On Schmidt's Subspace Theorem

Jan-Hendrik Evertse
Universiteit Leiden

Conference on Arithmetic Algebraic Geometry
on the occasion of Gerd Faltings' 60th birthday

June 10, 2014, Bonn

Faltings' Product Theorem

Faltings' Product Theorem is a powerful non-vanishing result from algebraic geometry with many applications to Diophantine geometry, in particular Schmidt's Subspace Theorem.

It explains the structure of the set of points at which a given multihomogeneous polynomial has not too small index (weighted multiplicity).

The index

Let $m, n_1, \ldots, n_m \geq 2$. Write $\mathbf{X}_i = (X_{i1}, \ldots, X_{i,n_i})$, and let $F \in \overline{\mathbb{Q}}[\mathbf{X}_1, \ldots, \mathbf{X}_m]$ be homogeneous of degree d_i in $\mathbf{X}_i$, for $i = 1, \ldots, m$.

For a tuple $\mathbf{a} = (a_{ij})_{i=1,\ldots,m,\, j=1,\ldots,n_i}$ of non-negative integers, define the partial derivative

$$D_{\mathbf{a}} := \prod_{i=1}^{m} \prod_{j=1}^{n_i} \frac{\partial^{a_{ij}}}{\partial X_{ij}^{a_{ij}}}$$

and its weighted order

$$(\mathbf{a}/\mathbf{d}) := \sum_{i=1}^{m} \frac{1}{d_i} \Big(\sum_{j=1}^{n_i} a_{ij} \Big).$$

Then the *index* of F at $\mathbf{x} \in \mathbb{P}^{n_1-1} \times \cdots \times \mathbb{P}^{n_m-1}$ is given by

$$I(F, \mathbf{x}) := \max \Big\{ \sigma : \quad D_{\mathbf{a}} F(\mathbf{x}) = 0 \ \text{ for all } \mathbf{a} \text{ with } (\mathbf{a}/\mathbf{d}) \leq \sigma \Big\}.$$

Heights

For $\mathbf{x} \in \mathbb{P}^n(\overline{\mathbb{Q}})$, denote by $h(\mathbf{x})$ the (absolute logarithmic Weil) height of $\mathbf{x}$.

For a polynomial F with coefficients in $\overline{\mathbb{Q}}$, denote by $h(F)$ the height of the vector of coefficients of F.

For an algebraic subvariety X of $\mathbb{P}^n$ defined over $\overline{\mathbb{Q}}$ denote by $h(X)$ the height of the Chow form of X.

Statement of Faltings' Product Theorem

Let $F \in \overline{\mathbb{Q}}[\mathbf{X}_1, \dots, \mathbf{X}_m]$, $\mathbf{X}_i$ block of n_i variables, $\deg_{\mathbf{X}_i} F = d_i$ for $i = 1, \dots, m$, and let $\epsilon > 0$.

Theorem (Faltings' Product Theorem, 1991)

There are $\omega_1, \omega_2, \omega_3 > 0$, depending only on $M := n_1 + \cdots + n_m$ and ϵ, with the following property: assume that

$$d_1/d_2, \dots, d_{m-1}/d_m > \omega_1.$$

Then

$$\left\{ \mathbf{x} \in \mathbb{P}^{n_1 - 1} \times \cdots \times \mathbb{P}^{n_m - 1} : \, I(F, \mathbf{x}) \geq \epsilon \right\} \subseteq Z_1 \times \cdots \times Z_m$$

where $Z_1 \times \cdots \times Z_m$ is a proper product subvariety of $\mathbb{P}^{n_1 - 1} \times \cdots \times \mathbb{P}^{n_m - 1}$, defined over $\overline{\mathbb{Q}}$, with

$$\sum_{i=1}^{m} d_i h(Z_i) \leq \omega_2 \cdot \big(d_1 + \cdots + d_m + h(F) \big), \qquad \prod_{i=1}^{m} \deg Z_i \leq \omega_3.$$

Statement of Faltings' Product Theorem

Let $F \in \overline{\mathbb{Q}}[\mathbf{X}_1, \ldots, \mathbf{X}_m]$, $\mathbf{X}_i$ block of n_i variables, $\deg_{\mathbf{X}_i} F = d_i$ for $i = 1, \ldots, m$, and let $\epsilon > 0$.

Theorem (Faltings' Product Theorem, 1991)

There are $\omega_1, \omega_2, \omega_3 > 0$, depending only on $M := n_1 + \cdots + n_m$ and ϵ, with the following property: assume that

$$d_1/d_2, \ldots, d_{m-1}/d_m > \omega_1.$$

Then

$$\left\{\mathbf{x} \in \mathbb{P}^{n_1-1} \times \cdots \times \mathbb{P}^{n_m-1} : I(F, \mathbf{x}) \geq \epsilon\right\} \subseteq Z_1 \times \cdots \times Z_m$$

where $Z_1 \times \cdots \times Z_m$ is a proper product subvariety of $\mathbb{P}^{n_1-1} \times \cdots \times \mathbb{P}^{n_m-1}$, defined over $\overline{\mathbb{Q}}$, with

$$\sum_{i=1}^{m} d_i h(Z_i) \leq \omega_2 \cdot \big(d_1 + \cdots + d_m + h(F)\big), \qquad \prod_{i=1}^{m} \deg Z_i \leq \omega_3.$$

van der Put, E., Ferretti, Rémond: quantitative versions of PT with explicit $\omega_1, \omega_2, \omega_3$.

