cdots L_n(\mathbf{x})| \leq \|\mathbf{x}\|^{-\delta} \text{ in } \mathbf{x} \in \mathbb{Z}^n$$

is contained in finitely many proper linear subspaces of $\mathbb{Q}^n$.

There are generalizations where the unknowns are taken from a number field and various archimedean and non-archimedean absolute values are involved. (Schmidt, Schlickewei)

Systems of inequalities

By a combinatorial argument, the inequality (2) $|L_1(\mathbf{x}) \cdots L_n(\mathbf{x})| \le \|\mathbf{x}\|^{-\delta}$ can be reduced to finitely many systems of inequalities of the shape

$$(3) \qquad |L_1(\mathbf{x})| \le \|\mathbf{x}\|^{c_1}, \ldots, |L_n(\mathbf{x})| \le \|\mathbf{x}\|^{c_n} \quad \text{in } \mathbf{x} \in \mathbb{Z}^n,$$

where $c_1 + \cdots + c_n < 0$.

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where $c_1 + \cdots + c_n < 0$.

Idea.

Let $\mathbf{x} \in \mathbb{Z}^n$ be a solution of (2). Then

$$|L_1(\mathbf{x})| \leq \|\mathbf{x}\|^{c_1(\mathbf{x})}, \ldots, |L_n(\mathbf{x})| \leq \|\mathbf{x}\|^{c_n(\mathbf{x})}$$

with

$$\mathbf{c}(\mathbf{x}) := (c_1(\mathbf{x}), \ldots, c_n(\mathbf{x})) \in \text{bounded set } S.$$

Cover S by a very fine, finite grid. Then $\mathbf{x}$ satisfies (3) with $\mathbf{c} = (c_1, \ldots, c_n)$ a grid point very close to $\mathbf{c}(\mathbf{x})$. □

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where $c_1 + \cdots + c_n < 0$.

Thus, the following is equivalent to the Subspace Theorem:

Theorem

The solutions of (3) *lie in finitely many proper linear subspaces of* $\mathbb{Q}^n$.

A refinement of the Subspace Theorem

Let again $L_1, \dots, L_n$ be linearly independent linear forms in $X_1, \dots, X_n$ with coefficients in $\overline{\mathbb{Q}}$ and $c_1, \dots, c_n$ reals with $c_1 + \cdots + c_n < 0$. Consider again

$$(3) \qquad |L_1(\mathbf{x})| \le \|\mathbf{x}\|^{c_1}, \dots, |L_n(\mathbf{x})| \le \|\mathbf{x}\|^{c_n} \quad \text{in } \mathbf{x} \in \mathbb{Z}^n.$$

Theorem (Vojta (1989), Schmidt (1993), Faltings-Wüstholz (1994))

There is an effectively computable, proper linear subspace T^{exc} of $\mathbb{Q}^n$ such that (3) has only finitely many solutions outside T^{exc}.

The space T^{exc} belongs to a finite collection, depending only on $L_1, \dots, L_n$ and independent of $c_1, \dots, c_n$.

This refinement can be deduced from Schmidt's basic Subspace Theorem, so it is in fact equivalent to Schmidt's basic Subspace Theorem.

About the exceptional subspace

Assume for simplicity that $L_1, \ldots, L_n$ have real algebraic coefficients.

For a linear subspace T of $\mathbb{Q}^n$, we say that a subset $\{L_{i_1}, \ldots, L_{i_m}\}$ of $\{L_1, \ldots, L_n\}$ is linearly independent on T if no non-trivial $\mathbb{R}$-linear combination of $L_{i_1}, \ldots, L_{i_m}$ vanishes identically on T.

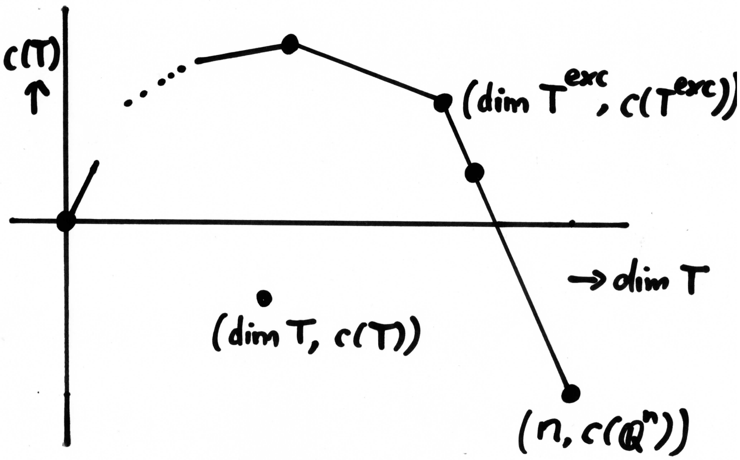
For a linear subspace T of $\mathbb{Q}^n$ define $c(T)$ to be the minimum of the quantities $c_{i_1} + \cdots + c_{i_m}$, taken over all subsets $\{L_{i_1}, \ldots, L_{i_m}\}$ of $\{L_1, \ldots, L_n\}$ of cardinality $m = \dim T$ that are linearly independent on T.

