ASYMPTOTIC FORMULAS FOR SUMS OF ELEMENTS FROM A MULTIPLICATIVE GROUP

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To the memory of Graham Everest (1957–2010)

Abstract. Let K be a number field, $k \geq 2$ an integer, $(K^*)^k$ the k-fold direct product of K^* with coordinatewise multiplication, and Γ a finitely generated subgroup of rank r of $(K^*)^k$. Further, let $H(\alpha)$ denote the absolute exponential height of an algebraic number α. Fix non-zero elements $a_1, \dots, a_k \in K$. We give asymptotic formulas for the number of $\mathbf{x} = (x_1, \dots, x_k) \in \Gamma$ with $H(a_1x_1 + \cdots + a_kx_k) \leq X$ as $X \to \infty$ such that no non-empty subsum of $a_1x_1 + \cdots + a_kx_k$ vanishes. By the same method of proof, we obtain an asymptotic formula as $X \to \infty$ for the number of non-negative integers n with $H(u_n) \leq X$, where $\{u_n\}$ is a linear recurrence sequence.

1. Introduction

Let K be a number field, $k \geq 2$ an integer, and S a finite set of places of K, containing all infinite places. Suppose S has cardinality $s + 1$. Denote by U_S the group of S-units of K; then U_S has rank s. Fix non-zero $a_1, \dots, a_k \in K$. Denote by $H(\alpha)$ the absolute exponential Weil height of an algebraic number α.

This paper builds further on work of G. Everest. By a slight modification of the arguments in his papers [4, 5], one can deduce an asymptotic formula for the number of tuples $(x_1, \dots, x_k)$ with $x_1, \dots, x_k \in U_S$ that satisfy

$$H(a_1x_1 + \cdots + a_kx_k) \leq X$$

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and are non-degenerate in the sense that no non-trivial subsum of $\sum_{i=1}^{k} a_i x_i$ vanishes. In fact, Everest's arguments yield for this number an asymptotic formula

$$c(U_S, k)(\log X)^{ks} + O((\log X)^{ks-1}) \ \text{ as } X \to \infty, \tag{1.1}$$

where $c(U_S, k) > 0$. Everest's main tools are a result from [6] that compares $H(a_1x_1+\cdots+a_kx_k)$ with the height of the vector $(x_1, \dots, x_k)$ (which in turn was deduced from Schmidt's and Schlickewei's p-adic Subspace theorem [15, 13]), and logarithmic forms estimates.

Let $(K^*)^k$ denote the k-fold direct product of K^* with coordinatewise multiplication and inversion. We consider more generally sums

$$a_1x_1 + \cdots + a_kx_k \ \text{ with } (x_1, \dots, x_k) \in \Gamma,$$

where Γ is a finitely generated subgroup of rank r of $(K^*)^k$. We assume that for each distinct $i, j \in \{1, \dots, k\}$, there is $(x_1, \dots, x_k) \in \Gamma$ such that x_i/x_j is not a root of unity. The purpose of this paper is to deduce an asymptotic formula for the number of non-degenerate $(x_1, \dots, x_k) \in \Gamma$ with $H(a_1x_1 + \cdots + a_kx_k) \leq X$. Our first result (see Theorem 2.1 below) is, that this number equals

$$c(\Gamma)(\log X)^r \cdot \left(1 + O\Big(\frac{(\log\log X)^2}{\log X}\Big)^{1/(r+4)}\right) \ \text{ as } X \to \infty, \tag{1.2}$$

with $c(\Gamma) > 0$ depending only on Γ. In this general set-up, we could not make Everest's method work, so we used instead a different method, based on a suitable quantitative version of the Parametric Subspace Theorem [7].

By a similar argument, we obtain an asymptotic formula for non-degenerate linear recurrence sequences $u_n = \sum_{i=1}^{k} a_i(n)\alpha_i^n$, i.e., the a_i are non-zero polynomials in $K[X]$ and the α_i non-zero elements of K, such that for any two distinct i, j, α_i/α_j is not a root of unity. In fact, we show that the number of non-negative integers n with $H(u_n) \leq X$ equals

$$c\log X \left(1 + O\Big(\frac{(\log\log X)^2}{\log X}\Big)^{1/4}\right) \ \text{ as } X \to \infty, \tag{1.3}$$

where $c > 0$, see Theorem 2.2 below.

We can improve on (1.2) in the special case

$$\Gamma = \Gamma_1^k = \{(x_1, \dots, x_k) : x_1, \dots, x_k \in \Gamma_1\},$$

where Γ_1 is a finitely generated subgroup of K^*. Indeed, in this situation, Everest's method is applicable, and we obtain instead of (1.2),

$$c(\Gamma)(\log X)^r + O((\log X)^{r-1}) \quad \text{as } X \to \infty, \tag{1.4}$$

see Theorem 2.4 below. In fact, our method of proof is a simplification of Everest's, which avoids the logarithmic forms estimates. In the case $\Gamma_1 = U_S$ we obtain (1.1).

Finally, we deduce an asymptotic formula where we do not count tuples $(x_1, \dots, x_k)$, but instead algebraic numbers α with $H(\alpha) \le X$ that are representable in the form $a_1x_1 + \dots + a_kx_k$ with $x_1, \dots, x_k \in \Gamma_1$, see Theorem 2.5 below. This extends work of Ádám, Hajdu, and Luca [1], and of Frei, Tichy, and Ziegler [11].

In all our theorems, the constants implied by the O-symbols in the asymptotic formulas cannot be effectively determined by our method of proof, except for the one in Theorem 2.2 on linear recurrence sequences, where the implied constant is effectively computable.

In the next section, we introduce the necessary notation and state the above results in a precise form. In Section 3 we have collected our tools, and in the subsequent sections, we prove our theorems.

2. Notation and results

We denote by $|\mathcal{A}|$ the cardinality of a set $\mathcal{A}$. Throughout this paper, K will be an algebraic number field of degree d and $k \ge 2$ an integer. Let K^* denote the group $K \setminus \{0\}$ with multiplication, and $(K^*)^k$ its k-fold direct product. Denote by O_K the ring of integers of K. The real infinite places of K are its real embeddings $\sigma_i : K \hookrightarrow \mathbb{R}$ $(i = 1, \dots, r_1)$, the complex infinite places of K are its pairs of conjugate complex embeddings

$$\{\sigma_i, \overline{\sigma_i} : K \hookrightarrow \mathbb{C}\} \quad (i = r_1 + 1, \dots, r_1 + r_2),$$

and the finite places of K are the prime ideals of O_K. The set of places M_K of K consists of the (real and complex) infinite places and the finite places of K. We define normalized absolute values $\|\cdot\|_v$ $(v \in M_K)$ on