Applications

1. Proof of Lang's conjecture (Faltings, 1991):
   Let A be an abelian variety, and X a subvariety of A, both defined over a number field K. Assume that X does not contain a translate of an abelian subvariety of A. Then $X(K)$ is finite.

2. Extensions to semi-abelian varieties (Vojta, 1996); explicit upper bound for $\#X(K)$ (Rémond, 2000).

3. New proof of Schmidt's Subspace Theorem (Faltings and Wüstholz, 1994).

4. Stronger quantitative versions of Roth's Theorem on the approximation of algebraic numbers by rationals (E., 1996; Bugeaud&E., 2008).

5. Stronger quantitative versions of Schmidt's Subspace theorem (E., 1996; Schlickewei &E., 2000; Ferretti &E., 2013).

Roth's Theorem

For $\xi \in \mathbb{Q}$ we define $H(\xi) = \max(|p|, |q|)$, where $\xi = p/q$, $p, q \in \mathbb{Z}$, $\gcd(p, q) = 1$.

Theorem (Roth, 1955)

Let α be a real algebraic number and $\delta > 0$. Then there are only finitely many $\xi \in \mathbb{Q}$ such that

$$|\alpha - \xi| \leq H(\xi)^{-2-\delta}.$$

Quantitative versions (upper bounds for the number of solutions) were given by Davenport& Roth (1955), Mignotte (1972), Bombieri&van der Poorten (1987), Schmidt (1988), E. (1996), Bugeaud& E. (2008).

A quantitative Roth's Theorem

Let α be a real algebraic number of degree d and $0 < \delta < 1$.

Define $H(\alpha)$ to be the maximum of the absolute values of the coefficients of the minimal polynomials of α.

A solution ξ of

(1) $$|\alpha - \xi| \leq H(\xi)^{-2-\delta} \quad \text{in } \xi \in \mathbb{Q}$$

is called *large* if $H(\xi) \geq \max(4^{1/\delta}, H(\alpha))$ and *small* otherwise.

Theorem (Bugeaud, E., 2008)

Inequality (1) *has at most*

$A := 2^{25}\delta^{-3} \log 4d \log(\delta^{-1} \log 4d)$ *large solutions, and at most*

$B := 10\delta^{-1} \log\left(\delta^{-1} \log(4H(\alpha))\right)$ *small solutions.*

Outline of proof

- Assume that $|\alpha - \xi| \leq H(\xi)^{-2-\delta}$ has 'too many' large solutions, and select from those $\xi_1, \ldots, \xi_m$ with $h(\xi_1),\ h(\xi_2)/h(\xi_1), \ldots, h(\xi_m)/h(\xi_{m-1})$ large (here $h = \log H$).
- Let $\mathbf{x} = ((\xi_1 : 1), \ldots, (\xi_m : 1)) \in \mathbb{P}^1(\mathbb{Q}) \times \cdots \times \mathbb{P}^1(\mathbb{Q})$.
  Let $d_1, \ldots, d_m$ be integers with $d_1 h(\xi_1) \approx \cdots \approx d_m h(\xi_m)$, and construct $F \in \mathbb{Z}[\mathbf{X}_1, \ldots, \mathbf{X}_m]$ of not too large height, such that F has degree d_i in $\mathbf{X}_i = (X_{i1}, X_{i2})$, and F is divisible by a high power of $X_{i1} - \alpha X_{i2}$ for $i = 1, \ldots, m$. Then show that $I(F, \mathbf{x}) \gg m\delta$.
- Apply a strong version of Roth's Lemma (= Quantitative Product Theorem over $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$), and derive a contradiction.

Roth's Lemma

Theorem

Let $0 < \epsilon < 1$, $m \geq 2$. There are $\omega_1, \omega_2 > 0$ depending only on m, ϵ with the following property:
Let $F \in \overline{\mathbb{Q}}[\mathbf{X}_1, \ldots, \mathbf{X}_m]$ be a multihomogeneous polynomial of degree d_i in $\mathbf{X}_i = (X_{1i}, X_{2i})$ for $i = 1, \ldots, m$, and let
$\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_m) \in \mathbb{P}^1(\overline{\mathbb{Q}}) \times \cdots \times \mathbb{P}^1(\overline{\mathbb{Q}})$ for $i = 1, \ldots, m$.

Assume that $d_1/d_2, \ldots, d_{m-1}/d_m > \omega_1$ and that $I(F, \mathbf{x}) \geq m\epsilon$. Then

$$\min_i d_i h(\mathbf{x}_i) \leq \omega_2 \cdot \big(d_1 + \cdots + d_m + h(F)\big).$$

Roth (1955): $\omega_1 = \omega_2 = (\epsilon^{-1})^{c^m} \quad (c > 1)$
QPT: $\omega_1 = 2m^2/\epsilon, \quad \omega_2 = (3m^2/\epsilon)^m$.

Application to Roth's Theorem: $m = c_1\delta^{-2} \log 2d$, $\epsilon = c_2\delta$,

Number of large approximants ξ of α: $\leq c_3(m \log \omega_1 + \log \omega_2)$.

Schmidt's Subspace Theorem

Let $\overline{\mathbb{Q}}$ denote the field of algebraic numbers in $\mathbb{C}$ and let

$$L_i(\mathbf{X}) = \alpha_{i1}X_1 + \cdots + \alpha_{in}X_n \quad (i = 1, \ldots, n)$$

be linearly independent linear forms with coefficients $\alpha_{ij} \in \overline{\mathbb{Q}}$.

For $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{Z}^n$, put $\|\mathbf{x}\| := \max(|x_1|, \ldots, |x_n|)$.