T^{exc} is the unique proper linear subspace T of $\mathbb{Q}^n$ such that

$$\frac{c(\mathbb{Q}^n) - c(T)}{n - \dim T} \text{ is minimal,}$$

subject to this condition, $\dim T$ is minimal.

About the exceptional subspace



An effective estimate

Lemma (E., Ferretti, 2013)

Suppose that the coefficients of $L_1, \ldots, L_n$ have heights $\leq H$.

Then T^{exc} has a basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_m\} \subset \mathbb{Z}^n$ with

$$\|\mathbf{x}_i\| \leq (\sqrt{n}H^n)^{4^n} \quad (i = 1, \ldots, m).$$

Open problem

Is there an efficient method to determine T^{exc} in general?

Easy combinatorial expression of T^{exc} in terms of $L_1, \ldots, L_n$, $c_1, \ldots, c_n$?

Remarks

With the present methods of proof it is not possible to determine effectively the solutions of

$$(3) \qquad |L_1(\mathbf{x})| \le \|\mathbf{x}\|^{c_1}, \ldots, |L_n(\mathbf{x})| \le \|\mathbf{x}\|^{c_n} \quad \text{in } \mathbf{x} \in \mathbb{Z}^n$$

outside T^{exc}.

It is possible to give an explicit upper bound for the minimal number of proper linear subspaces of $\mathbb{Q}^n$ whose union contains all solutions of (3).

This bound depends on n, $\delta := -(c_1 + \cdots + c_n)$ and on the heights and degrees of the coefficients of $L_1, \ldots, L_n$ (Schmidt (1989),..., E. and Ferretti (2013)).

With the present methods it is not possible to estimate from above the number of solutions of (3) outside T^{exc}.

A semi-effective version of the Subspace Theorem

Let $L_1, \ldots, L_n$ be linearly independent linear forms in $X_1, \ldots, X_n$ and $c_1, \ldots, c_n$ reals such that:

- the coefficients of $L_1, \ldots, L_n$ have heights $\leq H$ and generate a number field K of degree d;
- $c_1 + \cdots + c_n = -\delta < 0$.

Theorem

For every solution $\mathbf{x}$ *of*

$$(4) \qquad |L_1(\mathbf{x})| \leq \|\mathbf{x}\|^{c_1}, \ldots, |L_n(\mathbf{x})| \leq \|\mathbf{x}\|^{c_n} \quad \text{with } \mathbf{x} \in \mathbb{Z}^n \setminus T^{\mathrm{exc}}$$

we have $\|\mathbf{x}\| \leq \max\left(B^{\mathrm{ineff}}(n, \delta, K), H^{c^{\mathrm{eff}}(n,\delta,d)}\right)$.

Proof.

Small modification in the proof of the Subspace Theorem. □

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we have $\|\mathbf{x}\| \leq \max\left(B^{\mathrm{ineff}}(n, \delta, K), H^{c^{\mathrm{eff}}(n,\delta,d)}\right)$.

We may take $c^{\mathrm{eff}}(n, \delta, d) = \exp\left(10^{6n}(1 + \delta^{-3}) \log 4d \log\log 4d\right)$.

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we have $\|\mathbf{x}\| \leq \max\left(B^{\mathrm{ineff}}(n, \delta, K), H^{c^{\mathrm{eff}}(n,\delta,d)}\right)$.

This may be viewed as a version of the Subspace Theorem with moving targets, where we have only finitely many tuples $(\mathbf{x}, L_1, \ldots, L_n)$ with (4), such that the coefficients of $L_1, \ldots, L_n$ lie in a given number field K and have small heights with respect to $\mathbf{x}$.
(How to compare this with a result of Ru and Vojta?)

A conjectural improvement

Keep the assumptions

- $L_1, \dots, L_n$ are linearly independent linear forms, whose coefficients have heights $\leq H$ and generate a number field of degree d;
- $c_1 + \cdots + c_n = -\delta < 0$.