K by

$$\|\cdot\|_v := |\sigma(\cdot)|^{1/d} \quad \text{if } v=\sigma \text{ is real infinite},$$
$$\|\cdot\|_v := |\sigma(\cdot)|^{2/d} \quad \text{if } v=\{\sigma,\overline{\sigma}\} \text{ is complex infinite},$$
$$\|\cdot\|_v := N\mathfrak{p}^{-\mathrm{ord}_{\mathfrak{p}}(\cdot)/d} \quad \text{if } v=\mathfrak{p} \text{ is finite},$$

where $N\mathfrak{p} = |O_K/\mathfrak{p}|$ is the norm of $\mathfrak{p}$, and $\mathrm{ord}_{\mathfrak{p}}(\alpha)$ is the exponent of $\mathfrak{p}$ in the unique prime ideal factorization of $\alpha \in K$, with $\mathrm{ord}_{\mathfrak{p}}(0)=\infty$. These absolute values satisfy the product formula

$$\prod_{v\in M_K} \|\alpha\|_v = 1 \quad \text{for } \alpha \in K^*.$$

The absolute, exponential height and logarithmic height of $\alpha \in K$ are given by

$$H(\alpha) := \prod_{v\in M_K} \max(1, \|\alpha\|_v), \quad h(\alpha) := \log H(\alpha).$$

These heights depend only on α, i.e., are independent of the choice of a number field K containing α. For later use, note that (as is well-known), we have

$$H(\alpha_1\cdots\alpha_n) \le H(\alpha_1)\cdots H(\alpha_n)$$

and

$$H(\alpha_1+\cdots+\alpha_n) \le nH(\alpha_1)\cdots H(\alpha_n)$$

for any $\alpha_1,\dots,\alpha_n \in K$. Further, for any non-zero $\alpha \in K$ and $m \in \mathbb{Z}$ we have

$$H(\alpha^m) = H(\alpha)^{|m|}.$$

More generally, for a vector $\mathbf{x} = (x_1,\dots,x_k) \in K^k$ we define

$$\|\mathbf{x}\|_v := \max(\|x_1\|_v,\dots,\|x_k\|_v) \quad \text{for } v \in M_K$$

and subsequently its exponential and logarithmic height

$$H(\mathbf{x}) = H(x_1,\dots,x_k) := \prod_{v\in M_K} \max(1, \|\mathbf{x}\|_v), \quad h(\mathbf{x}) := \log H(\mathbf{x}).$$

The product of $\mathbf{x} = (x_1,\dots,x_k)$, $\mathbf{y} = (y_1,\dots,y_k) \in K^k$ is defined coordinatewise, i.e., $\mathbf{x}\cdot\mathbf{y} = (x_1y_1,\dots,x_ky_k)$. Further, if $\mathbf{x} \in (K^*)^k$, i.e., $x_1\cdots x_k \neq 0$ we define $\mathbf{x}^{-1} := (x_1^{-1},\dots,x_k^{-1})$. One easily shows that

$$H(\mathbf{x}\cdot\mathbf{y}) \le H(\mathbf{x})\cdot H(\mathbf{y}) \quad \text{for } \mathbf{x},\mathbf{y} \in K^k, \tag{2.1}$$

$$H(\mathbf{x}^m) = H(\mathbf{x})^m \quad \text{for } \mathbf{x} \in K^k,\ m \in \mathbb{Z}_{\ge 0}. \tag{2.2}$$

Further,

$$H(\mathbf{x}^{-1}) \leq H(\mathbf{x})^k \quad \text{for } \mathbf{x} \in (K^*)^k. \tag{2.3}$$

Indeed, if $\mathbf{x} = (x_1, \dots, x_k)$ one may write for $v \in M_K$,

$$\begin{aligned}
\max(1, \|\mathbf{x}^{-1}\|_v) &= \max(1, \|x_1\|_v^{-1}, \dots, \|x_k\|_v^{-1}) \\
&= \|x_1 \cdots x_k\|_v^{-1} \max\Big(\|x_1 \cdots x_k\|_v, \|\prod_{j \neq 1} x_j\|_v, \dots, \|\prod_{j \neq k} x_j\|_v\Big) \\
&\leq \|x_1 \cdots x_k\|_v^{-1} \max(1, \|\mathbf{x}\|_v)^k
\end{aligned}$$

and by taking the product over all v and applying the product formula, one obtains (2.3).

Let S be a finite subset of M_K with $|S| \geq 2$, containing all infinite places. The multiplicative group of S-units in K is defined by

$$U_S := \{u \in K : \ \|u\|_v = 1 \text{ for } v \in M_K \setminus S\}.$$

It is well-known that U_S is finitely generated and that its rank is $s := |S| - 1$. Further, the torsion group of U_S is the group W_K of roots of unity in K. Denote by ω_K the order of W_K.

Let $S = \{v_0, , \dots, v_s\}$, and let $\{\gamma_1, \dots, \gamma_s\}$ be a fundamental set of S-units, or equivalently, a basis of U_S/W_K, i.e., every $u \in U_S$ can be expressed uniquely as $\zeta\gamma_1^{z_1} \cdots \gamma_r^{z_r}$ with $\zeta \in W_K$ and $z_1, \dots, z_r \in \mathbb{Z}$. Define the quantity

$$R_S := \left|\det \begin{pmatrix} \log \|\gamma_1\|_{v_1} & \cdots & \log \|\gamma_s\|_{v_1} \\ \vdots & \cdots & \vdots \\ \log \|\gamma_1\|_{v_s} & \cdots & \log \|\gamma_s\|_{v_s} \end{pmatrix}\right|. \tag{2.4}$$

This is independent of the choice of $v_1, \dots, v_s, \gamma_1, \dots, \gamma_s$. For properties of the introduced notions, see e.g. Chapters 2.1 and 2.2 of Evertse and Győry [8]. The S-regulator as defined in [8] is $d^s R_S$.

Now let Γ be a finitely generated subgroup of rank r of $(K^*)^k$. Write elements of Γ as $\mathbf{x} = (x_1, \dots, x_k)$. Choose a basis $\mathbf{u}_1, \dots, \mathbf{u}_r$ of $\Gamma/\Gamma_{\text{tors}}$, by which we mean that $\mathbf{u}_1, \dots, \mathbf{u}_r \in \Gamma$ and every $\mathbf{x} \in \Gamma$ can be expressed uniquely as

$$\zeta \cdot \mathbf{u}_1^{z_1} \cdots \mathbf{u}_r^{z_r} \quad \text{with } \zeta \in \Gamma_{\text{tors}}, \quad z_1, \dots, z_r \in \mathbb{Z}. \tag{2.5}$$

Write $\mathbf{u}_i = (u_{i1}, \dots, u_{ik})$ for $i = 1, \dots, k$. Let S be the smallest set of places of K such that $\|u_{ij}\|_v = 1$ for $v \in M_K \setminus S$, $i = 1, \dots, r$,

$j = 1, \dots, k$. Then S is finite, and we have

(2.6)
$$S = \{v \in M_K : \text{there is } \mathbf{x} \in \Gamma \text{ with } \|x_i\|_v \neq 1 \text{ for at least one } i\}.$$

Now for the logarithmic height of $\mathbf{x} \in \Gamma$, we have

$$h(\mathbf{x}) = \sum_{v \in S} \max\big(0, \ell_{v1}(\mathbf{z}), \dots, \ell_{vk}(\mathbf{z})\big), \tag{2.7}$$
$$\text{where } \ell_{vj}(\mathbf{z}) := \sum_{i=1}^{r} z_i \log \|u_{ij}\|_v \quad (v \in S,\ j = 1, \dots, k).$$