Theorem (W.M. Schmidt, 1972)

Let $\delta > 0$. Then the set of $\mathbf{x} \in \mathbb{Z}^n$ with

$$|L_1(\mathbf{x}) \cdots L_n(\mathbf{x})| \leq \|\mathbf{x}\|^{-\delta}$$

is contained in a union of finitely many proper linear subspaces of $\mathbb{Q}^n$.

Methods of proof

Schmidt's method, 1972

- Geometry of numbers
- Construction of auxiliary polynomial
- Application of Roth's Lemma

Faltings' and Wüstholz' method, 1994

- Inductive argument
- In each step construction of a global section of a line bundle on a product of projective varieties of large degree and an application of the Product Theorem.

Schmidt's Quantitative Subspace Theorem

Theorem (Schmidt, 1989, slight variation)

Assume that the coefficients of the linear forms L_i have heights at most H and degrees at most D. Put $\Delta := |\det(L_1, \dots, L_n)|$.
Then the solutions $\mathbf{x} \in \mathbb{Z}^n$ of

$$(2) \qquad |L_1(\mathbf{x}) \cdots L_n(\mathbf{x})| \le \Delta \|\mathbf{x}\|^{-\delta}$$

with

$$\|\mathbf{x}\| \ge \max\left(H, (n!)^{10/\delta}\right)$$

lie in at most $(2D)^{2^{27n}\delta^{-2}}$ proper linear subspaces of $\mathbb{Q}^n$.

Schmidt's bound has been improved by E. (1996), Schlickewei& E. (2002) and Ferretti& E. (2013). These improvements all use the improvement of Roth's Lemma following from the QPT.

Systems of inequalities

By a combinatorial argument, inequality (2) can be reduced to $(cn/\delta)^n$ systems of inequalities of the shape

$$(3) \qquad |L_1(\mathbf{x})| \le \Delta^{1/n}\|\mathbf{x}\|^{c_1}, \dots, |L_n(\mathbf{x})| \le \Delta^{1/n}\|\mathbf{x}\|^{c_n} \ \text{ in } \mathbf{x} \in \mathbb{Z}^n,$$

where $c_1 + \cdots + c_n \approx -\delta$.

The number of subspaces that contain the solutions of an individual system (3) is much smaller than $(cn/\delta)^n$.

Our quantitative results will be formulated for systems (3).

Technical assumptions

We consider

$$(3) \qquad |L_1(\mathbf{x})| \le \Delta^{1/n}\|\mathbf{x}\|^{c_1}, \ldots, |L_n(\mathbf{x})| \le \Delta^{1/n}\|\mathbf{x}\|^{c_n} \quad \text{in } \mathbf{x} \in \mathbb{Z}^n,$$

where $c_1, \ldots, c_n$ and

$$L_i = \alpha_{i1}X_1 + \cdots + \alpha_{in}X_n \quad (i = 1, \ldots, n)$$

satisfy the following:

$$(4) \qquad \begin{cases} H(\alpha_{ij}) \le H, \ \deg \alpha_{ij} \le D \quad \forall i, j; \\ \Delta = |\det(L_1, \ldots, L_n)| > 0; \\ \\ c_1 + \cdots + c_n \le -\delta \text{ with } 0 < \delta \le 1; \\ \max(c_1, \ldots, c_n) = 1. \end{cases}$$

An improved quantitative Subspace Theorem

Assume (4) and consider the system

(3) $\quad |L_1(\mathbf{x})| \leq \Delta^{1/n}\|\mathbf{x}\|^{c_1}, \ldots, |L_n(\mathbf{x})| \leq \Delta^{1/n}\|\mathbf{x}\|^{c_n}$ in $\mathbf{x} \in \mathbb{Z}^n$.

A solution $\mathbf{x}$ is called *large* if

$$\|\mathbf{x}\| \geq \max\left(n^{n/\delta}, H\right)$$

and *small* otherwise.

Theorem 1 (Ferretti, E., 2013)

The large solutions of (3) *lie in at most*

$$A := 2^{2n}(10n)^{20}\delta^{-3}\log\delta^{-1}\log 4D\log(\delta^{-1}\log 4D)$$

proper linear subspaces of $\mathbb{Q}^n$*, and the small solutions in at most*

$$B := 10^{3n}\delta^{-1}\log(\delta^{-1}\log 4H)$$

subspaces.

A refinement of the Subspace Theorem

Consider again

(3) $\quad |L_1(\mathbf{x})| \leq \Delta^{1/n}\|\mathbf{x}\|^{c_1}, \ldots, |L_n(\mathbf{x})| \leq \Delta^{1/n}\|\mathbf{x}\|^{c_n}$ in $\mathbf{x} \in \mathbb{Z}^n$

where $c_1 + \cdots + c_n < 0$.

Theorem (Vojta, 1989; Schmidt, 1993; Faltings&Wüstholz, 1994)

There is an effectively determinable, proper linear subspace T^{exc} of $\mathbb{Q}^n$, that can be chosen from a finite set depending only on $L_1, \ldots, L_n$, such that (3) *has only finitely many solutions outside T^{exc}.*

Very difficult open problem

Determine an upper bound for the number of solutions outside T^{exc}.

