

Asymptotic formulas for sums of elements from a multiplicative group

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Debrecen Online Number Theory Seminar

October 9, 2026

Introduction

Let K be a number field, $m \geq 2$, $a_1, \dots, a_m \in K^*$, and Γ a finitely generated subgroup of $(K^*)^m = \{x = (x_1, \dots, x_m) : x_i \in K^*\}$ with coordinatewise multiplication.

Theorem (E.; van der Poorten and Schlickewei, 1980s)

For each $\alpha \in K^*$, there are only finitely many $x \in \Gamma$ with $a_1x_1 + \dots + a_mx_m = \alpha$ and with

$$(1) \quad \sum_{i \in I} a_i x_i \neq 0 \text{ for each non-empty } I \subseteq \{1, \dots, m\}.$$

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We present and sketch a proof of an asymptotic formula as $X \rightarrow \infty$ for

$$\#\{x \in \Gamma : x \text{ satisfies (1), } H(a_1x_1 + \dots + a_mx_m) \leq X\},$$

where $H(\cdot)$ denotes the absolute (exponential) height.

By the above theorem, this quantity is finite.

This extends work of Everest (1989/90), who proved such a formula in the case $\Gamma = U_S^m$, where U_S is the group of S -units.

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where $H(\cdot)$ denotes the absolute (exponential) height.

Likewise, we present an asymptotic formula for

$$\#\{n \in \mathbb{Z}_{\geq 0} : H(u_n) \leq X\},$$

where $\{u_n\}$ is a linear recurrence sequence with algebraic terms.

Notation

Absolute values on $\mathbb{Q}$:

Let $M_{\mathbb{Q}} := \{\infty\} \cup \{\text{prime numbers}\}$ be the set of places of $\mathbb{Q}$.

Define absolute values $|\cdot|_p$ ($p \in M_{\mathbb{Q}}$) on $\mathbb{Q}$ by

$|\cdot|_{\infty} = |\cdot|$ the standard absolute value,

$|\cdot|_p$ for prime numbers p , the p -adic absolute value with $|p|_p = p^{-1}$.

Absolute values on K , K number field

Let M_K be the set of places of K .

Define absolute values $|\cdot|_v$ ($v \in M_K$) on K by taking the continuations to K of $|\cdot|_p$ ($p \in M_{\mathbb{Q}}$).

Define normalized absolute values $\|\cdot\|_v$ ($v \in M_K$) on K by

$$\|\cdot\|_v = |\cdot|_v^{[K_v:\mathbb{Q}_p]/[K:\mathbb{Q}]} \quad \text{for } v \in M_K \text{ with } v|p.$$

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Product formula: $\prod_{v \in M_K} \|\alpha\|_v = 1$ for $\alpha \in K^*$.

Absolute height: $H(\alpha) := \prod_{v \in M_K} \max(1, \|\alpha\|_v)$ for $\alpha \in K$.

Group of S -units $U_S := \{\alpha \in K : \|\alpha\|_v = 1 \text{ for } v \in M_K \setminus S\}$
($S \subset M_K$ finite, contains all infinite places, i.e. places $v|\infty$).

Everest's result

The following result is an easy consequence of work of Everest:

Theorem (Everest, 1989/90)

Let K be a number field, and S a finite set of places of K , containing all infinite places. Suppose $\#S = s + 1$. Let $m \geq 2$ be an integer, and $\mathbf{a} = (a_1, \dots, a_m) \in (K^*)^m$.

Then the number of $\mathbf{x} = (x_1, \dots, x_m) \in U_S^m$ such that

$$H(a_1x_1 + \dots + a_mx_m) \leq X,$$

$$\sum_{i \in I} a_i x_i \neq 0 \text{ for each non-empty } I \subseteq \{1, \dots, m\}$$

is $c(U_S, m)(\log X)^{ms} + O((\log X)^{ms-1})$ as $X \rightarrow \infty$,

where $c(U_S, m) > 0$, and the implied constant depends on K , m , $\mathbf{a}$, and S .

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Everest's proof depends on the (non-quantitative) p -adic Subspace thm.

Kerber, Tichy, and Weitzer (2021) gave an explicit expression for $c(U_S, m)$. The implied constant in the error term is ineffective.