Let μ denote the Lebesgue measure on $\mathbb{R}$ with $\mu([0,1]) = 1$, and μ^r the product measure on $\mathbb{R}^r$. Define

$$\mathcal{C}(\Gamma) := \big\{\xi \in \mathbb{R}^r : \sum_{v \in S} \max\big(0, \ell_{v1}(\xi), \dots, \ell_{vk}(\xi)\big) \leq 1\big\}, \tag{2.8}$$

$$c(\Gamma) := |\Gamma_{\text{tors}}| \cdot \mu^r(\mathcal{C}(\Gamma)). \tag{2.9}$$

The product formula implies $\sum_{v \in S} \ell_{vj} = 0$ for $j = 1, \dots, r$, and from this one easily deduces that $\mathcal{C}(\Gamma)$ is a bounded set. Hence, $c(\Gamma)$ is finite. The set $\mathcal{C}(\Gamma)$ depends on the choice of $\mathbf{u}_1, \dots, \mathbf{u}_r$, but the quantity $c(\Gamma)$ is independent of this choice.

Henceforth we write $\mathbf{x} = (x_1, \dots, x_k)$, $\mathbf{a} = (a_1, \dots, a_k)$, etc. For $\mathbf{a} \in (K^*)^k$ and $X > 1$, define the set

$$\mathcal{V}_{\Gamma,\mathbf{a}}(X) := \Big\{\mathbf{x} \in \Gamma : H(a_1x_1 + \cdots + a_kx_k) \leq X,$$
$$\sum_{i \in I} a_ix_i \neq 0 \text{ for each non-empty } I \subseteq \{1, \dots, k\}\Big\}.$$

Our first result is as follows.

Theorem 2.1. *Let K be a number field, $k \geq 2$ an integer, and Γ a finitely generated subgroup of $(K^*)^k$ of rank $r \geq 1$. Assume that*

(2.10) *for each $i \neq j \in \{1, \dots, k\}$ there is $\mathbf{x} \in \Gamma$ such that x_i/x_j is not a root of unity.*

Let $\mathbf{a} = (a_1, \dots, a_k) \in (K^*)^k$. Then

$$|\mathcal{V}_{\Gamma,\mathbf{a}}(X)| = c(\Gamma)(\log X)^r \cdot \left(1 + O\Big(\Big(\frac{(\log\log X)^2}{\log X}\Big)^{1/(r+4)}\Big)\right)$$
$$\text{as } X \to \infty,$$

where $c(\Gamma)$ *is defined by* (2.9). *Here, the implied constant depends only on* Γ *and* $\mathbf{a}$.

By specializing to the case $\Gamma = \{(\alpha_1^n, \dots, \alpha_k^n) : n \in \mathbb{Z}\}$, i.e., $r = 1$, we can deduce a result for linear recurrence sequences $u_n = \sum_{i=1}^k a_i \alpha_i^n$. By modifying the proof of Theorem 2.1 we managed to improve and generalize this.

Theorem 2.2. *Let* K *be a number field,* $k \geq 2$ *an integer,* $a_1, \dots, a_k \in K[X]$ *non-zero polynomials, and* $\alpha_1, \dots, \alpha_k \in K^*$. *Assume that*

$$\text{for each } i \neq j \in \{1, \dots, k\},\ \alpha_i/\alpha_j \text{ is not a root of unity.} \tag{2.11}$$

Put $u_n := \sum_{i=1}^k a_i(n)\alpha_i^n$ *for* $n \in \mathbb{Z}_{\geq 0}$. *Then*

$$|\{n \in \mathbb{Z}_{\geq 0} : H(u_n) \leq X\}| = \frac{\log X}{\log H} \cdot \left(1 + O\Big(\Big(\frac{(\log\log X)^2}{\log X}\Big)^{1/4}\Big)\right) \quad \text{as } X \to \infty,$$

where

$$H := H(\alpha_1, \dots, \alpha_k) = \prod_{v \in M_K} \max(1, \|\alpha_1\|_v, \dots, \|\alpha_k\|_v),$$

and the implied constant depends only on $a_1, \dots, a_k, \alpha_1, \dots, \alpha_k$ *and can be determined effectively.*

Further potential applications of our method of proof are to linear combinations of linear recurrence sequences $\{b_1 u_m + b_2 v_n : m, n \in \mathbb{Z}_{\geq 0}\}$ (see [17] for a related result) or perhaps even $\{b_1 u_{n_1,1} + \cdots + b_l u_{n_l,l} : n_1, \dots, n_l \in \mathbb{Z}_{\geq 0}\}$. Most generally, one might consider exponential polynomials

$$\Big\{\sum_{i=1}^k f_i(n_1, \dots, n_l)\alpha_{i1}^{n_1} \cdots \alpha_{il}^{n_l} : n_1, \dots, n_l \in \mathbb{Z}\Big\},$$

where $f_i \in K[X_1, \dots, X_l], \alpha_{i1}, \dots, \alpha_{il} \in K^*$ for $i = 1, \dots, k$.

We can sharpen Theorem 2.1 if we strengthen (2.10). As before, we write $\mathbf{x} = (x_1, \dots, x_k)$, and put $x_0 := 1$.

Theorem 2.3. *Let* K *be a number field,* $k \geq 2$ *an integer, and* Γ *a finitely generated subgroup of* $(K^*)^k$ *of rank* $r \geq 1$. *Let* S *be given by*

(2.6). *Assume that*

(2.12) *for each* $v \in S$, $i \neq j \in \{0, \dots, k\}$ *there is* $\mathbf{x} \in \Gamma$ *with* $\|x_i\|_v \neq \|x_j\|_v$.

Let $\mathbf{a} \in (K^*)^k$. *Then*

$$|\mathcal{V}_{\Gamma,\mathbf{a}}(X)| = c(\Gamma)(\log X)^r + O((\log X)^{r-1}) \quad \text{as } X \to \infty.$$

Here, the implied constant depends only on Γ *and* $\mathbf{a}$.

An important special case is, when

$$\Gamma = \Gamma_1^k = \{(x_1, \dots, x_k) : x_1, \dots, x_k \in \Gamma_1\},$$

where Γ_1 is a finitely generated subgroup of K^*. Note that in this case, $\operatorname{rank}\Gamma = k \cdot \operatorname{rank}\Gamma_1$.

Theorem 2.4. *Let* K *be a number field,* $k \geq 2$ *an integer,* Γ_1 *a subgroup of* K^* *of rank* $r_1 \geq 1$, *and* $\mathbf{a} \in (K^*)^k$. *Then*

$$|\mathcal{V}_{\Gamma_1^k,\mathbf{a}}(X)| = c(\Gamma_1^k)(\log X)^{kr_1} + O((\log X)^{kr_1-1}) \quad \text{as } X \to \infty.$$

Here, the implied constant depends only on Γ_1, k, *and* $\mathbf{a}$.

