On Schmidt's Subspace Theorem

Jan-Hendrik Evertse
Universiteit Leiden

Winter school on Lang's and Vojta's conjectures

March 3, 2014, CIRM, Luminy

Slides can be downloaded from
http://www.math.leidenuniv.nl/∼evertse/lectures.shtml

Roth's Theorem

For $\xi \in \mathbb{Q}$ we define

$$H(\xi) = \max(|p|, |q|), \text{ where } \xi = p/q, \ p, q \in \mathbb{Z}, \ \gcd(p, q) = 1.$$

Theorem (Roth, 1955)

Let α be a real algebraic number and $\delta > 0$. Then the inequality

$$|\alpha - \xi| \leq H(\xi)^{-2-\delta} \text{ in } \xi \in \mathbb{Q} \tag{1}$$

has only finitely many solutions.

Proof.

Roth machinery: construction of auxiliary polynomial, application of Roth's Lemma. □

Roth's Theorem

For $\xi \in \mathbb{Q}$ we define

$$H(\xi) = \max(|p|, |q|), \text{ where } \xi = p/q,\ p, q \in \mathbb{Z},\ \gcd(p, q) = 1.$$

Theorem (Roth, 1955)

Let α be a real algebraic number and $\delta > 0$. Then the inequality

$$(1) \qquad |\alpha - \xi| \leq H(\xi)^{-2-\delta} \text{ in } \xi \in \mathbb{Q}$$

has only finitely many solutions.

Quantitative versions, i.e., upper bounds for the number of solutions of (1), were given by Davenport& Roth (1955), Mignotte (1972), Bombieri&van der Poorten (1987), Schmidt (1988), Ev. (1996), Bugeaud& Ev. (2008).

A quantitative Roth's Theorem

Let α be a real algebraic number of degree d and $0 < \delta < 1$. Denote by $H(\alpha)$ the absolute, multiplicative height of α.

We call $\xi \in \mathbb{Q}$ *large* if $H(\xi) \geq \max(4^{1/\delta}, H(\alpha))$ and *small* otherwise.

Theorem (Bugeaud, Ev., 2008)

The number of large solutions of

$$|\alpha - \xi| \leq H(\xi)^{-2-\delta} \text{ in } \xi \in \mathbb{Q}$$

is at most

$$A := 2^{25}\delta^{-3}\log 4d \log(\delta^{-1}\log 4d)$$

and the number of small solutions at most

$$B := 10\delta^{-1}\log\left(\delta^{-1}\log(4H(\alpha))\right).$$

Roth's Lemma

Let $F \in \mathbb{Z}[X_1, \ldots, X_m]$ with $\deg_{X_h} F \leq d_h$ for $h = 1, \ldots, m$.
Put $H(F) := \max(|\text{coeff. of F}|)$.

Define the *index* of F at $\mathbf{x}$ w.r.t. $\mathbf{d} = (d_1, \ldots, d_m)$ by

$$I(F; \mathbf{x}, \mathbf{d}) := \max \Big\{ \sum_{h=1}^{m} \frac{i_h}{d_h} : \quad \frac{\partial^{i_1 + \cdots + i_m} F}{\partial X_1^{i_1} \cdots \partial X_m^{i_m}}(\mathbf{x}) = 0 \Big\}.$$

Roth's Lemma

Let $F \in \mathbb{Z}[X_1, \ldots, X_m]$ with $\deg_{X_h} F \le d_h$ for $h = 1, \ldots, m$.
Put $H(F) := \max(|\text{coeff. of F}|)$.

Define the *index* of F at $\mathbf{x}$ w.r.t. $\mathbf{d} = (d_1, \ldots, d_m)$ by

$$I(F; \mathbf{x}, \mathbf{d}) := \max\Big\{ \sum_{h=1}^{m} \frac{i_h}{d_h} : \quad \frac{\partial^{i_1+\cdots+i_m} F}{\partial X_1^{i_1} \cdots \partial X_m^{i_m}}(\mathbf{x}) = 0 \Big\}.$$

Theorem (Roth, 1955)

Let $0 < \varepsilon < 1$. There are numbers $\omega_1, \omega_2 > 0$ depending on m, ε, such that if F and $\mathbf{x} = (\xi_1, \ldots, \xi_m) \in \mathbb{Q}^m$ satisfy

$$d_1/d_2, d_2/d_3, \ldots, d_{m-1}/d_m > \omega_1,$$
$$H(\xi_h)^{d_h} \ge \big(e^{d_1+\cdots+d_m} H(F)\big)^{\omega_2} \text{ for } h = 1, \ldots, m,$$

then $I(F; \mathbf{x}, \mathbf{d}) < m\varepsilon$.