A generalization

Everest's result can be generalized from U_S to an arbitrary finitely generated multiplicative subgroup Γ_1 of K^* . The argument is essentially the same.

Theorem 1 (EGHLR, 2026)

Let K be a number field, Γ_1 a f.g. subgroup of K^* of rank $r_1 \geq 1$, $m \geq 2$ an integer, and $\mathbf{a} = (a_1, \dots, a_m) \in (K^*)^m$.

Then the number of $\mathbf{x} = (x_1, \dots, x_m) \in \Gamma_1^m$ such that

$$H(a_1x_1 + \dots + a_mx_m) \leq X,$$

$$\sum_{i \in I} a_i x_i \neq 0 \text{ for each non-empty } I \subseteq \{1, \dots, m\}$$

is $c(\Gamma_1, m)(\log X)^{mr_1} + O((\log X)^{mr_1-1})$ as $X \rightarrow \infty$,

where $c(\Gamma_1, m) > 0$, and the implied constant depends on K , m , $\mathbf{a}$, and Γ_1 .

The constant $c(\Gamma_1, m)$ is computable in terms of Γ_1 and m , but we don't have an explicit expression for it.

The implied constant in the error term is ineffective.

Counting sums $a_1x_1 + \cdots + a_mx_m$

Instead of tuples $(x_1, \dots, x_m)$, we count the number of α with $H(\alpha) \leq X$ representable in the form $a_1x_1 + \cdots + a_mx_m$ with $x_1, \dots, x_m \in \Gamma_1$.

For a tuple $\mathbf{a} = (a_1, \dots, a_m)$ and a permutation σ of $\{1, \dots, m\}$, define $\sigma(\mathbf{a}) := (a_{\sigma(1)}, \dots, a_{\sigma(m)})$. Let $G(\mathbf{a}) := \{\sigma : \sigma(\mathbf{a}) = \mathbf{a}\}$.

Corollary (EGHLR, 2026)

Let K be a number field, and Γ_1 a f.g. subgroup of K^* of rank $r_1 \geq 1$. Let $m \geq 2$ be an integer, and $\mathbf{a} = (a_1, \dots, a_m) \in (K^*)^m$. Assume that either $a_i = a_j$ or $a_i/a_j \notin \Gamma_1$ for $i, j = 1, \dots, m$.

Then the number of $\alpha \in K$ such that

$$(2) \quad H(\alpha) \leq X, \quad \exists x_1, \dots, x_m \in \Gamma_1 \text{ with } \alpha = a_1x_1 + \cdots + a_mx_m$$

$$\text{is } \frac{c(\Gamma_1, m)}{\#G(\mathbf{a})} (\log X)^{mr_1} + O((\log X)^{mr_1-1}) \text{ as } X \rightarrow \infty.$$

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$$\text{is } \frac{c(\Gamma_1, m)}{\#G(\mathbf{a})} (\log X)^{mr_1} + O((\log X)^{mr_1-1}) \text{ as } X \rightarrow \infty.$$

Idea (Frei, Tichy, Ziegler, 2014)

For each α with (2), with at most $O((\log X)^{mr_1-1})$ exceptions, there are precisely $\#G(\mathbf{a})$ tuples $(x_1, \dots, x_m) \in \Gamma_1^m$ with $a_1x_1 + \cdots + a_mx_m = \alpha$.

A more general situation

We now take any f.g. subgroup Γ of $(K^*)^m$ of rank $r \geq 1$ and consider the number of $x = (x_1, \dots, x_m) \in \Gamma$ with $H(a_1x_1 + \dots + a_mx_m) \leq X$.

This is vastly more general than the case Γ_1^m with Γ_1 a f.g. subgroup of K^* considered before.

For instance, it includes the rank 1 case $\Gamma = \{(\alpha_1^n, \dots, \alpha_m^n) : n \in \mathbb{Z}\}$.

In this general situation, Everest's method does not work anymore, and we use another method, based on a quantitative version of the Subspace Theorem.

This leads to a larger error term.

The general theorem

Write $x = (x_1, \dots, x_m)$, $a = (a_1, \dots, a_m)$.