In general, it seems to be a difficult problem to derive an explicit expression for $c(\Gamma)$, but this can be done in the case that $\Gamma = U_S^k$, where S is a finite set of places of K, containing all infinite places. Write $S = \{v_0, \dots, v_s\}$ and pick a basis $\{\gamma_1, \dots, \gamma_s\}$ of U_S/W_K. Using the product formula in the form $\log\|u\|_{v_0} = -\sum_{j=1}^s \log\|u\|_{v_j}$ for $u \in U_S$, together with (2.4), one can show that $\mathcal{C}(U_S^k)$ is the image under a linear transformation of determinant R_S^{-k} of the set

$$\mathcal{E}_{s,k} := \Big\{\xi = (\xi_{v_i,j})_{i=1,\dots,s,j=1,\dots,k} \in \mathbb{R}^{ks} : \sum_{i=0}^{s} \max(0, \xi_{v_i,1}, \dots, \xi_{v_i,k}) \leq 1\Big\},$$

where $\xi_{v_0,j} = -\sum_{i=1}^s \xi_{v_i,j}$ for $j = 1, \dots, k$. Kerber, Tichy, and Weitzer [12] proved that this set (called by them 'Everest polytope' since it was introduced by Everest [4]) has measure $((k+1)s)!/(ks)!(s!)^{k+1}$. Noticing that $(U_S^k)_{\text{tors}} = W_K^k$ and thus, has cardinality ω_K^k, it follows that

$$c(U_S^k) = \frac{\omega_K^k}{R_S^k} \cdot \frac{((k+1)s)!}{(ks)!(s!)^{k+1}}.$$

In our next result, we do not count tuples $(x_1, \dots, x_k) \in \Gamma_1^k$, but instead numbers α representable in the form $a_1x_1 + \dots + a_kx_k$. Further, we do not take a fixed tuple of coefficients $(a_1, \dots, a_k)$, but instead a finite set of such tuples.

Denote by S_k the permutation group of $\{1, \dots, k\}$. Henceforth, we write

$$\mathbf{a} = (a_1, \dots, a_k), \quad \mathbf{b} = (b_1, \dots, b_k), \quad \mathbf{a}/\mathbf{b} = (a_1/b_1, \dots, a_k/b_k),$$
$$\sigma(\mathbf{a}) = (a_{\sigma(1)}, \dots, a_{\sigma(k)}) \quad (\sigma \in S_k).$$

Let $\mathcal{A}$ be a finite set of tuples in $(K^*)^k$ with the following properties:

(2.13)

for each $\mathbf{a} \in \mathcal{A}$, $i, j \in \{1, \dots, k\}$ we have $a_i = a_j$ or $a_i/a_j \notin \Gamma_1$;

(2.14)

for each $\mathbf{a}, \mathbf{b} \in \mathcal{A}$ we have $\mathbf{a} = \mathbf{b}$ or $\mathbf{a}/\mathbf{b} \notin \Gamma_1^k$;

(2.15)

for each $\mathbf{a} \in \mathcal{A}$, $\sigma \in S_k$ we have $\sigma(\mathbf{a}) \in \mathcal{A}$.

From Theorem 2.4 we deduce the following result. In the deduction we closely follow Frei, Tichy, and Ziegler [11].

Theorem 2.5. *Let K be a number field, k an integer ≥ 1, and Γ_1 a finitely generated subgroup of K^* of rank $r_1 > 0$. Further, let $\mathcal{A}$ be a finite set of tuples in $(K^*)^k$ with* (2.13), (2.14), (2.15). *Let $\mathcal{T}_{\mathcal{A}}(X)$ be the set of $\alpha \in K^*$ with the following properties:*

$$(2.16) \qquad \begin{cases} H(\alpha) \leq X; \\ \textit{there are } (a_1, \dots, a_k) \in \mathcal{A},\ x_1, \dots, x_k \in \Gamma_1 \\ \textit{such that } \alpha = a_1x_1 + \dots + a_kx_k. \end{cases}$$

Then

$$|\mathcal{T}_{\mathcal{A}}(X)| = \frac{|\mathcal{A}| \cdot c(\Gamma_1^k)}{k!} (\log X)^{kr_1} + O((\log X)^{kr_1 - 1}) \quad \textit{as } X \to \infty.$$

Here, the implied constant depends only on Γ_1, k, and $\mathcal{A}$.

Note that in (2.16) we allow subsums of $a_1x_1 + \dots + a_kx_k$ to be 0.

Two special cases of Theorem 2.5 are of interest:

(1) Let $\mathcal{A} = \mathcal{B}^k$ (k times cartesian product), where $\mathcal{B}$ is a finite subset of K^* such that for all $b, b' \in \mathcal{B}$ with $b \neq b'$ we have $b/b' \notin \Gamma_1$. Note that $\mathcal{B}^k$ satisfies (2.13), (2.14), (2.15). Hence,

$$|\mathcal{T}_{\mathcal{B}^k}(X)| = \frac{|\mathcal{B}|^k \cdot c(\Gamma_1^k)}{k!} (\log X)^{kr_1} + O((\log X)^{kr_1 - 1}).$$

(2) Let $\mathcal{A} = \{\mathbf{a}\}$ where $\mathbf{a} \in (K^*)^k$ is a tuple with (2.13). Of course, (2.15) need not be satisfied. But let $\mathcal{A}' := \{\sigma(\mathbf{a}) : \sigma \in S_k\}$; then $\mathcal{T}_{\{\mathbf{a}\}}(X) = \mathcal{T}_{\mathcal{A}'}(X)$, and $\mathcal{A}'$ satisfies (2.13), (2.14), (2.15). Define $G(\mathbf{a})$ to be the subgroup of $\sigma \in S_k$ such that $\sigma(\mathbf{a}) = \mathbf{a}$. Then $|\mathcal{A}'| = k!/|G(\mathbf{a})|$. So,

$$|\mathcal{T}_{\{\mathbf{a}\}}(X)| = |\mathcal{T}_{\mathcal{A}'}(X)| = \frac{c(\Gamma_1^k)}{|G(\mathbf{a})|}(\log X)^{kr_1} + O((\log X)^{kr_1-1}).$$

3. Auxiliary results

In this section, we have collected the main tools for our proofs. We start with a quantitative version of the so-called Parametric Subspace Theorem. The first result of this type was proved in 1996 by Schlickewei [14], then it was improved and generalized in 2002 by Evertse and Schlickewei [9]. The up to now sharpest version was obtained in 2013 by Evertse and Ferretti [7]. The next result is a consequence of their Theorem 2.1. The following notation is used:

- K is a number field, S a finite set of places of K (not necessarily containing all infinite places) and $k \geq 2$ an integer;
- $\mathcal{L} = (L_{iv} : v \in S, i = 1, \dots, k)$ is a tuple of linear forms in $K[X_1, \dots, X_k]$ such that for each $v \in S$, $\{L_{iv} : i = 1, \dots, k\}$ is linearly independent;
- $\mathbf{c} = (c_{iv} : v \in S, i = 1, \dots, k)$ is a tuple of reals.