Sharpest version (Ev., 1995): $\omega_1 = 2m^2/\varepsilon$, $\omega_2 = (3m^2/\varepsilon)^m$.

Outline of the proof of Roth's Theorem

- Assume $|\alpha - \xi| < H(\xi)^{-2-\delta}$ has more than A large solutions $\xi \in \mathbb{Q}$.
- Choose ε small in terms of δ and m large in terms of ε and select solutions $\xi_1 = p_1/q_1, \ldots, \xi_m = p_m/q_m$ with $H(\xi_1)$ large and $H(\xi_{h+1}) > H(\xi_h)^{\omega_1}$ for $h = 1, \ldots, m-1$.

Outline of the proof of Roth's Theorem

- Assume $|\alpha - \xi| < H(\xi)^{-2-\delta}$ has more than A large solutions $\xi \in \mathbb{Q}$.

- Choose ε small in terms of δ and m large in terms of ε and select solutions $\xi_1 = p_1/q_1, \ldots, \xi_m = p_m/q_m$ with $H(\xi_1)$ large and $H(\xi_{h+1}) > H(\xi_h)^{\omega_1}$ for $h = 1, \ldots, m-1$.

- Using Siegel's Lemma, construct an auxiliary polynomial $F \in \mathbb{Z}[X_1, \ldots, X_m]$ of degree at most d_h at X_h for $h = 1, \ldots, m$ with $H(\xi_1)^{d_1} \approx \cdots \approx H(\xi_m)^{d_m}$ such that F has large index at $(\alpha, \ldots, \alpha)$.

  Then F and $\mathbf{x} = (\xi_1, \ldots, \xi_m)$ satisfy the conditions of Roth's Lemma.

Outline of the proof of Roth's Theorem

- Assume $|\alpha - \xi| < H(\xi)^{-2-\delta}$ has more than A large solutions $\xi \in \mathbb{Q}$.
- Choose ε small in terms of δ and m large in terms of ε and select solutions $\xi_1 = p_1/q_1, \dots, \xi_m = p_m/q_m$ with $H(\xi_1)$ large and $H(\xi_{h+1}) > H(\xi_h)^{\omega_1}$ for $h = 1, \dots, m-1$.
- Using Siegel's Lemma, construct an auxiliary polynomial $F \in \mathbb{Z}[X_1, \dots, X_m]$ of degree at most d_h at X_h for $h = 1, \dots, m$ with $H(\xi_1)^{d_1} \approx \cdots \approx H(\xi_m)^{d_m}$ such that F has large index at $(\alpha, \dots, \alpha)$.

  Then F and $\mathbf{x} = (\xi_1, \dots, \xi_m)$ satisfy the conditions of Roth's Lemma.
- Show for all $(i_1, \dots, i_m)$ with $\sum_{h=1}^{m}(i_h/d_h) \leq m\varepsilon$ that the integer $q_1^{d_1} \cdots q_m^{d_m} \left| \frac{1}{i_1! \cdots i_m!} \frac{\partial^{i_1+\cdots+i_m}}{\partial X_1^{i_1} \cdots \partial X_m^{i_m}} F(\mathbf{x}) \right| < 1$, hence $= 0$, and derive a contradiction with Roth's Lemma.

Schmidt's Subspace Theorem

Let $\overline{\mathbb{Q}}$ denote the field of algebraic numbers in $\mathbb{C}$ and let

$$L_i(\mathbf{X}) = \alpha_{i1}X_1 + \cdots + \alpha_{in}X_n \quad (i = 1, \ldots, n)$$

be linearly independent linear forms with coefficients $\alpha_{ij} \in \overline{\mathbb{Q}}$.

For $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{Z}^n$, put $\|\mathbf{x}\| := \max_i |x_i|$.