Theorem 2 (EGHLR, 2026)

- Let K be a number field, $m \geq 2$, $a \in (K^*)^m$,
- let Γ be a f.g. subgroup of $(K^*)^m$ of rank $r \geq 1$ with the following property:

$$(++) \quad \left\{ \begin{array}{l} \text{for each } i, j \text{ in } \{1, \dots, m\} \text{ with } i \neq j, \text{ there is } x \in \Gamma \\ \text{such that } x_i/x_j \text{ is not a root of unity.} \end{array} \right.$$

Define

$$\mathcal{V}_{\Gamma, a}(X) := \left\{ x \in \Gamma : \begin{array}{l} H(a_1 x_1 + \dots + a_m x_m) \leq X \\ \sum_{i \in I} a_i x_i \neq 0 \text{ for each non-empty } I \subseteq \{1, \dots, m\} \end{array} \right\}.$$

Then

$$\#\mathcal{V}_{\Gamma, a}(X) = c(\Gamma)(\log X)^r + O((\log X)^{r-1/(r+4)}(\log \log X)^{2/(r+4)}) \text{ as } X \rightarrow \infty$$

where $c(\Gamma) > 0$, and the implied constant depends on K , a , and Γ .

The implied constant cannot be effectively determined from the method of proof.

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Write $x = (x_1, \dots, x_m)$, $a = (a_1, \dots, a_m)$.

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where $c(\Gamma) > 0$, and the implied constant depends on K , a , and Γ .

Problem: Get the same result with error term $O((\log X)^{r-1})$.

Main idea of the proof of Theorem 2

Let K be a number field. Define the height of $x = (x_1, \dots, x_m) \in K^m$ by

$$H(x) := \prod_{v \in M_K} \max(1, \|x_1\|_v, \dots, \|x_m\|_v).$$

Let Γ be a f.g. subgroup of $(K^*)^m$ of rank $r \geq 1$ and put

$$\mathcal{H}_\Gamma(X) := \{x \in \Gamma : H(x) \leq X\}.$$

The idea of the proof of Theorem 2 is as follows:

- (1) derive an asymptotic formula for $\#\mathcal{H}_\Gamma(X)$ (which is elementary lattice point counting);
- (2) show that $\#\mathcal{H}_\Gamma(X) - \#\mathcal{V}_{\Gamma,a}(X)$ is small, where $\mathcal{V}_{\Gamma,a}(X)$ is the set from Theorem 2 (this requires several auxiliary results, all following from the Subspace theorem).

We present the lattice point counting result from (1) and the auxiliary results from (2), and finally sketch the proof of Theorem 2.

Lattice point counting

Let K be a number field, and Γ a f.g. subgroup of $(K^*)^m$ of rank $r \geq 1$. Define $\mathcal{H}_\Gamma(X) := \{x = (x_1, \dots, x_m) \in \Gamma : H(x) \leq X\}$.

Proposition 1

$$\#\mathcal{H}_\Gamma(X) = c(\Gamma)(\log X)^r + O((\log X)^{r-1}) \quad \text{as } X \rightarrow \infty.$$

Here $c(\Gamma)$ is the same constant as in Theorem 2.

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Proof.

Choose $u_1, \dots, u_r \in \Gamma$ such that every $x \in \Gamma$ can be written as $x = \zeta u_1^{z_1} \cdots u_r^{z_r}$ with unique $\zeta \in \Gamma_{\text{tors}}$ and $z = (z_1, \dots, z_r) \in \mathbb{Z}^r$.

Thus, $\log \|x_i\|_v = \ell_{iv}(z)$ with ℓ_{iv} a linear form, and $\log H(x) = \psi(z)$, with $\psi(z) = \sum_{v \in M_K} \max(0, \max_{1 \leq i \leq m} \ell_{iv}(z))$.

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$\varphi : x \mapsto z$ is $\#\Gamma_{\text{tors}}$ to 1, and $\varphi(\mathcal{H}_\Gamma(X)) = \{z \in \mathbb{Z}^r : \psi(z) \leq \log X\}$.