For every parameter value $Q > 1$ we define the twisted height

(3.1)
$$H_{\mathcal{L},\mathbf{c},Q}(\mathbf{x}) := \prod_{v \in S} \Big(\max_{1 \leq i \leq k} \|L_{iv}(\mathbf{x})\|_v Q^{-c_{iv}} \Big) \cdot \prod_{v \in M_K \setminus S} \|\mathbf{x}\|_v \quad \text{for } \mathbf{x} \in K^n.$$

Lemma 3.1. *There are constants $C_1, C_2 > 0$, depending only on k, K and $\mathcal{L}$, with the following property. Let $0 < \delta < 1$ and assume that*

$$\sum_{i=1}^{k} c_{iv} = 0 \text{ for } v \in S, \quad \sum_{v \in S} \max(c_{1v}, \dots, c_{kv}) \leq 1. \tag{3.2}$$

Then there are proper linear subspaces $T_1, \dots, T_t$ of K^k, with

$$t \leq C_1 \delta^{-3} (\log \delta^{-1})^2$$

such that for every real Q with

$$Q \geq C_2^{1/\delta}$$

there is $T_i \in \{T_1, \dots, T_t\}$ *with* $\{\mathbf{x} \in K^n : H_{\mathcal{L},\mathbf{c},Q}(\mathbf{x}) \le Q^{-\delta}\} \subset T_i$.

In fact, Theorem 2.1 of [7] gives a version with fully explicit C_1, C_2. Moreover, it gives an absolute version for twisted heights defined on the algebraic closure of $\mathbb{Q}$ instead of just on a number field. For our purposes, we need only the explicit dependence on δ.

Condition (3.2) is a normalization. For applications, the following corollary, where this normalization has been removed, is more convenient.

Corollary 3.2. *Let* C_1, C_2 *be the constants from Lemma 3.1. Let* λ, μ, θ *be reals with* $\lambda < \mu < \theta$. *Further, let* $\mathbf{d} = (d_{iv} : v \in S,\, i = 1, \dots, k)$ *be a tuple of reals such that*

$$\frac{1}{k} \sum_{v \in S} \sum_{i=1}^{k} d_{iv} \le \lambda, \quad \sum_{v \in S} \max(d_{1v}, \dots, d_{kv}) \le \theta. \tag{3.3}$$

Put $\delta := \frac{\mu - \lambda}{\theta - \lambda}$. *Then there are proper linear subspaces* $T_1, \dots, T_t$ *of* K^n, *with*

$$t \le C_1 \delta^{-3} (\log \delta^{-1})^2 \tag{3.4}$$

such that for every real Q *with*

$$Q \ge C_2^{1/(\mu - \lambda)} \tag{3.5}$$

there is $T_i \in \{T_1, \dots, T_t\}$ *with* $\{\mathbf{x} \in K^n : H_{\mathcal{L},Q,\mathbf{d}}(\mathbf{x}) \le Q^{-\mu}\} \subset T_i$.

Proof. We reduce to Lemma 3.1. We first observe that if we increase λ, then both the upper bound for t in (3.4), as well as the lower bound for Q in (3.5) increase. Hence we may, and shall, assume that

$$\frac{1}{k} \sum_{v \in S} \sum_{i=1}^{k} d_{iv} = \lambda.$$

Clearly, $0 < \delta < 1$. Define

$$Q' := Q^{\theta - \lambda},$$

$$c_{iv} := \frac{1}{\theta - \lambda} \cdot \Big(d_{iv} - \frac{1}{k} \sum_{i=1}^{k} d_{iv} \Big) \quad \text{for } v \in S,\ i = 1, \dots, k.$$

Notice that these c_{iv} satisfy (3.2). Further, $H_{\mathcal{L},\mathbf{c},Q'}(\mathbf{x}) = Q^{\lambda} H_{\mathcal{L},\mathbf{d},Q}(\mathbf{x})$. Hence, $H_{\mathcal{L},\mathbf{d},Q}(\mathbf{x}) \le Q^{-\mu}$ is equivalent to $H_{\mathcal{L},\mathbf{c},Q'}(\mathbf{x}) \le Q'^{-\delta}$. Also, the condition $Q \geq C_2^{1/(\mu-\lambda)}$ is equivalent to $Q' \geq C_2^{1/\delta}$. Thus, Lemma 3.1 implies Corollary 3.2. □

Henceforth, K is an algebraic number field, and Γ a finitely generated subgroup of rank $r \geq 1$ of $(K^*)^k$, where $k \geq 2$. Elements of Γ are written as $\mathbf{x} = (x_1, \dots, x_k)$. We fix a basis $\mathbf{u}_1 = (u_{11}, \dots, u_{k1}), \dots, \mathbf{u}_r = (u_{r1}, \dots, u_{rk})$ of $\Gamma/\Gamma_{\text{tors}}$. Let S be given by (2.6).

Corollary 3.2 (applied with $Q = H(\mathbf{x})$ for $\mathbf{x} \in \Gamma$) is the main tool in the proof of Theorem 2.1. It will be used in combination with the following covering lemma, which is Lemma 6.3.6 of [8].

Lemma 3.3. *Let $\eta > 0$. There is a subset $\mathcal{E}$ of $\mathbb{R}^{k|S|}$ of cardinality*

$$|\mathcal{E}| \leq (1 + 4k/\eta)^r$$

such that for every $\mathbf{x} \in \Gamma$ there is $\mathbf{e} = (e_{iv} : v \in S, i = 1, \dots, k) \in \mathcal{E}$ with

$$\sum_{v \in S} \sum_{i=1}^{k} \left| e_{iv} - \frac{\log \|x_i\|_v}{h(\mathbf{x})} \right| \leq \eta. \tag{3.6}$$

The following lemma compares the height $H(a_1x_1 + \cdots + a_kx_k)$ with $H(\mathbf{x})$, for $\mathbf{x} = (x_1, \dots, x_k) \in \Gamma$.

Lemma 3.4. *Let Γ be a finitely generated subgroup of $(K^*)^k$ of rank $r \geq 1$ and let $\mathbf{a} = (a_1, \dots, a_k) \in (K^*)^k$, $\varepsilon > 0$. There is a constant $C(\Gamma, \mathbf{a}, \varepsilon) > 0$, depending only on $\Gamma, \mathbf{a}, \varepsilon$, such that*

$$H(\mathbf{x}) \leq C(\Gamma, \mathbf{a}, \varepsilon) \cdot H(a_1x_1 + \cdots + a_kx_k)^{1+\varepsilon} \tag{3.7}$$

for all $\mathbf{x} \in \Gamma$ with $\sum_{i \in I} a_ix_i \neq 0$ for each non-empty subset I of $\{1, \dots, k\}$.

Remark. The constant $C(\Gamma, \mathbf{a}, \varepsilon)$ is not effectively computable from the method of proof. This lemma is used in the proofs of all theorems, except Theorem 2.2. Consequently, the implied constants in the asymptotic formulas in all theorems except Theorem 2.2 are not effectively computable.