Theorem (W.M. Schmidt, 1972)

Let $\delta > 0$. Then the set of solutions of

(2) $$|L_1(\mathbf{x}) \cdots L_n(\mathbf{x})| \leq \|\mathbf{x}\|^{-\delta} \text{ in } \mathbf{x} \in \mathbb{Z}^n$$

is contained in a union of finitely many proper linear subspaces of $\mathbb{Q}^n$.

There are generalizations involving p-adic absolute values (Schlickewei's p-adic Subspace Theorem) and versions over number fields involving both archimedean and non-archimedean absolute values (Schmidt, Schlickewei)

Subspace Theorem $\Longrightarrow$ Roth's Theorem

Let α be a real algebraic number and $\delta > 0$, and consider the solutions x_1/x_2 with $x_1, x_2 \in \mathbb{Z}$ of

$$|\alpha - x_1/x_2| < H(x_1/x_2)^{-2-\delta} = \max(|x_1|, |x_2|)^{-2-\delta}.$$

These satisfy

$$|(x_1 - \alpha x_2)x_2| < \max(|x_1|, |x_2|)^{-\delta}.$$

By the Subspace Theorem, the pairs (x_1, x_2) lie in finitely many one-dimensional subspaces of $\mathbb{Q}^2$.

These give rise to finitely many fractions x_1/x_2. □

Systems of inequalities

By a combinatorial argument, the inequality (2) $|L_1(\mathbf{x})\cdots L_n(\mathbf{x})| \le \|\mathbf{x}\|^{-\delta}$ can be reduced to finitely many systems of inequalities of the shape

$$(3) \qquad |L_1(\mathbf{x})| \le C\|\mathbf{x}\|^{c_1}, \ldots, |L_n(\mathbf{x})| \le C\|\mathbf{x}\|^{c_n} \quad \text{in } \mathbf{x} \in \mathbb{Z}^n,$$

where $C > 0$, $c_1 + \cdots + c_n < 0$.

Thus, the following is equivalent to the Subspace Theorem:

Theorem

The solutions of (3) *lie in a finite union of proper linear subspaces of* $\mathbb{Q}^n$.

Systems of inequalities

By a combinatorial argument, the inequality (2) $|L_1(\mathbf{x}) \cdots L_n(\mathbf{x})| \leq \|\mathbf{x}\|^{-\delta}$ can be reduced to finitely many systems of inequalities of the shape

$$(3) \qquad |L_1(\mathbf{x})| \leq C\|\mathbf{x}\|^{c_1}, \ldots, |L_n(\mathbf{x})| \leq C\|\mathbf{x}\|^{c_n} \quad \text{in } \mathbf{x} \in \mathbb{Z}^n,$$

where $C > 0$, $c_1 + \cdots + c_n < 0$.

Idea.

if $\mathbf{x} \in \mathbb{Z}^n$ satisfies (2) then

$$\mathbf{c}(\mathbf{x}) := \Big(\frac{\log |L_1(\mathbf{x})|}{\log \|\mathbf{x}\|}, \ldots, \frac{\log |L_n(\mathbf{x})|}{\log \|\mathbf{x}\|} \Big) \in S,$$

where S is a bounded set independent of $\mathbf{x}$. Cover S by a sufficiently fine finite grid. Then $\mathbf{x}$ satisfies (3) with $\mathbf{c} = (c_1, \ldots, c_n)$ a grid point close to $\mathbf{c}(\mathbf{x})$. □

We proceed further with systems (3).

A refinement of the Subspace Theorem

Let again $L_1, \ldots, L_n$ be linearly independent linear forms in $X_1, \ldots, X_n$ with coefficients in $\overline{\mathbb{Q}}$ and $C, c_1, \ldots, c_n$ reals with $C > 0$, $c_1 + \cdots + c_n < 0$. Consider again

$$(3) \qquad |L_1(\mathbf{x})| \leq C\|\mathbf{x}\|^{c_1}, \ldots, |L_n(\mathbf{x})| \leq C\|\mathbf{x}\|^{c_n} \quad \text{in } \mathbf{x} \in \mathbb{Z}^n.$$

Theorem (Vojta (1989), Schmidt (1993), Faltings-Wüstholz (1994))

There is an effectively computable, proper linear subspace T^{exc} of $\mathbb{Q}^n$ such that (3) has only finitely many solutions outside T^{exc}.