Lattice point counting gives

$$\begin{aligned} \#\mathcal{H}_\Gamma(X) &= \#\Gamma_{\text{tors}} \cdot \#\{z \in \mathbb{Z}^r : \psi(z) \leq \log X\} \\ &= c(\Gamma)(\log X)^r + O((\log X)^{r-1}) \end{aligned}$$

with $c(\Gamma) = (\#\Gamma_{\text{tors}}) \cdot \text{vol}(\{z \in \mathbb{R}^r : \psi(z) \leq 1\})$.



Applications of the Subspace Theorem (I)

We state a number of auxiliary results needed in the proof of Theorem 2, all following from the Subspace Theorem.

Let K be a number field, and Γ a f.g. subgroup of $(K^*)^m$ of rank $r \geq 1$. Let $m \geq 2$ and $\mathbf{a} = (a_1, \dots, a_m) \in (K^*)^m$.

It is easy to prove that $H(a_1x_1 + \dots + a_mx_m) \ll_{K,m,\mathbf{a}} H(\mathbf{x})$ for $\mathbf{x} = (x_1, \dots, x_m) \in K^m$.

For $\mathbf{x} \in \Gamma$ there are inequalities in the other direction.

Proposition 2 (E., 1984)

For every ϵ with $0 < \epsilon < 1$ we have

$$H(a_1x_1 + \dots + a_mx_m) \gg_{K,\Gamma,\mathbf{a},\epsilon} H(\mathbf{x})^{1-\epsilon}$$

for all $\mathbf{x} \in \Gamma$ with $\sum_{i \in I} a_ix_i \neq 0$ for each non-empty $I \subseteq \{1, \dots, m\}$.

The proof is based on the p-adic Subspace Theorem of Schmidt and Schlickewei (1972–1977). Consequently, the implied constant is ineffective.

Applications of the Subspace Theorem (II)

Let K be a number field, and Γ a f.g. subgroup of $(K^*)^m$ of rank $r \geq 1$. Let $m \geq 2$ and $\mathbf{a} = (a_1, \dots, a_m) \in (K^*)^m$.

Proposition 3 (EGHLR, 2026)

Let ϵ with $0 < \epsilon < 1$. Then the set of $\mathbf{x} \in \Gamma$ such that

$$H(a_1 x_1 + \dots + a_m x_m) \leq H(\mathbf{x})^{1-\epsilon}$$

is contained in the union of

$$\ll_{K, \Gamma, \mathbf{a}} (\epsilon^{-1})^{r+3} (\log(1 + \epsilon^{-1}))^2$$

proper linear subspaces of K^m .

The proof is based on the most recent quantitative version of the Subspace Theorem, due to E. and Ferretti (2013). The implied constant can be computed explicitly, however the subspaces cannot be computed from the method of proof.

Points in a subspace (I)

Let K be a number field, $m \geq 2$, and Γ a subgroup of $(K^*)^m$ of rank $r \geq 1$ with the following property:

$$(++) \quad \begin{cases} \text{for each } i, j \in \{1, \dots, m\} \text{ with } i \neq j, \text{ there is } x \in \Gamma \\ \text{such that } x_i/x_j \text{ is not a root of unity.} \end{cases}$$

Recall $\mathcal{H}_\Gamma(X) = \{x \in \Gamma : H(x) \leq X\}$.

Proposition 4 (EGHLR, 2026)

For every proper linear subspace T of K^m , we have

$$\#\mathcal{H}_\Gamma(X) \cap T \ll_{K,\Gamma} (\log X)^{r-1},$$

where the implied constant is independent of T .

The independence of T of the upper bound is crucial for the proof of Theorem 2.

To prove this result, we used a quantitative result for S -unit equations. Assumption $(++)$ is needed to get an upper bound $\ll (\log X)^{r-1}$.

Points in a subspace (II)

Let K be a number field, $m \geq 2$, $\mathbf{a} = (a_1, \dots, a_m) \in (K^*)^m$, and Γ a f.g. subgroup of $(K^*)^m$ of rank $r \geq 1$ with property $(++)$. Let

$$\mathcal{V}_{\Gamma, \mathbf{a}}(X) := \left\{ \mathbf{x} \in \Gamma : \begin{array}{l} H(a_1 x_1 + \dots + a_m x_m) \leq X \\ \sum_{i \in I} a_i x_i \neq 0 \text{ for each non-empty } I \subseteq \{1, \dots, m\} \end{array} \right\}.$$

Proposition 5

For every proper linear subspace T of K^m we have

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Proof.