Conjecture 1

There are an effectively computable constant $c^{\mathrm{eff}}(n, \delta, d) > 0$ and a constant $B'(n, \delta, d) > 0$ such that for every solution $\mathbf{x}$ of

$$(4) \qquad |L_1(\mathbf{x})| \leq \|\mathbf{x}\|^{c_1}, \dots, |L_n(\mathbf{x})| \leq \|\mathbf{x}\|^{c_n} \quad \text{with } \mathbf{x} \in \mathbb{Z}^n \setminus T^{\mathrm{exc}}$$

we have $\|\mathbf{x}\| \leq \max\left(B'(n, \delta, d), H^{c^{\mathrm{eff}}(n,\delta,d)}\right)$.

(In moving targets terms: there are only finitely many tuples $(\mathbf{x}, L_1, \dots, L_n)$ with (4) such that the coefficients of $L_1, \dots, L_n$ have bounded degree and have heights small compared with $\mathbf{x}$).

A conjectural improvement

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- $L_1, \ldots, L_n$ are linearly independent linear forms, whose coefficients have heights $\leq H$ and generate a number field of degree d;
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we have $\|\mathbf{x}\| \leq \max\left(B'(n, \delta, d), H^{c^{\mathrm{eff}}(n,\delta,d)}\right)$.

This is hopeless with B' effective. But what if we allow B' to be ineffective?

abc-type inequalities

Let K be an algebraic number field of degree d and discriminant D_K.
Let a, b, c be non-zero elements of O_K with $a + b = c$.
Put $H_K(a,b,c) := \prod_{\sigma: K \hookrightarrow \mathbb{C}} \max(|\sigma(a)|, |\sigma(b)|, |\sigma(c)|)$.

Theorem 1 (Effective abc-inequality, Győry, 1978)

We have $H_K(a,b,c) \leq (2|N_{K/\mathbb{Q}}(abc)|)^{c_1(d)|D_K|^{c_2(d)}}$ with $c_1(d), c_2(d)$ effectively computable in terms of d.

Proof.

Baker-type logarithmic forms estimates. □

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Proof.

Baker-type logarithmic forms estimates. □

Theorem 2 (Semi-effective abc-inequality, well-known)

For every $\delta > 0$ *we have* $H_K(a, b, c) \leq C^{\text{ineff}}(K, \delta)|N_{K/\mathbb{Q}}(abc)|^{1+\delta}$.

Proof.

Roth's Theorem over number fields. □

A very weak abc-conjecture

Let again K be a number field of degree d and discriminant D_K.

Conjecture 2 (Very weak abc-conjecture)

There are a constant $C(d, \delta) > 0$ and an effectively computable constant $c^{\mathrm{eff}}(d, \delta) > 0$ with the following property:
for every non-zero $a, b, c \in O_K$ with $a + b = c$ and every $\delta > 0$ we have

$$H_K(a, b, c) \leq C(d, \delta)|D_K|^{c^{\mathrm{eff}}(d,\delta)}|N_{K/\mathbb{Q}}(abc)|^{1+\delta}.$$

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$$H_K(a,b,c) \leq C(d,\delta)|D_K|^{c^{\mathrm{eff}}(d,\delta)}|N_{K/\mathbb{Q}}(abc)|^{1+\delta}.$$

Conjecture 1 $\Longrightarrow$ Conjecture 2 (idea).

Choose a $\mathbb{Z}$-basis $\{\omega_1, \ldots, \omega_d\}$ of O_K with conjugates bounded from above in terms of D_K. Write $a = \sum_{i=1}^{d} x_i\omega_i$, $b = \sum_{i=1}^{d} y_i\omega_i$ with $x_i, y_i \in \mathbb{Z}$. Then $\mathbf{x} = (x_1, \ldots, y_d)$ satisfies one of finitely many systems of inequalities of the type

$$|L_i(\mathbf{x})| \leq \|\mathbf{x}\|^{c_i} \quad (i = 1, \ldots, 2d)$$

where the L_i are linear forms whose coefficients lie in the Galois closure of K and have heights bounded above in terms of $|D_K|$. □

Discriminants of binary forms

Definition

The discriminant of a binary form

$$F = a_0X^n + a_1X^{n-1}Y + \cdots + a_nY^n = \prod_{i=1}^{n}(\alpha_i X - \beta_i Y)$$

is given by $D(F) = \prod_{1 \leq i < j \leq n} (\alpha_i\beta_j - \alpha_j\beta_i)^2$.

This is a homogeneous polynomial in $\mathbb{Z}[a_0, \ldots, a_n]$ of degree $2n - 2$.

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For a matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ we define $F_A(X, Y) = F(aX + bY, cX + dY)$.

Two binary forms $F, G \in \mathbb{Z}[X, Y]$ are called equivalent if $G = \pm F_A$ for some $A \in \mathrm{GL}(2, \mathbb{Z})$.