The space T^{exc} can be chosen from a finite collection, depending only on $L_1, \ldots, L_n$ and independent of $c_1, \ldots, c_n$.

This refinement can be deduced from Schmidt's basic Subspace Theorem.

About the exceptional subspace

Assume for simplicity that $L_1, \ldots, L_n$ have real algebraic coefficients.

For a linear subspace T of $\mathbb{Q}^n$, we say that a subset $\{L_{i_1}, \ldots, L_{i_m}\}$ of $\{L_1, \ldots, L_n\}$ is linearly independent on T if there are no reals $\alpha_{i_1}, \ldots, \alpha_{i_m}$, not all 0, such that $\alpha_{i_1} L_{i_1} + \cdots + \alpha_{i_m} L_{i_m}$ vanishes identically on T.

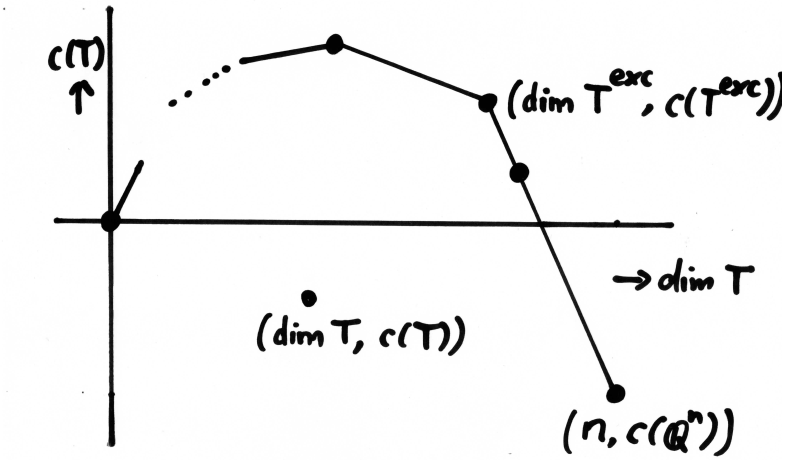
For a linear subspace T of $\mathbb{Q}^n$ define $c(T)$ to be the minimum of the quantities $c_{i_1} + \cdots + c_{i_m}$, taken over all subsets $\{L_{i_1}, \ldots, L_{i_m}\}$ of $\{L_1, \ldots, L_n\}$ of cardinality $m = \dim T$ that are linearly independent on T.

T^{exc} is the unique proper linear subspace T of $\mathbb{Q}^n$ such that

$$\frac{c(\mathbb{Q}^n) - c(T)}{n - \dim T} \text{ is minimal,}$$

subject to this condition, $\dim T$ is minimal.

About the exceptional subspace



An effective estimate

Lemma (Ev., Ferretti ,2013)

Suppose that the coefficients of $L_1, \ldots, L_n$ have absolute (multiplicative) heights $\leq H$.

Then T^{exc} has a basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_m\} \subset \mathbb{Z}^n$ with

$$\|\mathbf{x}_i\| \leq (\sqrt{n} H^n)^{4^n} \quad (i = 1, \ldots, m).$$

Open problem

Is there an efficient method to determine T^{exc} in general?

Example

Let $\alpha_2, \ldots, \alpha_n$ be real algebraic numbers, and $c_1, \ldots, c_n$ reals with

$$c_1 + \cdots + c_n < 0, \quad 0 \leq c_2, \ldots, c_n \leq 1$$

and consider

$$(4) \quad |x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n| \leq \|\mathbf{x}\|^{c_1}, \;\; |x_2| \leq \|\mathbf{x}\|^{c_2}, \ldots, |x_n| \leq \|\mathbf{x}\|^{c_n}$$

in $\mathbf{x} \in \mathbb{Z}^n$.

Exercise: $T^{\mathrm{exc}} = \{\mathbf{x} \in \mathbb{Q}^n : \; x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0\}$.

Corollary (Schmidt, 1970)

(4) *has only finitely many solutions with* $x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n \neq 0$.