By Proposition 2, there is a constant $c' > 0$ such that for each $\mathbf{x} \in \mathcal{V}_{\Gamma, \mathbf{a}}(X)$ we have $H(a_1 x_1 + \dots + a_m x_m) \geq c' H(\mathbf{x})^{1/2}$.

Together with Proposition 4 this implies

$$\#\mathcal{V}_{\Gamma, \mathbf{a}}(X) \cap T \leq \#\{\mathbf{x} \in \Gamma : H(\mathbf{x}) \leq c'^{-2} X^2\} \cap T \ll_{K, \Gamma, \mathbf{a}} (\log X)^{r-1}.$$

Proof of Theorem 2

Let K be a number field, $m \geq 2$, $\mathbf{a} = (a_1, \dots, a_m) \in (K^*)^m$, and Γ a f.g. subgroup of $(K^*)^m$ of rank $r \geq 1$ with the following property:

$$(++) \quad \begin{cases} \text{for each } i, j \in \{1, \dots, m\} \text{ with } i \neq j, \text{ there is } x \in \Gamma \\ \text{such that } x_i/x_j \text{ is not a root of unity.} \end{cases}$$

Let

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Our aim is to prove that

$$\begin{aligned} \#\mathcal{V}_{\Gamma, \mathbf{a}}(X) &= c(\Gamma)(\log X)^r + O((\log X)^{r-1/(r+4)}(\log \log X)^{2/(r+4)}) \\ &\text{as } X \rightarrow \infty. \end{aligned}$$

We first derive a lower bound and then an upper bound for $\#\mathcal{V}_{\Gamma, \mathbf{a}}(X)$.

Lower bound for $\#\mathcal{V}_{\Gamma,a}(X)$

Let K be a number field, $m \geq 2$, $a = (a_1, \dots, a_m) \in (K^*)^m$, and Γ a f.g. subgroup of $(K^*)^m$ of rank $r \geq 1$ with property $(++)$. Let

$$\mathcal{V}_{\Gamma,a}(X) := \left\{ x \in \Gamma : \begin{array}{l} H(a_1x_1 + \dots + a_mx_m) \leq X \\ \sum_{i \in I} a_ix_i \neq 0 \text{ for each non-empty } I \subseteq \{1, \dots, m\} \end{array} \right\}.$$

For $x \in \mathcal{V}_{\Gamma,a}(X)$ we have $H(a_1x_1 + \dots + a_mx_m) \leq cH(x)$ for some $c > 0$, hence

$$\mathcal{V}_{\Gamma,a}(X) \supseteq \left\{ x \in \Gamma, H(x) \leq c^{-1}X, \quad \forall I \quad \sum_{i \in I} a_ix_i \neq 0 \right\}.$$

Using

$$\#\{x \in \Gamma, H(x) \leq c^{-1}X\} = c(\Gamma)(\log X)^r + O((\log X)^{r-1}) \quad \text{by Prop. 1,}$$

$$\#\left\{ x \in \Gamma, H(x) \leq c^{-1}X, \sum_{i \in I} a_ix_i = 0 \right\} \ll (\log X)^{r-1} \quad \forall I \text{ by Prop. 4.}$$

we obtain

$$\#\mathcal{V}_{\Gamma,a}(X) \geq c(\Gamma)(\log X)^r + O((\log X)^{r-1}) \quad \text{as } X \rightarrow \infty.$$

Upper bound for $\#\mathcal{V}_{\Gamma,a}(X)$: splitting $\mathcal{V}_{\Gamma,a}(X)$

Let K be a number field, $m \geq 2$, $\mathbf{a} = (a_1, \dots, a_m) \in (K^*)^m$, and Γ a f.g. subgroup of $(K^*)^m$ of rank $r \geq 1$ with property $(++)$.