Proof. We apply [8, Theorem 6.1.1] (which is equivalent to [6, Theorem 1]), which can be stated as follows. Let T be a finite set of places of K, containing all infinite places. Let $y_0, \dots, y_m$ be T-integers, i.e., $\|y_i\|_v \leq 1$ for $v \in M_K \setminus T$, $i = 0, \dots, m$ and suppose that $y_0 + \cdots + y_m = 0$

and that $\sum_{i\in I} y_i \neq 0$ for each proper, non-empty subset I of $\{0,\dots,m\}$. Let $\varepsilon > 0$. Then

$$\prod_{v\in T} \max(\|y_0\|_v,\dots,\|y_m\|_v) \ll \Big(\prod_{v\in T}\prod_{i=0}^{m} \|y_i\|_v\Big)^{1+\varepsilon}, \tag{3.8}$$

where the constant implied by the Vinogradov symbol $\ll$ depends on m, K, T and ε.

Now choose for T the set of places obtained by adding to S all infinite places of K and all places v of K such that $\|a_i\|_v \neq 1$ for at least one i. Then k, K, T are determined by Γ and $\mathbf{a}$. Choose a set $\mathcal{N}$ of cardinality 2^k, consisting of positive rational integers n such that $\|n\|_v = 1$ for all finite places $v \in T$. We can choose $\mathcal{N}$ with an upper bound depending only on k, K, T, hence only on Γ, $\mathbf{a}$. Let $\mathbf{x} \in \Gamma$ be such that $\sum_{i\in I} a_i x_i \neq 0$ for each non-empty $I \subseteq \{1,\dots,k\}$. There is $n_0 \in \mathcal{N}$ such that $n_0 + \sum_{i\in I} a_i x_i \neq 0$ for each non-empty $I \subseteq \{1,\dots,k\}$. Let $m = k+1$, $y_0 = n_0$, $y_i = a_i x_i$ for $i = 1,\dots,k$, $y_{k+1} = -n_0 - \sum_{i=1}^{k} y_i$. Then all conditions of the above stated [8, Theorem 6.1.1] are satisfied, so we can apply (3.8). Since $\max_i \|y_i\|_v \geq 1$ for all $v \in T$, and equal to 1 for $v \in M_K \setminus T$, the left-hand side of (3.8) is $\gg H(\mathbf{x})$, while the right-hand side of (3.8) is

$$\ll \Big(\prod_{v\in T} \|n_0 + a_1x_1 + \cdots + a_kx_k\|_v\Big)^{1+\varepsilon} \ll H(a_1x_1 + \cdots + a_kx_k)^{1+\varepsilon},$$

where the implied constants depend on $\Gamma, \mathbf{a}$, and ε. This implies Lemma 3.4. □

Lemma 3.5. *Let Γ be a finitely generated subgroup of $(K^*)^k$ of rank $r \geq 1$ and $\mathbf{a} = (a_1,\dots,a_k) \in (K^*)^k$. Then the equation*

$$a_1x_1 + \cdots + a_kx_k = 1 \quad \textit{in } \mathbf{x} \in \Gamma$$

has at most $c(k,r)$ solutions, with $\sum_{i\in I} a_i x_i \neq 0$ for each proper, non-empty subset of $\{1,\dots,k\}$, where $c(k,r)$ depends on k and r only.

This is the main result of [10], with $c(k,r)$ exponential in r and doubly exponential in k. Amoroso and Viada [2] improved this to $c(k,r) = (8k)^{4k^4(k+r+1)}$. □

We also need a quantitative result for linear recurrence sequences.

Lemma 3.6. *Let $a_1, \dots, a_k$ be non-zero polynomials in $K[X]$, and $\alpha_1, \dots, \alpha_k \in K^*$. Suppose that α_i/α_j is not a root of unity for each pair $i \neq j \in \{1, \dots, k\}$. Let $u_n := a_1(n)\alpha_1^n + \dots + a_k(n)\alpha_k^n$ for $n \in \mathbb{Z}_{\geq 0}$. Then the number of $n \in \mathbb{Z}_{\geq 0}$ such that $u_n = 0$ is at most $c'(\ell)$, the latter depending only on $\ell := k + \deg a_1 + \dots + \deg a_k$.*

This was proved by Schmidt [16] with $c'(\ell)$ triply exponential in ℓ. Amoroso and Viada [3] improved this to $c'(\ell) = \exp\exp(70\ell)$. □

We now prove some asymptotic formulas. Define the set

$$\mathcal{H}_\Gamma(X) := \{\mathbf{x} \in \Gamma : H(\mathbf{x}) \leq X\}.$$

Let $c(\Gamma)$ be the quantity defined by (2.9). In the results stated below, the implied constants depend on Γ (in fact, a priori they depend also on the choice of a basis $\{\mathbf{u}_1, \dots, \mathbf{u}_r\}$ of $\Gamma/\Gamma_{\text{tors}}$, but we can choose such a basis depending on Γ, e.g., we can choose a total ordering on the set of tuples $(\mathbf{u}_1, \dots, \mathbf{u}_k) \in (K^*)^k$, and choose the smallest basis of $\Gamma/\Gamma_{\text{tors}}$ in this total ordering).

Lemma 3.7. *Let Γ be a finitely generated subgroup of $(K^*)^k$ of rank $r \geq 1$. Then*

$$|\mathcal{H}_\Gamma(X)| = c(\Gamma)(\log X)^r + O((\log X)^{r-1}) \quad \text{as } X \to \infty, \tag{3.9}$$

where the implied constant depends on Γ.

Proof. We can represent elements of Γ in the form (2.5). Define the map

$$\varphi : \Gamma \to \mathbb{Z}^r : \zeta \mathbf{u}_1^{z_1} \cdots \mathbf{u}_r^{z_r} \mapsto \mathbf{z} = (z_1, \dots, z_r). \tag{3.10}$$

Notice that φ is a surjective homomorphism, with kernel Γ_{tors}. By (2.7) we have

$$\begin{aligned} \varphi(\mathcal{H}_\Gamma(X)) &= \Big\{\mathbf{z} \in \mathbb{Z}^r : \sum_{v \in S} \max\big(0, \ell_{v1}(\mathbf{z}), \dots, \ell_{vk}(\mathbf{z})\big) \leq \log X\Big\} \\ &= (\log X) \cdot \mathcal{C}(\Gamma) \cap \mathbb{Z}^r, \end{aligned}$$

where $\mathcal{C}(\Gamma)$ is the set defined by (2.8). To each $\mathbf{z} \in \mathbb{Z}^r$ we associate a cube of measure 1,

$$\mathcal{K}_{\mathbf{z}} := \{\xi := (\xi_1, \dots, \xi_r) \in \mathbb{R}^r : -\tfrac{1}{2} < \xi_i - z_i \leq \tfrac{1}{2} \text{ for } i = 1, \dots, r\}. \tag{3.11}$$

Notice that these cubes cover $\mathbb{R}^r$ and are pairwise disjoint. There is a constant $c > 0$, depending only on Γ, such that

$$(\log X - c)\mathcal{C}(\Gamma) \subseteq \bigcup_{\mathbf{z} \in (\log X)\mathcal{C}(\Gamma) \cap \mathbb{Z}^r} \mathcal{K}_{\mathbf{z}} \subseteq (\log X + c)\mathcal{C}(\Gamma).$$