An interval result for the refinement

Assume (4) (i.e., $H(\mathrm{coeff}\, L_i) \le H$, $\deg \mathrm{coeff}\, L_i \le D$,
$\Delta := |\det(L_1, \ldots, L_n)| > 0$, $\sum_i c_i \le -\delta$ with $0 < \delta \le 1$ and $\max_i c_i = 1$)
and consider again

$$(3) \qquad |L_1(\mathbf{x})| \le \Delta^{1/n} \|\mathbf{x}\|^{c_1}, \ldots, |L_n(\mathbf{x})| \le \Delta^{1/n} \|\mathbf{x}\|^{c_n} \quad \text{in } \mathbf{x} \in \mathbb{Z}^n.$$

Theorem 2 (Ferretti, E., 2013)

Let

$$A := 2^{2n}(10n)^{20} \delta^{-3} \log \delta^{-1} \log 4D \log(\delta^{-1} \log 4D).$$

Then there are positive reals $Q_1 < \cdots < Q_{[A]-1}$ *such that for every solution* $\mathbf{x} \in \mathbb{Z}^n$ *of* (3) *with* $\mathbf{x} \notin T^{\mathrm{exc}}$ *we have*

$$\|\mathbf{x}\| < \max(n^{n/\delta}, H) \text{ or}$$
$$\|\mathbf{x}\| \in \big[Q_1, Q_1^{1+\delta/n}\big) \cup \cdots \cup \big[Q_{[A]-1}, Q_{[A]-1}^{1+\delta/n}\big).$$

$Q_1, \ldots, Q_{[A]-1}$ cannot be effectively determined from the proof.

An interval result for the refinement

Assume (4) and consider

(3) $|L_1(\mathbf{x})| \le \Delta^{1/n}\|\mathbf{x}\|^{c_1}, \ldots, |L_n(\mathbf{x})| \le \Delta^{1/n}\|\mathbf{x}\|^{c_n}$ in $\mathbf{x} \in \mathbb{Z}^n$.

Theorem 3 (Ferretti, E., 2013)

Let

$$A := 2^{2n}(10n)^{20}\delta^{-3}\log\delta^{-1}\log 4D\log(\delta^{-1}\log 4D).$$

Then there are positive reals $Q_1 < \cdots < Q_{[A]-1}$ *such that for every solution* $\mathbf{x} \in \mathbb{Z}^n$ *of* (3) *with* $\mathbf{x} \notin T^{\mathrm{exc}}$ *we have*

$$\|\mathbf{x}\| < \max(n^{n/\delta}, H) \text{ or}$$
$$\|\mathbf{x}\| \in \left[Q_1, Q_1^{1+\delta/n}\right) \cup \cdots \cup \left[Q_{[A]-1}, Q_{[A]-1}^{1+\delta/n}\right).$$

Fact

For every $Q \ge n^{n/\delta}$, the solutions of (3) with $\|\mathbf{x}\| \in \left[Q, Q^{1+\delta/n}\right)$ lie in **one** proper linear subspace of $\mathbb{Q}^n$.

About the proof of the interval result

The proof follows Schmidt's method, based on geometry of numbers, a construction of an auxiliary polynomial, and the sharp version of Roth's Lemma that follows from the Quantitative Product Theorem.

But the construction of the auxiliary polynomial is taken from the proof of Faltings' and Wüstholz.

With the method of Faltings and Wüstholz it is also possible to prove an interval result (worked out by Ferretti)
but the resulting upper bound A for the number of intervals is much larger, because of the algebraic varieties of large degree that occur in the inductive argument.

The number of solutions outside the exceptional space

An observation of Schmidt (1988) suggests, that a general explicit upper bound for the number of solutions of

$$|L_1(\mathbf{x})| \leq \Delta^{1/n}\|\mathbf{x}\|^{c_1}, \ldots, |L_n(\mathbf{x})| \leq \Delta^{1/n}\|\mathbf{x}\|^{c_n} \quad \text{in } \mathbf{x} \in \mathbb{Z}^n \text{ with } \mathbf{x} \notin T^{\mathrm{exc}}$$

would imply very strong *effective* Diophantine inequalities.

Proving such strong effective Diophantine inequalities is believed to be very difficult.

Effective approximation of algebraic numbers by rationals

Let α be a real algebraic number of degree $d \geq 3$.

Theorem (Liouville, 1844)

There is an effectively computable number $C(\alpha) > 0$ such that

$$|\alpha - \xi| \geq C(\alpha) \cdot H(\xi)^{-d} \quad \text{for } \xi \in \mathbb{Q}.$$

Theorem (A. Baker, Fel'dman, 1971)

There are effectively computable $C_1(\alpha), \epsilon(\alpha) > 0$ such that

$$|\alpha - \xi| \geq C_1(\alpha) H(\xi)^{-d+\epsilon(\alpha)} \quad \text{for } \xi \in \mathbb{Q}.$$

Here $C_1(\alpha), \epsilon(\alpha) \to 0$ as $d \to \infty$ or $H(\alpha) \to \infty$.

General effective improvements with instead of $-d + \epsilon(\alpha)$ something independent of d or $H(\alpha)$ seem to be out of reach (let alone effective Roth $|\alpha - \xi| \geq C^{\text{eff}}(\alpha, \delta) H(\xi)^{-2-\delta}$).

Connection with the Subspace Theorem

Let α be a real algebraic number of degree ≥ 4 and $\delta > 0$. Consider the system of inequalities

$$(5) \qquad |x_1 + \alpha x_2 + \alpha^2 x_3| \leq \|\mathbf{x}\|^{-2-\delta}, \quad |x_2| \leq \|\mathbf{x}\|, \ |x_3| \leq \|\mathbf{x}\|$$
$$\text{in } \mathbf{x} = (x_1, x_2, x_3) \in \mathbb{Z}^3.$$

For this system, $T^{\mathrm{exc}} = \{0\}$. Hence it has only finitely many solutions.