Equivalent binary forms have the same discriminant.

A finiteness result for binary forms of given discriminant

Theorem (Lagrange ($n = 2$, 1773), Hermite ($n = 3$, 1851), Birch and Merriman ($n \geq 4$, 1972))

For every $n \geq 2$ and $D \neq 0$, there are only finitely many equivalence classes of binary forms $F \in \mathbb{Z}[X, Y]$ of degree n and discriminant D.

The proofs of Lagrange and Hermite are effective (in that they allow to compute a full system of representatives for the equivalence classes), that of Birch and Merriman is ineffective.

An effective finiteness result

Define the height of $F = a_0X^n + a_1X^{n-1}Y + \cdots + a_nY^n \in \mathbb{Z}[X, Y]$ by $H(F) := \max_i |a_i|$.

Theorem 3 (E., Győry, recent improvement of result from 1991)

Let $F \in \mathbb{Z}[X, Y]$ be a binary form of degree $n \geq 4$ and discriminant $D \neq 0$. Then F is equivalent to a binary form G for which

$$H(G) \leq \exp\left((16n^3)^{25n^2}|D|^{5n-3}\right).$$

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More precise versions of the arguments of Lagrange and Hermite give a bound $H(G) \leq \text{constant} \cdot |D|$ in case that F has degree ≤ 3.

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Proof (idea).

Let L be the splitting field of F. Assume for convenience that $F = \prod_{i=1}^{n}(\alpha_i X - \beta_i Y)$ with $\alpha_i, \beta_i \in O_L\ \forall i$. Put $\Delta_{ij} := \alpha_i\beta_j - \alpha_j\beta_i$ and apply an explicit version of the effective abc-inequality (Theorem 1) to the identities

$$\Delta_{ij}\Delta_{kl} + \Delta_{jk}\Delta_{il} = \Delta_{ik}\Delta_{jl} \quad (1 \leq i, j, k, l \leq n).$$

□

A semi-effective finiteness result

Theorem 4 (E., 1993)

Let $F \in \mathbb{Z}[X,Y]$ be a binary form of degree $n \geq 4$, discriminant $D \neq 0$ and splitting field L. Then F is equivalent to a binary form G for which

$$H(G) \leq C^{\mathrm{ineff}}(n,L) \cdot |D|^{21/(n-1)}.$$

Proof (idea).

Apply the semi-effective abc-inequality Theorem 2 to the identities

$$\Delta_{ij}\Delta_{kl} + \Delta_{jk}\Delta_{il} = \Delta_{ik}\Delta_{jl} \quad (1 \leq i,j,k,l \leq n).$$

□

A conjecture

Conjecture 3

Let $F \in \mathbb{Z}[X, Y]$ be a binary form of degree $n \geq 4$ and discriminant $D \neq 0$. Then F is equivalent to a binary form G for which

$$H(G) \leq C_1(n)|D|^{c_2^{\mathrm{eff}}(n)}.$$

Conjecture 2 $\Longrightarrow$ Conjecture 3.

Let L be the splitting field of F. Following the proof of Theorem 4 and using the very weak abc-conjecture, one obtains that there is G equivalent to F such that

$$H(G) \leq C_3(n)|D_L|^{c_4^{\mathrm{eff}}(n)}|D|^{21/(n-1)}.$$

Use that D_L divides $D^{n!}$. □

A conjecture

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$$H(G) \leq C_1(n)|D|^{c_2^{\mathrm{eff}}(n)}.$$

Problem

What is the right value of the exponent on $|D|$?

A function field analogue

Let k be an algebraically closed field of characteristic 0, $K = k(t)$, $R = k[t]$.
Define $|\cdot|$ on $k(t)$ by $|f/g| := e^{\deg f - \deg g}$ for $f, g \in R$.

Define the height of $F = a_0X^n + a_1X^{n-1}Y + \cdots + a_nY^n \in R[X, Y]$ by $H(F) := \max_i |a_i|$.

Call two binary forms $F, G \in R[X, Y]$ equivalent if $G = uF_A$ for some $u \in k^*$, $A \in \mathrm{GL}(2, R)$.

Theorem 5 (W. Zhuang)

Let $F \in R[X, Y]$ be a binary form of degree $n \geq 4$ and discriminant $D \neq 0$. Then F is equivalent to a binary form G for which

$$H(G) \leq e^{n^2+4n+14}|D|^{20+7/(n-2)}.$$

Proof.

Follow the proof over $\mathbb{Z}$ and apply Mason's abc-theorem for function fields. □

Thank you for your attention!