By comparing measures, we see that

$$(\log X - c)^r \mu^r(\mathcal{C}(\Gamma)) \leq |(\log X)\mathcal{C}(\Gamma) \cap \mathbb{Z}^r| \leq (\log X + c)^r \mu^r(\mathcal{C}(\Gamma)).$$

Recalling that the map φ defined above is $|\Gamma_{\text{tors}}|$ to 1, we obtain

$$(\log X - c)^r c(\Gamma) \leq |\mathcal{H}_\Gamma(X)| \leq (\log X + c)^r c(\Gamma).$$

This implies Lemma 3.7. $\square$

Lemma 3.8. *Let Γ be a finitely generated subgroup of $(K^*)^k$ of rank $r \geq 1$, and assume that Γ satisfies* (2.10)*. Let T be a linear subspace of K^n. Then*

$$|\mathcal{H}_\Gamma(X) \cap T| \ll (\log X)^{r-1} \quad \text{as } X \to \infty,$$

where the implied constant is independent of T and depends only on Γ.

Proof. Assume that $\mathcal{H}_\Gamma(X) \cap T \neq \emptyset$. Choose a non-zero tuple $\mathbf{b} = (b_1, \ldots, b_k) \in K^k$ such that $b_1 x_1 + \cdots + b_k x_k = 0$ for $\mathbf{x} \in T$. Then at least two of the entries of $\mathbf{b}$ are non-zero.

We first show that there is a finite subset $\mathcal{T}$ of K^* (possibly depending on $\mathbf{b}$) of cardinality $\leq c'(k, r)$ depending only on k, r with the following property. For every $\mathbf{x} \in \Gamma$ with $b_1 x_1 + \cdots + b_k x_k = 0$ there are $i \neq j \in \{1, \ldots, k\}$ and $\lambda \in \mathcal{T}$ such that $b_i b_j \neq 0$ and $x_i / x_j = \lambda$. Indeed, let $\mathbf{x} \in \Gamma$ be such that $\sum_{i=1}^k b_i x_i = 0$. Take a minimal subset I of $\{1, \ldots, k\}$ such that $b_i \neq 0$ for $i \in I$ and $\sum_{i \in I} b_i x_i = 0$. Then I has cardinality at least 2. Pick $j \in I$. Then

$$\sum_{i \in I \setminus \{j\}} (-b_i / b_j)(x_i / x_j) = 1,$$

and no proper subsum of the left-hand side is 0. The existence of $\mathcal{T}$ as above follows by applying Lemma 3.5 to the latter identity.

Fix i, j, λ, and let $\mathcal{T}_{i,j,\lambda}$ be the set of $\mathbf{x} \in \mathcal{H}_\Gamma(X)$ such that $x_i / x_j = \lambda$. Further, for $i \neq j$, let $\Gamma_{i,j}$ be the group of $\mathbf{x} \in \Gamma$ with $x_i = x_j$. By (2.10), this group has rank smaller than r. Notice that λ may depend on T.

Assume that $\mathcal{T}_{i,j,\lambda}$ is non-empty. Pick $\mathbf{x}_0$ from this set, and for each $\mathbf{x} \in \mathcal{T}_{i,j,\lambda}$, write $\mathbf{x} = \mathbf{x}_0 \cdot \mathbf{y}$. Then $\mathbf{y} \in \Gamma_{i,j}$. Further, for $\mathbf{x} \in \mathcal{T}_{i,j,\lambda}$ we have, by (2.1), (2.3),

$$H(\mathbf{y}) \leq H(\mathbf{x}_0^{-1}\mathbf{x}) \leq X^{k+1}.$$

Lemma 3.7 implies that if $\mathbf{x}$ runs through $\mathcal{T}_{i,j,\lambda}$, then $\mathbf{y}$ runs through a set of cardinality

$$\ll (\log X)^{\operatorname{rank}\Gamma_{i,j}} \ll (\log X)^{r-1},$$

where the implied constant depends only on Γ, and is independent of λ. This is clearly also true if $\mathcal{T}_{i,j,\lambda}$ is empty. Hence, $|\mathcal{T}_{i,j,\lambda}| \ll (\log X)^{r-1}$. Since the number of triples (i,j,λ) is at most $k^2c'(k,r)$, we have

$$|\mathcal{H}_\Gamma(X) \cap T| \leq \sum_{i,j,\lambda} |\mathcal{T}_{i,j,\lambda}| \ll (\log X)^{r-1} \quad \text{as } X \to \infty,$$

where the implied constant depends only on Γ and is independent of T. □

Lemma 3.9. *Let Γ be a finitely generated subgroup of $(K^*)^k$ of rank $r \geq 1$ and let $C_2 > C_1 > 0$. Put $x_0 := 1$.*
Let $w \in S$ and $i \neq j \in \{0, \dots, k\}$ be such that there are $\mathbf{x} \in \Gamma$ with

$$\|x_i\|_w \neq \|x_j\|_w.$$

Then

(3.12)
$$\left|\left\{\mathbf{x} \in \mathcal{H}_\Gamma(X) :\ C_1 \leq \frac{\|x_i\|_w}{\|x_j\|_w} \leq C_2\right\}\right| \ll (\log X)^{r-1} \quad \text{as } X \to \infty,$$

where the implied constant depends on Γ, C_1 and C_2.

Remark. Notice that we allow here that one of i, j is 0.

Proof. Put $c_i := \log C_i$ for $i = 1, 2$. We use the notation from (2.7), (2.8) and in addition, put $\ell_{0w}(\xi) := 0$. Further, put $m_1(\xi) := \ell_{iw}(\xi) - \ell_{jw}(\xi)$. By our assumptions on Γ, i, j, w, the linear form m_1 is not identically 0. Let $\mathcal{H}$ denote the set given in (3.12). The map φ defined by (3.10) maps $\mathcal{H}$ to

$$\mathcal{A} \cap \mathbb{Z}^r, \quad \text{where } \mathcal{A} := \{\xi \in (\log X) \cdot \mathcal{C}(\Gamma),\ c_1 \leq m_1(\xi) \leq c_2\},$$

and is $|\Gamma_{\text{tors}}|$ to 1. We replace $\mathcal{A}$ by a larger set that is easier to handle. Choose linear forms $m_2, \dots, m_r$ from ℓ_{vh} $(v \in S,\ h = 1, \dots, k)$ such that $m_1, \dots, m_r$ are linearly independent. There is a constant $C > 0$, depending only on Γ, such that for $\xi \in \mathbb{R}^r$ we have

$$\sum_{v \in S} \max\big(0, \ell_{v1}(\xi), \dots, \ell_{vk}(\xi)\big) \geq C \cdot \max\big(|m_1(\xi)|, \dots, |m_r(\xi)|\big).$$