Proposition (E., 2010)

Let N be an upper bound for the number of solutions of (5) with $\gcd(x_1, x_2, x_3) = 1$. Then for every $\xi \in \mathbb{Q}$ we have

$$|\alpha - \xi| \geq (1 + |\alpha|)^{-1}(2N)^{-3-\delta} \cdot H(\xi)^{-3-\delta}.$$

Proof

Choose $\xi = p/q$ with $p, q \in \mathbb{Z}$, $\gcd(p, q) = 1$ and let

$$C := |\alpha - \xi| \cdot H(\xi)^{3+\delta}.$$

For any $y \in \mathbb{Z}$ define $\mathbf{x} = (x_1, x_2, x_3)$ by

$$(1 + y\alpha)(q\alpha - p) = -p + (q - py)\alpha + qy\alpha^2 = x_1 + x_2\alpha + x_3\alpha^2$$

and put $A := \frac{1}{2}\big((1 + |\alpha|)C\big)^{-1/(3+\delta)}$.

Proof

Choose $\xi = p/q$ with $p, q \in \mathbb{Z}$, $\gcd(p,q) = 1$ and let

$$C := |\alpha - \xi| \cdot H(\xi)^{3+\delta}.$$

For any $y \in \mathbb{Z}$ define $\mathbf{x} = (x_1, x_2, x_3)$ by

$$(1 + y\alpha)(q\alpha - p) = -p + (q - py)\alpha + qy\alpha^2 = x_1 + x_2\alpha + x_3\alpha^2$$

and put $A := \frac{1}{2}\big((1 + |\alpha|)C\big)^{-1/(3+\delta)}$.

Considering the integers y with $|y| \le A$ we obtain $\ge A$ tuples $\mathbf{x} \in \mathbb{Z}^3$ for which $\gcd(x_1, x_2, x_3) = 1$, and

$$\begin{aligned} |x_1 + x_2\alpha + x_3\alpha^2| &\le (1 + |\alpha|)A \cdot q \cdot C \max(|p|, |q|)^{-3-\delta} \\ &\le (1 + |\alpha|)A^{3+\delta} \max(1, |y|)^{-2-\delta} \cdot C \max(|p|, |q|)^{-2-\delta} \\ &\le (1 + |\alpha|)(2A)^{3+\delta} C \cdot \|\mathbf{x}\|^{-2-\delta} \le \|\mathbf{x}\|^{-2-\delta}. \end{aligned}$$

Hence $A \le N$, and

$$C \ge (1 + |\alpha|)^{-1}(2N)^{-3-\delta}.$$

A semi-effective version of the Subspace Theorem

Assume (4) (i.e., $H(\mathrm{coeff}\, L_i) \le H$, $\deg \mathrm{coeff}\, L_i \le D$,
$\Delta := |\det(L_1, \dots, L_n)| > 0$, $\sum_i c_i \le -\delta$ with $0 < \delta \le 1$ and $\max_i c_i = 1$)
and consider again

(3) $\quad |L_1(\mathbf{x})| \le \Delta^{1/n} \|\mathbf{x}\|^{c_1}, \dots, |L_n(\mathbf{x})| \le \Delta^{1/n} \|\mathbf{x}\|^{c_n} \quad$ in $\mathbf{x} \in \mathbb{Z}^n$.

Theorem

Let K be the n.f. generated by the coefficients of $L_1, \dots, L_n$.

There are an effectively computable constant $c^{\mathrm{eff}}(n, \delta, D) > 0$ and an ineffective constant $B^{\mathrm{ineff}}(n, \delta, K) > 0$ such that for every solution $\mathbf{x} \in \mathbb{Z}^n$ of (3) with $\mathbf{x} \notin T^{\mathrm{exc}}$ we have

$$\|\mathbf{x}\| \le \max\left(B^{\mathrm{ineff}}(n, \delta, K), H^{c^{\mathrm{eff}}(n,\delta,D)} \right).$$

Proof.

Small modification in the proof of the Subspace Theorem (well-known). □

A conjecture

Assume again (4) and consider

$$(3) \qquad |L_1(\mathbf{x})| \le \Delta^{1/n}\|\mathbf{x}\|^{c_1}, \ldots, |L_n(\mathbf{x})| \le \Delta^{1/n}\|\mathbf{x}\|^{c_n} \quad \text{in } \mathbf{x} \in \mathbb{Z}^n.$$

Conjecture 4

There are an effectively computable constant $c^{\mathrm{eff}}(n,\delta,D) > 0$ and a constant $B'(n,\delta,D) > 0$ such that for every solution $\mathbf{x} \in \mathbb{Z}^n$ of (3) with $\mathbf{x} \notin T^{\mathrm{exc}}$ we have

$$\|\mathbf{x}\| \le \max\left(B'(n,\delta,D), H^{c^{\mathrm{eff}}(n,\delta,D)}\right).$$

This is hopeless with B' effective. But what if we allow B' to be ineffective?

abc-type inequalities

Let K be an algebraic number field of degree d, $x^{(i)}$ ($i = 1, \ldots, d$) the conjugates of $x \in K$, O_K the ring of integers and D_K the discriminant of K.

Define $H_K(a,b,c) := \prod_{i=1}^{d} \max(|a^{(i)}|, |b^{(i)}|, |c^{(i)}|)$ for $a, b, c \in O_K$.

Theorem

Let a, b, c be non-zero elements of O_K with $a + b = c$. Then

(i) we have $H_K(a,b,c) \leq (2|N_{K/\mathbb{Q}}(abc)|)^{c_1(d)|D_K|^{c_2(d)}}$ with $c_1(d), c_2(d)$ effectively computable in terms of d.