Methods of proof

1. **Schmidt's method (1972)**
   - Geometry of numbers
   - Construction of an auxiliary polynomial and application of Roth's Lemma

2. **The method of Faltings and Wüstholz (1994)**
   - Inductive argument, which involves Diophantine approximation on projective varieties of large degree
   - In the induction step a construction of an auxiliary polynomial (i.e., global section of appropriate line bundle) on a product of large degree projective varieties and an application of Faltings' Product Theorem (a deep generalization of Roth's Lemma)

Quantitative versions

The Quantitative Subspace Theorem gives an upper bound for the number of subspaces, containing the solutions of

$$(3) \qquad |L_1(\mathbf{x})| \leq C\|\mathbf{x}\|^{c_1}, \ldots, |L_n(\mathbf{x})| \leq C\|\mathbf{x}\|^{c_n} \quad \text{in } \mathbf{x} \in \mathbb{Z}^n.$$

The first such bound was given by Schmidt (1989).
His bounds were improved by Ev. (1995), Ev.& Schlickewei (2002), Ev.& Ferretti (2013).

Quantitative versions

The Quantitative Subspace Theorem gives an upper bound for the number of subspaces, containing the solutions of

$$(3) \qquad |L_1(\mathbf{x})| \le C\|\mathbf{x}\|^{c_1}, \ldots, |L_n(\mathbf{x})| \le C\|\mathbf{x}\|^{c_n} \quad \text{in } \mathbf{x} \in \mathbb{Z}^n.$$

The first such bound was given by Schmidt (1989).
His bounds were improved by Ev. (1995), Ev.& Schlickewei (2002), Ev.& Ferretti (2013).

With the present methods it is possible only to give an explicit upper bound for the number of subspaces containing the solutions.
Estimating the number of solutions outside $T^{\rm exc}$ is out of reach.

Quantitative versions

The Quantitative Subspace Theorem gives an upper bound for the number of subspaces, containing the solutions of

$$(3) \qquad |L_1(\mathbf{x})| \le C\|\mathbf{x}\|^{c_1}, \ldots, |L_n(\mathbf{x})| \le C\|\mathbf{x}\|^{c_n} \quad \text{in } \mathbf{x} \in \mathbb{Z}^n.$$

The first such bound was given by Schmidt (1989).
His bounds were improved by Ev. (1995), Ev.& Schlickewei (2002), Ev.& Ferretti (2013).

With the present methods it is possible only to give an explicit upper bound for the number of subspaces containing the solutions.
Estimating the number of solutions outside T^{exc} is out of reach.

Schmidt's method gives much better upper bounds for the number of subspaces, since it involves only linear varieties, whereas in the method of Faltings and Wüstholz one has to work with non-linear varieties of large degree.

The Quantitative Subspace Theorem

Let $L_1, \ldots, L_n$ be linear forms in $X_1, \ldots, X_n$ with coefficients in $\overline{\mathbb{Q}}$, and $C, c_1, \ldots, c_n$ reals such that

- the coefficients of $L_1, \ldots, L_n$ have abs. heights $\leq H$ and degrees $\leq D$;
- $0 < C \leq |\det(L_1, \ldots, L_n)|^{1/n}$;
- $c_1 + \cdots + c_n \leq -\delta$ with $0 < \delta < 1$, $\max(c_1, \ldots, c_n) = 1$.

Call a solution $\mathbf{x} \in \mathbb{Z}^n$ of

(3) $$|L_1(\mathbf{x})| \leq C\|\mathbf{x}\|^{c_1}, \ldots, |L_n(\mathbf{x})| \leq C\|\mathbf{x}\|^{c_n}$$

large if $\|\mathbf{x}\| \geq \max(n^{n/\delta}, H)$ and *small* otherwise.

The Quantitative Subspace Theorem

Let $L_1, \ldots, L_n$ be linear forms in $X_1, \ldots, X_n$ with coefficients in $\overline{\mathbb{Q}}$, and $C, c_1, \ldots, c_n$ reals such that

- the coefficients of $L_1, \ldots, L_n$ have abs. heights $\leq H$ and degrees $\leq D$;
- $0 < C \leq |\det(L_1, \ldots, L_n)|^{1/n}$;
- $c_1 + \cdots + c_n \leq -\delta$ with $0 < \delta < 1$, $\max(c_1, \ldots, c_n) = 1$.