We choose $\epsilon = \epsilon(X)$ a function of X to be determined later, decreasing to 0 as $X \rightarrow \infty$, and split $\mathcal{V}_{\Gamma,a}(X)$ into the following two sets:

$$\begin{aligned}\mathcal{V}_{\Gamma,a}^{(1)}(X) &= \{x \in \mathcal{V}_{\Gamma,a}(X) : H(a_1x_1 + \dots + a_mx_m) > H(x)^{1-\epsilon}\}, \\ \mathcal{V}_{\Gamma,a}^{(2)}(X) &= \{x \in \mathcal{V}_{\Gamma,a}(X) : H(a_1x_1 + \dots + a_mx_m) \leq H(x)^{1-\epsilon}\}.\end{aligned}$$

We estimate from above the cardinalities of both sets and add them together.

Upper bound for $\#\mathcal{V}_{\Gamma,a}^{(1)}(X)$

Recall

$$\mathcal{V}_{\Gamma,a}(X) := \left\{ x \in \Gamma : \begin{array}{l} H(a_1x_1 + \cdots + a_mx_m) \leq X \\ \sum_{i \in I} a_ix_i \neq 0 \text{ for each non-empty } I \subseteq \{1, \dots, m\} \end{array} \right\}$$

$$\text{and } \mathcal{V}_{\Gamma,a}^{(1)}(X) = \{x \in \mathcal{V}_{\Gamma,a}(X) : H(a_1x_1 + \cdots + a_mx_m) > H(x)^{1-\epsilon}\}.$$

By Proposition 1,

$$\begin{aligned} \#\mathcal{V}_{\Gamma,a}^{(1)}(X) &\leq \#\{x \in \Gamma : H(x) \leq X^{1/(1-\epsilon)}\} \\ &= c(\Gamma)(\log X^{1/(1-\epsilon)})^r + O((\log X)^{r-1}) \\ &= c(\Gamma)(\log X)^r + O(\epsilon \cdot (\log X)^r + (\log X)^{r-1}). \end{aligned}$$

Upper bound for $\#\mathcal{V}_{\Gamma,a}^{(2)}(X)$

Recall

$$\mathcal{V}_{\Gamma,a}(X) := \left\{ x \in \Gamma : \begin{array}{l} H(a_1x_1 + \cdots + a_mx_m) \leq X \\ \sum_{i \in I} a_ix_i \neq 0 \text{ for each non-empty } I \subseteq \{1, \dots, m\} \end{array} \right\}$$

$$\text{and } \mathcal{V}_{\Gamma,a}^{(2)}(X) = \{x \in \mathcal{V}_{\Gamma,a}(X) : H(a_1x_1 + \cdots + a_mx_m) \leq H(x)^{1-\epsilon}\}.$$

By Proposition 3, $\mathcal{V}_{\Gamma,a}^{(2)}(X)$ is contained in the union of

$$\ll_{K,\Gamma,a} (\epsilon^{-1})^{r+3} (\log(1 + \epsilon^{-1}))^2$$

proper linear subspaces of K^m .

Upper bound for $\#\mathcal{V}_{\Gamma,a}^{(2)}(X)$

Recall

$$\mathcal{V}_{\Gamma,a}(X) := \left\{ x \in \Gamma : \begin{array}{l} H(a_1x_1 + \cdots + a_mx_m) \leq X \\ \sum_{i \in I} a_ix_i \neq 0 \text{ for each non-empty } I \subseteq \{1, \dots, m\} \end{array} \right\}$$

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proper linear subspaces of K^m .

By Proposition 5, for each of these subspaces T , we have

$$\#\mathcal{V}_{\Gamma,a}^{(2)}(X) \cap T \ll_{K,\Gamma,a} (\log X)^{r-1}.$$

Hence

$$\#\mathcal{V}_{\Gamma,a}^{(2)}(X) \ll_{K,\Gamma,a} (\log X)^{r-1} (\epsilon^{-1})^{r+3} (\log(1 + \epsilon^{-1}))^2.$$

Completion of the proof of Theorem 2

Recall

$$\mathcal{V}_{\Gamma,a}(X) := \left\{ x \in \Gamma : \begin{array}{l} H(a_1x_1 + \cdots + a_mx_m) \leq X \\ \sum_{i \in I} a_ix_i \neq 0 \text{ for each non-empty } I \subseteq \{1, \dots, m\} \end{array} \right\}.$$