(ii) for every $\delta > 0$ we have $H_K(a,b,c) \leq C^{\text{ineff}}(K,\delta)|N_{K/\mathbb{Q}}(abc)|^{1+\delta}$.

Proof.

(i) Baker theory; (ii) Roth's theorem over number fields. □

abc-type inequalities: application of Conjecture 4

Let again K be a number field of degree d.

Conjecture 5

There are a constant $C(d,\delta) > 0$ and an effectively computable constant $c^{\mathrm{eff}}(d,\delta) > 0$ with the following property:
for any non-zero $a, b, c \in O_K$ with $a + b = c$, and any $\delta > 0$ we have

$$H_K(a,b,c) \leq C(d,\delta)|D_K|^{c^{\mathrm{eff}}(d,\delta)}|N_{K/\mathbb{Q}}(abc)|^{1+\delta}.$$

abc-type inequalities: application of Conjecture 4

Let again K be a number field of degree d.

Conjecture 5

There are a constant $C(d,\delta) > 0$ and an effectively computable constant $c^{\mathrm{eff}}(d,\delta) > 0$ with the following property:
for any non-zero $a, b, c \in O_K$ with $a + b = c$, and any $\delta > 0$ we have

$$H_K(a,b,c) \leq C(d,\delta)|D_K|^{c^{\mathrm{eff}}(d,\delta)}|N_{K/\mathbb{Q}}(abc)|^{1+\delta}.$$

Conjecture 4 $\Longrightarrow$ Conjecture 5 (idea).

Choose a $\mathbb{Z}$-basis $\{\omega_1, \ldots, \omega_d\}$ of O_K with heights bounded above in terms of $|D_K|$.

Write $a = \sum_{j=1}^{d} x_j\omega_j$, $b = \sum_{j=1}^{d} y_j\omega_j$ with $x_j, y_j \in \mathbb{Z}$.

Then $\mathbf{x} = (x_1, \ldots, y_d)$ satisfies one of a finite number of systems of inequalities as considered in Conjecture 4 with linear forms with heights bounded above in terms of d and D_K. □

Approximation of algebraic numbers by algebraic numbers of bounded degree

For $\xi \in \overline{\mathbb{Q}} \subset \mathbb{C}$, define $H(\xi)$ to be the maximum of the absolute values of the coefficients of the minimal polynomial of α.

Theorem (Schmidt, 1971)

Let $\alpha \in \overline{\mathbb{Q}}$, $d \in \mathbb{Z}_{>0}$ and $\delta > 0$. Then there are only finitely many $\xi \in \overline{\mathbb{Q}}$ of degree d such that

$$|\alpha - \xi| \leq H(\xi)^{-d-1-\delta}.$$

Proof.

Consequence of the Subspace Theorem. □

Wirsing (1969) proved this earlier with $-2d - \delta$ instead of $-d - 1 - \delta$, by a different method.

Approximation of algebraic numbers by algebraic numbers of bounded degree: Vojta's conjecture

For $\xi \in \overline{\mathbb{Q}}$, define $D(\xi)$ to be the discriminant of the number field $\mathbb{Q}(\xi)$.

Conjecture (Vojta, 1987)

Let $\alpha \in \overline{\mathbb{Q}}$, $d \in \mathbb{Z}_{>0}$ and $\delta > 0$. Then there are only finitely many $\xi \in \overline{\mathbb{Q}}$ of degree d such that

$$|\alpha - \xi| \leq |D(\xi)|^{-1} H(\xi)^{-2-\delta}.$$

This implies Wirsing's Theorem since $|D(\xi)| \ll H(\xi)^{2d-2}$.

Restricting to algebraic numbers ξ in a given number field K, we obtain Roth's Theorem over number fields.

Approximation of algebraic numbers by algebraic numbers of bounded degree: application of Conjecture 4

Our Conjecture 4 implies the following:

Conjecture 6

Let $\alpha \in \overline{\mathbb{Q}}$, $d \in \mathbb{Z}_{>0}$, $\delta > 0$, *and put* $m := [\mathbb{Q}(\alpha) : \mathbb{Q}]$. *There is an effectively computable number* $c^{\mathrm{eff}}(m, d, \delta) > 0$, *such that the inequality*

$$|\alpha - \xi| \leq |D(\xi)|^{-c^{\mathrm{eff}}(m,d,\delta)} H(\xi)^{-2-\delta}$$

has only finitely many solutions in algebraic numbers ξ *of degree* d.

The method of Faltings and Wüstholz

Consider the system

$$(3) \qquad |L_1(\mathbf{x})| \leq \Delta^{1/n}\|\mathbf{x}\|^{c_1}, \ldots, |L_n(\mathbf{x})| \leq \Delta^{1/n}\|\mathbf{x}\|^{c_n} \quad \text{in } \mathbf{x} \in \mathbb{Z}^n.$$

- Assume (3) has "too many solutions" outside T^{exc} and select from those $\mathbf{x}_1, \ldots, \mathbf{x}_m$ with $h(\mathbf{x}_1), h(\mathbf{x}_2)/h(\mathbf{x}_1), \ldots, h(\mathbf{x}_m)/h(\mathbf{x}_{m-1})$ large, where $h(\mathbf{x}_i) = \log\|\mathbf{x}_i\|$. Put $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_m)$.