Call a solution $\mathbf{x} \in \mathbb{Z}^n$ of

$$(3) \qquad |L_1(\mathbf{x})| \leq C\|\mathbf{x}\|^{c_1}, \ldots, |L_n(\mathbf{x})| \leq C\|\mathbf{x}\|^{c_n}$$

large if $\|\mathbf{x}\| \geq \max(n^{n/\delta}, H)$ and *small* otherwise.

Theorem (Ev., Ferretti, 2013)

The large solutions $\mathbf{x} \in \mathbb{Z}^n$ of (3) lie in at most

$$A := 2^{2n}(10n)^{20}\delta^{-3}\log\delta^{-1}\log 4D\log(\delta^{-1}\log 4D)$$

proper linear subspaces of $\mathbb{Q}^n$, and the small solutions in at most

$$B := 10^{3n}\delta^{-1}\log(\delta^{-1}\log 4H)$$

subspaces.

Outline of the proof: Geometry of numbers

Consider again

$$(3) \qquad |L_1(\mathbf{x})| \le C\|\mathbf{x}\|^{-c_1}, \ldots, |L_n(\mathbf{x})| \le C\|\mathbf{x}\|^{-c_n} \quad \text{in } \mathbf{x} \in \mathbb{Z}^n.$$

Using geometry of numbers, for every solution $\mathbf{x}$ we can construct, for some $N \le 2^n$, a parallelepiped

$$\Pi(\mathbf{x}) := \left\{ \mathbf{z} \in \mathbb{R}^N : \, |M_i(\mathbf{z})| \le A_i(\mathbf{x}) \;\; (i = 1, \ldots, N) \right\}$$

such that $\Pi(\mathbf{x}) \cap \mathbb{Z}^N$ generates a linear subspace $T(\mathbf{x})$ of $\mathbb{Q}^N$ of dimension $N - 1$.

Here $M_1, \ldots, M_N$ are linearly independent linear forms in N variables with real algebraic coefficients, which are independent of $\mathbf{x}$. The $A_i(\mathbf{x})$ are positive reals that do depend on $\mathbf{x}$.

Outline of the proof: Construction of an auxiliary polynomial

- Suppose the refinement of the Subspace Theorem is false and that there are infinitely many solutions outside T^{exc}.
  Select solutions $\mathbf{x}_1, \dots, \mathbf{x}_m$ outside T^{exc} with $\|\mathbf{x}_1\|$ large and $\log \|\mathbf{x}_{h+1}\| / \log \|\mathbf{x}_h\|$ large for $h = 1, \dots, m-1$.
- For $h = 1, \dots, m$, construct the parallelepipeds $\Pi(\mathbf{x}_h) = \{\mathbf{z} \in \mathbb{R}^N : |M_i(\mathbf{z})| \leq A_i(\mathbf{x}_h) \ (i = 1, \dots, N)\}$ from the previous slide, and let $T_h = T(\mathbf{x}_h)$ be the $(N-1)$-dimensional space spanned by $\Pi(\mathbf{x}_h) \cap \mathbb{Z}^N$.
- Let $d_1, \dots, d_m$ be positive integers with $\|\mathbf{x}_1\|^{d_1} \approx \cdots \approx \|\mathbf{x}_m\|^{d_m}$ and construct $F \in \mathbb{Z}[\mathbf{X}_1, \dots, \mathbf{X}_m]$ in the blocks of N variables $\mathbf{X}_1, \dots, \mathbf{X}_m$, which is homogeneous of degree d_h in block $\mathbf{X}_h$ for $h = 1, \dots, m$, and such that F is divisible by high powers of $M_1(\mathbf{X}_h), \dots, M_N(\mathbf{X}_h)$, for $h = 1, \dots, m$.

Outline of the proof: Application of Roth's Lemma

- Extrapolation: all partial derivatives of F of not too large order vanish identically on $T_1 \times \cdots \times T_m$.