Combining what we obtained before, we obtain

$$\begin{aligned} \#\mathcal{V}_{\Gamma,a}(X) &= \#\mathcal{V}_{\Gamma,a}^{(1)}(X) + \#\mathcal{V}_{\Gamma,a}^{(2)}(X) \\ &\leq c(\Gamma)(\log X)^r + O\left(\epsilon \cdot (\log X)^r + (\log X)^{r-1} \right. \\ &\quad \left. + (\log X)^{r-1}(\epsilon^{-1})^{r+3}(\log(1 + \epsilon^{-1}))^2\right). \end{aligned}$$

Completion of the proof of Theorem 2

Recall

$$\mathcal{V}_{\Gamma,a}(X) := \left\{ x \in \Gamma : \begin{array}{l} H(a_1x_1 + \cdots + a_mx_m) \leq X \\ \sum_{i \in I} a_ix_i \neq 0 \text{ for each non-empty } I \subseteq \{1, \dots, m\} \end{array} \right\}.$$

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The optimal choice $\epsilon = ((\log \log X)^2 / \log X)^{1/(r+4)}$ gives

$$\#\mathcal{V}_{\Gamma,a}(X) \leq c(\Gamma)(\log X)^r + O\left((\log X)^{r-1/(r+4)}(\log \log X)^{2/(r+4)}\right) \text{ as } X \rightarrow \infty.$$

Combined with the lower bound proved before, which has the same main term and a smaller error term, this proves Theorem 2. $\square$

Linear recurrence sequences

Let $\{u_n\}_{n \geq 0}$ be a linear recurrence sequence with algebraic terms, i.e.,

$$u_n = a_1(n)\alpha_1^n + \cdots + a_m(n)\alpha_m^n \text{ for } n \geq 0,$$

where $\alpha_1, \dots, \alpha_m$ are distinct non-zero algebraic numbers, and $a_1, \dots, a_m$ non-zero polynomials with algebraic coefficients.

Let K be the number field generated by $\alpha_1, \dots, \alpha_m$ and the coefficients of $a_1, \dots, a_m$. Define

$$H := H(\alpha_1, \dots, \alpha_m) = \prod_{v \in M_K} \max(1, \|\alpha_1\|_v, \dots, \|\alpha_m\|_v).$$

Theorem 3 (EGHLR, 2026)

Assume that $\{u_n\}$ is non-degenerate, i.e., α_i/α_j is not a root of unity for each $i, j \in \{1, \dots, m\}$ with $i \neq j$. Then

$$\#\{n \in \mathbb{Z}_{\geq 0} : H(u_n) \leq X\} = \frac{\log X}{\log H} + O\left((\log X)^{3/4}(\log \log X)^{1/2}\right) \\ \text{as } X \rightarrow \infty,$$

where the implied constant depends on the a_i and α_i and can be determined effectively.

Multi-recurrences

A multi-recurrence is an expression of the form

$$P(z) = \sum_{i=1}^m a_i(z_1, \dots, z_r) \alpha_{i1}^{z_1} \cdots \alpha_{ir}^{z_r} = \sum_{i=1}^m a_i(z) \underline{\alpha}_i^z$$

where the a_i are non-zero polynomials with algebraic coefficients, and the α_{ij} are non-zero algebraic numbers. Define the group

$$\Gamma := \{(\underline{\alpha}_1^z, \dots, \underline{\alpha}_m^z) : z \in \mathbb{Z}^r\}.$$

The following is ongoing work.

Theorem 4 (E., 2026)

Assume that

$$\{z \in \mathbb{Z}^r : \underline{\alpha}_i^z = \underline{\alpha}_j^z\} = \{0\} \text{ for each } i, j \in \{1, \dots, m\} \text{ with } i \neq j.$$

Then the number of $z \in \mathbb{Z}^r$ such that

$$H(P(z)) \leq X, \quad a_1(z) \cdots a_r(z) \neq 0$$

is $c(\Gamma)(\log X)^r + O((\log X)^{r-r/(r+4)}(\log \log X)^{2/(r+4)})$ as $X \rightarrow \infty$.

Reference:

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**Thank you for your
attention**