Consider the system

(3) $\quad |L_1(\mathbf{x})| \leq \Delta^{1/n} \|\mathbf{x}\|^{c_1}, \ldots, |L_n(\mathbf{x})| \leq \Delta^{1/n} \|\mathbf{x}\|^{c_n} \quad$ in $\mathbf{x} \in \mathbb{Z}^n$.

- Assume (3) has "too many solutions" outside T^{exc} and select from those $\mathbf{x}_1, \ldots, \mathbf{x}_m$ with $h(\mathbf{x}_1), h(\mathbf{x}_2)/h(\mathbf{x}_1), \ldots, h(\mathbf{x}_m)/h(\mathbf{x}_{m-1})$ large, where $h(\mathbf{x}_i) = \log \|\mathbf{x}_i\|$. Put $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_m)$.
- Let $d_1, \ldots, d_m$ be integers with $d_1 h(\mathbf{x}_1) \approx \cdots \approx d_m h(\mathbf{x}_m)$, and construct a multi-homogeneous $F \in \mathbb{Z}[\mathbf{X}_1, \ldots, \mathbf{X}_m]$ of not too large height such that F has degree d_i in $\mathbf{X}_i = (X_{i1}, \ldots, X_{in})$ and is divisible by high powers of $L_k(\mathbf{X}_i)$ for $k = 1, \ldots, n$, $i = 1, \ldots, m$. For this polynomial, $I(F, \mathbf{x})$ is not too small.

The method of Faltings and Wüstholz

Consider the system

$$(3) \qquad |L_1(\mathbf{x})| \le \Delta^{1/n}\|\mathbf{x}\|^{c_1}, \ldots, |L_n(\mathbf{x})| \le \Delta^{1/n}\|\mathbf{x}\|^{c_n} \quad \text{in } \mathbf{x} \in \mathbb{Z}^n.$$

- Assume (3) has "too many solutions" outside T^{exc} and select from those $\mathbf{x}_1, \ldots, \mathbf{x}_m$ with $h(\mathbf{x}_1), h(\mathbf{x}_2)/h(\mathbf{x}_1), \ldots, h(\mathbf{x}_m)/h(\mathbf{x}_{m-1})$ large, where $h(\mathbf{x}_i) = \log \|\mathbf{x}_i\|$. Put $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_m)$.
- Let $d_1, \ldots, d_m$ be integers with $d_1 h(\mathbf{x}_1) \approx \cdots \approx d_m h(\mathbf{x}_m)$, and construct a multi-homogeneous $F \in \mathbb{Z}[\mathbf{X}_1, \ldots, \mathbf{X}_m]$ of not too large height such that F has degree d_i in $\mathbf{X}_i = (X_{i1}, \ldots, X_{in})$ and is divisible by high powers of $L_k(\mathbf{X}_i)$ for $k = 1, \ldots, n$, $i = 1, \ldots, m$. For this polynomial, $I(F, \mathbf{x})$ is not too small.
- By Faltings' Product Theorem, $\mathbf{x} \in Z_1 \times \cdots \times Z_m$ for some proper product subvariety $Z_1 \times \cdots \times Z_m$ of $\mathbb{P}^{n-1} \times \cdots \times \mathbb{P}^{n-1}$, with explicit upper bounds for the degrees and heights of the Z_i in terms of F.

The method of Faltings and Wüstholz

Consider the system

$$(3) \qquad |L_1(\mathbf{x})| \leq \Delta^{1/n}\|\mathbf{x}\|^{c_1}, \ldots, |L_n(\mathbf{x})| \leq \Delta^{1/n}\|\mathbf{x}\|^{c_n} \quad \text{in } \mathbf{x} \in \mathbb{Z}^n.$$

- Assume (3) has "too many solutions" outside $T^{\rm exc}$ and select from those $\mathbf{x}_1, \ldots, \mathbf{x}_m$ with $h(\mathbf{x}_1), h(\mathbf{x}_2)/h(\mathbf{x}_1), \ldots, h(\mathbf{x}_m)/h(\mathbf{x}_{m-1})$ large, where $h(\mathbf{x}_i) = \log\|\mathbf{x}_i\|$. Put $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_m)$.
- Let $d_1, \ldots, d_m$ be integers with $d_1 h(\mathbf{x}_1) \approx \cdots \approx d_m h(\mathbf{x}_m)$, and construct a multi-homogeneous $F \in \mathbb{Z}[\mathbf{X}_1, \ldots, \mathbf{X}_m]$ of not too large height such that F has degree d_i in $\mathbf{X}_i = (X_{i1}, \ldots, X_{in})$ and is divisible by high powers of $L_k(\mathbf{X}_i)$ for $k = 1, \ldots, n$, $i = 1, \ldots, m$. For this polynomial, $I(F, \mathbf{x})$ is not too small.
- By Faltings' Product Theorem, $\mathbf{x} \in Z_1 \times \cdots \times Z_m$ for some proper product subvariety $Z_1 \times \cdots \times Z_m$ of $\mathbb{P}^{n-1} \times \cdots \times \mathbb{P}^{n-1}$, with explicit upper bounds for the degrees and heights of the Z_i in terms of F.
- Choose suitable projections $\pi_i : Z_i \to \mathbb{P}^{\dim Z_i}$ and apply the above procedure to $\pi_1(\mathbf{x}_1), \ldots, \pi_m(\mathbf{x}_m)$. We find a proper product subvariety of $Z_1 \times \cdots \times Z_m$ containing $\mathbf{x}$. By continuously repeating this, we obtain a zero-dimensional product variety containing $\mathbf{x}$. This imposes impossible constraints on the heights of the $\mathbf{x}_i$.