  Sketch: For $h = 1, \ldots, m$, $i = 1, \ldots, N$, F is divisible by a high power of $M_i(\mathbf{X}_h)$.
  For $h = 1, \ldots, m$, $\Pi(\mathbf{x}_h) \cap \mathbb{Z}^N$ spans T_h, therefore $|M_i(\mathbf{z}_h)|$ is very small for many points $\mathbf{z}_h \in T_h$.
  This implies that the values of the partial derivatives of F of not too large order at the points $(\mathbf{z}_1, \ldots, \mathbf{z}_m) \in T_1 \times \cdots \times T_m$ have absolute values < 1, hence must be 0.
  So the partial derivatives of F of not too small order vanish at many points of $T_1 \times \cdots \times T_m$, hence are identically 0 on $T_1 \times \cdots \times T_m$.

- Derive a contradiction with Roth's Lemma.

A semi-effective version of the Subspace Theorem

Let again $L_1, \dots, L_n$ be linear forms in $X_1, \dots, X_n$ with coefficients in $\overline{\mathbb{Q}}$, and $C, c_1, \dots, c_n$ reals such that

- the coefficients of $L_1, \dots, L_n$ have abs. heights $\leq H$ and degrees $\leq D$;
- $0 < C \leq |\det(L_1, \dots, L_n)|^{1/n}$;
- $c_1 + \cdots + c_n \leq -\delta$ with $0 < \delta < 1$, $\max(c_1, \dots, c_n) = 1$.

Theorem

Let K be the n.f. generated by the coefficients of $L_1, \dots, L_n$.

There are an effectively computable constant $c^{\text{eff}}(n, \delta, D) > 0$ and an ineffective constant $B^{\text{ineff}}(n, \delta, K) > 0$ such that for every solution $\mathbf{x} \in \mathbb{Z}^n$ of

$$|L_1(\mathbf{x})| \leq C\|\mathbf{x}\|^{c_1}, \dots, |L_n(\mathbf{x})| \leq C\|\mathbf{x}\|^{c_n} \quad \text{with } \mathbf{x} \notin T^{\text{exc}}$$

we have $\|\mathbf{x}\| \leq \max\left(B^{\text{ineff}}(n, \delta, K), H^{c^{\text{eff}}(n,\delta,D)}\right)$.

Proof.

Small modification in the proof of the Subspace Theorem. □

A conjecture

Keep the assumptions

- the coefficients of $L_1, \ldots, L_n$ have abs. heights $\leq H$ and degrees $\leq D$;
- $0 < C \leq |\det(L_1, \ldots, L_n)|^{1/n}$;
- $c_1 + \cdots + c_n \leq -\delta$ with $0 < \delta < 1$, $\max(c_1, \ldots, c_n) = 1$.

Conjecture 1

There are an effectively computable constant $c^{\mathrm{eff}}(n, \delta, D) > 0$ and a constant $B'(n, \delta, D) > 0$ such that for every solution $\mathbf{x} \in \mathbb{Z}^n$ of

$$|L_1(\mathbf{x})| \leq C\|\mathbf{x}\|^{c_1}, \ldots, |L_n(\mathbf{x})| \leq C\|\mathbf{x}\|^{c_n} \quad \text{with } \mathbf{x} \notin T^{\mathrm{exc}}$$

we have $\|\mathbf{x}\| \leq \max\left(B'(n, \delta, D), H^{c^{\mathrm{eff}}(n,\delta,D)}\right)$.

This is hopeless with B' effective. But what if we allow B' to be ineffective?

abc-type inequalities

Let K be an algebraic number field of degree d, $x^{(i)}$ $(i = 1, \ldots, d)$ the conjugates of $x \in K$, O_K the ring of integers and D_K the discriminant of K.

Define $H_K(a, b, c) := \prod_{i=1}^{d} \max(|a^{(i)}|, |b^{(i)}|, |c^{(i)}|)$ for $a, b, c \in O_K$.

Theorem

Let a, b, c be non-zero elements of O_K with $a + b = c$. Then

(i) we have $H_K(a, b, c) \leq (2|N_{K/\mathbb{Q}}(abc)|)^{c_1(d)|D_K|^{c_2(d)}}$ with $c_1(d), c_2(d)$ effectively computable in terms of d.

(ii) for every $\delta > 0$ we have $H_K(a, b, c) \leq C^{\mathrm{ineff}}(K, \delta)|N_{K/\mathbb{Q}}(abc)|^{1+\delta}$.

Proof.

(i) Baker theory; (ii) Roth's theorem over number fields. □

abc-type inequalities: application of Conjecture 1

Let again K be a number field of degree d.