Schmidt's method

Consider again

(3) $$|L_1(\mathbf{x})| \le \Delta^{1/n}\|\mathbf{x}\|^{c_1}, \ldots, |L_n(\mathbf{x})| \le \Delta^{1/n}\|\mathbf{x}\|^{c_n} \quad \text{in } \mathbf{x} \in \mathbb{Z}^n.$$

- Assume (3) has "too many solutions" outside T^{exc} and select from those $\mathbf{x}_1, \ldots, \mathbf{x}_m$ with $h(\mathbf{x}_1), h(\mathbf{x}_2)/h(\mathbf{x}_1), \ldots, h(\mathbf{x}_m)/h(\mathbf{x}_{m-1})$ large.

Schmidt's method

Consider again

$$(3) \qquad |L_1(\mathbf{x})| \le \Delta^{1/n}\|\mathbf{x}\|^{c_1}, \ldots, |L_n(\mathbf{x})| \le \Delta^{1/n}\|\mathbf{x}\|^{c_n} \quad \text{in } \mathbf{x} \in \mathbb{Z}^n.$$

- Assume (3) has "too many solutions" outside T^{exc} and select from those $\mathbf{x}_1, \ldots, \mathbf{x}_m$ with $h(\mathbf{x}_1), h(\mathbf{x}_2)/h(\mathbf{x}_1), \ldots, h(\mathbf{x}_m)/h(\mathbf{x}_{m-1})$ large.
- Let $d_1, \ldots, d_m$ be positive integers with $d_1 h(\mathbf{x}_1) \approx \cdots \approx d_m h(\mathbf{x}_m)$, and construct, for some integers $N_1, \ldots, N_m \le 2^n$, a multi-homogeneous $F \in \mathbb{Z}[\mathbf{X}_1, \ldots, \mathbf{X}_m]$ of degree d_i in $\mathbf{X}_i = (X_{i1}, \ldots, X_{i,N_i})$ for $i = 1, \ldots, m$, such that for some not too small ϵ,
$$I(F, \mathbf{x}) \ge \epsilon \ \text{ for all } \mathbf{x} \in V_1 \times \cdots \times V_m$$
where V_i is a hyperplane of $\mathbb{P}^{N_i - 1}$ over $\overline{\mathbb{Q}}$ with $h(V_i) \gg\ll h(\mathbf{x}_i)$ for $i = 1, \ldots, m$.

Schmidt's method

Consider again

$$(3) \qquad |L_1(\mathbf{x})| \leq \Delta^{1/n}\|\mathbf{x}\|^{c_1}, \ldots, |L_n(\mathbf{x})| \leq \Delta^{1/n}\|\mathbf{x}\|^{c_n} \quad \text{in } \mathbf{x} \in \mathbb{Z}^n.$$

- Assume (3) has "too many solutions" outside T^{exc} and select from those $\mathbf{x}_1, \ldots, \mathbf{x}_m$ with $h(\mathbf{x}_1), h(\mathbf{x}_2)/h(\mathbf{x}_1), \ldots, h(\mathbf{x}_m)/h(\mathbf{x}_{m-1})$ large.
- Let $d_1, \ldots, d_m$ be positive integers with $d_1 h(\mathbf{x}_1) \approx \cdots \approx d_m h(\mathbf{x}_m)$, and construct, for some integers $N_1, \ldots, N_m \leq 2^n$, a multi-homogeneous $F \in \mathbb{Z}[\mathbf{X}_1, \ldots, \mathbf{X}_m]$ of degree d_i in $\mathbf{X}_i = (X_{i1}, \ldots, X_{i,N_i})$ for $i = 1, \ldots, m$, such that for some not too small ϵ,

  $$I(F, \mathbf{x}) \geq \epsilon \ \text{ for all } \mathbf{x} \in V_1 \times \cdots \times V_m$$

  where V_i is a hyperplane of $\mathbb{P}^{N_i - 1}$ over $\overline{\mathbb{Q}}$ with $h(V_i) \gg\ll h(\mathbf{x}_i)$ for $i = 1, \ldots, m$.
- Derive a contradiction by applying the Product Theorem or Schmidt's Lemma (next slide).

Schmidt's Lemma

Let $m, n_1, \ldots, n_m \geq 2$, $0 < \epsilon < 1$.

Let $F \in \overline{\mathbb{Q}}[\mathbf{X}_1, \ldots, \mathbf{X}_m]$ be homogeneous of degree d_i in $\mathbf{X}_i = (X_{i1}, \ldots, X_{i,n_i})$ for $i = 1, \ldots, m$, and let $V_i \subset \mathbb{P}^{n_i - 1}$ be a hyperplane over $\overline{\mathbb{Q}}$ for $i = 1, \ldots, m$.

Lemma

There are $\omega_1, \omega_2 > 0$ depending on $M := n_1 + \cdots + n_m$ and ϵ with the following property:
Assume that $d_1/d_2, \ldots, d_{m-1}/d_m > \omega_1$, and

$$I(F, \mathbf{x}) \geq m\epsilon \quad \text{for all } \mathbf{x} \in V_1 \times \cdots \times V_m.$$

Then $\min_i d_i h(V_i) \leq \omega_2 \big(d_1 + \cdots + d_m + h(F)\big)$.

Schmidt (1971): $\omega_1 = \omega_2 = M(\epsilon^{-1})^{c^m}$ $\;(c > 1)$,
Quantitative Product Theorem: $\omega_1 = 2m^2/\epsilon$, $\omega_2 = M \cdot (3m^2/\epsilon)^m$.

Thank you for your attention!