Conjecture 2

There are a constant $C(d,\delta) > 0$ and an effectively computable constant $c^{\mathrm{eff}}(d,\delta) > 0$ with the following property:
for any non-zero $a, b, c \in O_K$ with $a + b = c$, and any $\delta > 0$ we have

$$H_K(a,b,c) \leq C(d,\delta)|D_K|^{c^{\mathrm{eff}}(d,\delta)}|N_{K/\mathbb{Q}}(abc)|^{1+\delta}.$$

abc-type inequalities: application of Conjecture 1

Let again K be a number field of degree d.

Conjecture 2

There are a constant $C(d,\delta) > 0$ and an effectively computable constant $c^{\mathrm{eff}}(d,\delta) > 0$ with the following property:
for any non-zero $a, b, c \in O_K$ with $a + b = c$, and any $\delta > 0$ we have

$$H_K(a,b,c) \leq C(d,\delta)|D_K|^{c^{\mathrm{eff}}(d,\delta)}|N_{K/\mathbb{Q}}(abc)|^{1+\delta}.$$

Conjecture 1 $\Longrightarrow$ Conjecture 2 (idea).

Choose a $\mathbb{Z}$-basis $\{\omega_1, \ldots, \omega_d\}$ of O_K with heights bounded above in terms of $|D_K|$.

Write $a = \sum_{j=1}^{d} x_j\omega_j$, $b = \sum_{j=1}^{d} y_j\omega_j$ with $x_j, y_j \in \mathbb{Z}$.

Then $\mathbf{x} = (x_1, \ldots, y_d)$ satisfies one of a finite number of systems of inequalities as considered in Conjecture 1 with linear forms with heights bounded above in terms of d and D_K. □

Approximation of algebraic numbers by algebraic numbers of bounded degree

For $\xi \in \overline{\mathbb{Q}}$, define the naive height $H_{\mathrm{n}}(\xi)$ to be the maximum of the absolute values of the minimal polynomial of ξ.

Theorem (Schmidt, 1971)

Let $\alpha \in \overline{\mathbb{Q}}$, $d \in \mathbb{Z}_{>0}$ and $\delta > 0$. Then there are only finitely many $\xi \in \overline{\mathbb{Q}}$ of degree d such that

$$|\alpha - \xi| \leq H_{\mathrm{n}}(\xi)^{-d-1-\delta}.$$

Proof.

Consequence of the Subspace Theorem. □

Wirsing (1969) proved this earlier with $-2d-\delta$ instead of $-d-1-\delta$, by a different method.

Approximation of algebraic numbers by algebraic numbers of bounded degree: Vojta's conjecture

For $\xi \in \overline{\mathbb{Q}}$, define $D(\xi)$ to be the discriminant of the number field $\mathbb{Q}(\xi)$.

Conjecture (Vojta, 1987)

Let $\alpha \in \overline{\mathbb{Q}}$, $d \in \mathbb{Z}_{>0}$ and $\delta > 0$. Then there are only finitely many $\xi \in \overline{\mathbb{Q}}$ of degree d such that

$$|\alpha - \xi| \leq |D(\xi)|^{-1} H_{\mathrm{n}}(\xi)^{-2-\delta}.$$

This implies Wirsing's Theorem since $|D(\xi)| \ll H_{\mathrm{n}}(\xi)^{2d-2}$.

Restricting to algebraic numbers ξ in a given number field K, we obtain Roth's Theorem over number fields.

Approximation of algebraic numbers by algebraic numbers of bounded degree: application of Conjecture 1

Our Conjecture 1 implies the following:

Conjecture 3

Let $\alpha \in \overline{\mathbb{Q}}$, $d \in \mathbb{Z}_{>0}$, $\delta > 0$, and put $m := [\mathbb{Q}(\alpha) : \mathbb{Q}]$. There is an effectively computable number $c^{\mathrm{eff}}(m, d, \delta) > 0$, such that the inequality

$$|\alpha - \xi| \leq |D(\xi)|^{-c^{\mathrm{eff}}(m,d,\delta)} H_{\mathrm{n}}(\xi)^{-2-\delta}$$

has only finitely many solutions in algebraic numbers ξ of degree d.

Thank you for your attention!