This can be proved by observing first that it suffices to show this for the set of $\xi \in \mathbb{R}^r$ for which the maximum on the right-hand side is 1, second, that this set is compact, and third, that the left-hand side is a continuous and positive real function, hence assumes a minimum $C > 0$ on this set. Thus, it follows that

$$\mathcal{A} \subseteq \mathcal{B} := \{\xi \in \mathbb{R}^r : c_1 \leq m_1(\xi) \leq c_2, \\ |m_i(\xi)| \leq C^{-1} \log X \text{ for } i = 2, \dots, k\}.$$

We estimate $|\mathcal{B} \cap \mathbb{Z}^r|$ from above. Consider again the cubes $\mathcal{K}_{\mathbf{z}}$ defined by (3.11). There is $c > 0$ depending only on Γ such that

$$\bigcup_{\mathbf{z} \in \mathcal{B} \cap \mathbb{Z}^r} \mathcal{K}_{\mathbf{z}} \subseteq \{\xi \in \mathbb{R}^r : c_1 - c \leq m_1(\xi) \leq c_2 + c, \\ |m_i(\xi)| \leq c + C^{-1} \log X \text{ for } i = 2, \dots, k\}.$$

It follows that

$$|\mathcal{B} \cap \mathbb{Z}^r| \ll (c_2 - c_1 + 2c)(c + C^{-1} \log X)^{r-1} \ll (\log X)^{r-1} \quad \text{as } X \to \infty.$$

Hence, $|\mathcal{H}| \leq |\Gamma_{\mathrm{tors}}| \cdot |\mathcal{A} \cap \mathbb{Z}^r| \ll (\log X)^{r-1}$ as $X \to \infty$. □

4. Proof of Theorem 2.1

We keep the notation introduced in Sections 2 and 3. In particular, K is a number field, $k \geq 2$, $\mathbf{a} = (a_1, \dots, a_k) \in (K^*)^k$, and Γ is a finitely generated subgroup of $(K^*)^k$ of rank $r \geq 1$. Let S be the finite set of places of K given by (2.6).

Let $\varepsilon = \varepsilon(X)$ be a positive, decreasing function of X, to be specified later. We first estimate from above $|\mathcal{V}'(X)|$, where

$$\mathcal{V}'(X) := \{\mathbf{x} \in \mathcal{V}_{\Gamma,\mathbf{a}}(X) : \\ \exists w \in S \text{ with } \|a_1 x_1 + \cdots + a_k x_k\|_w \leq \|\mathbf{x}\|_w \cdot H(\mathbf{x})^{-\varepsilon}\}.$$

Write $\mathcal{V}'(X) = \bigcup_{w \in S,\, i=1,\dots,k} \mathcal{V}_{i,w}(X)$, where

$$\mathcal{V}_{i,w}(X) := \Big\{\mathbf{x} \in \mathcal{V}_{\Gamma,\mathbf{a}}(X) : \|\mathbf{x}\|_w = \|x_i\|_w, \\ \|a_1 x_1 + \cdots + a_k x_k\|_w \leq \|x_i\|_w \cdot H(\mathbf{x})^{-\varepsilon}\Big\}.$$

Fix i, w. To estimate from above $|\mathcal{V}_{i,w}(X)|$, we apply Corollary 3.2. Define the system of linear forms $\mathcal{L} = \{L_{jv} : v \in S, j = 1, \dots, k\}$, given by

$$L_{jv} := X_j \quad (v \in S,\ j = 1, \dots, k,\ (v, j) \neq (w, i)),$$
$$L_{iw} := a_1 X_1 + \cdots + a_k X_k.$$

Let $\mathbf{x} \in \mathcal{V}_{i,w}(X)$. For $v \in S$, $j = 1, \dots, k$, define $e_{jv}(\mathbf{x}) := \log \|x_j\|_v / h(\mathbf{x})$. Let $\mathcal{E}$ be the set from Lemma 3.3, with $\eta = \varepsilon/(2k+2)$. Then

$$|\mathcal{E}| \ll \varepsilon^{-r}, \tag{4.1}$$

where here and below, constants implied by $\ll$, $\gg$ and the O-symbols depend on $\Gamma, \mathbf{a}$ only. Choose $\mathbf{e} \in \mathcal{E}$ with

$$\sum_{v \in S} \sum_{j=1}^{k} |e_{jv} - e_{jv}(\mathbf{x})| \leq \eta. \tag{4.2}$$

Next, define the tuple $\mathbf{d} = (d_{jv} : v \in S,\ j = 1, \dots, k)$ by

$$d_{jv} := e_{jv}\ (v \in S,\ j = 1, \dots, k,\ (v, j) \neq (w, i)), \quad d_{iw} := e_{iw} - \varepsilon.$$

Consider the set of $\mathbf{x} \in \mathcal{V}_{i,w}(X)$ satisfying (4.2) for some fixed $\mathbf{e} \in \mathcal{E}$. Take $Q := H(\mathbf{x})$. We first estimate from above $H_{\mathcal{L},\mathbf{d},Q}(\mathbf{x})$. One first verifies that for $v \in S$,

$$\begin{aligned}
\max_{1 \leq j \leq k} \|L_{jv}(\mathbf{x})\|_v Q^{-d_{jv}} &\leq \max_{1 \leq j \leq k} \|x_j\|_v Q^{-e_{jv}} \\
&\leq \Big(\max_{1 \leq j \leq k} \|x_j\|_w Q^{-e_{jv}(\mathbf{x})} \Big) \cdot Q^{\max_{j=1}^{k} |e_{jv} - e_{jv}(\mathbf{x})|} \\
&\leq Q^{\sum_{j=1}^{k} |e_{jv} - e_{jv}(\mathbf{x})|},
\end{aligned}$$

while clearly, $\|\mathbf{x}\|_v = 1$ for $v \in M_K \setminus S$. This implies

$$H_{\mathcal{L},\mathbf{d},Q}(\mathbf{x}) \leq Q^{\eta}. \tag{4.3}$$

Next, since by the product formula, $\sum_{v \in S} \sum_{j=1}^{k} e_{jv}(\mathbf{x}) = 0$,

$$\begin{aligned}
\frac{1}{k} \sum_{v \in S} \sum_{j=1}^{k} d_{jv} &= \frac{1}{k} \Big(\sum_{v \in S} \sum_{j=1}^{k} e_{jv} - \varepsilon \Big) \\
&\leq \frac{1}{k} \Big(\sum_{v \in S} \sum_{j=1}^{k} |e_{jv} - e_{jv}(\mathbf{x})| - \varepsilon \Big) \leq \frac{1}{k} (\eta - \varepsilon).
\end{aligned} \tag{4.4}$$

Lastly,

$$\sum_{v\in S}\max(d_{1v},\dots,d_{kv}) \le \sum_{v\in S}\max(e_{1v},\dots,e_{kv}) \le \sum_{v\in S}\max(e_{1v}(\mathbf{x}),\dots,e_{kv}(\mathbf{x})) + \sum_{v\in S}\sum_{j=1}^{k}|e_{jv}-e_{jv}(\mathbf{x})| \le 1+\eta. \tag{4.5}$$

We apply Corollary 3.2 with $\lambda = (\eta-\varepsilon)/k$, $\mu = -\eta$, $\theta = 1+\eta$, $Q = H(\mathbf{x})$. With our choice $\eta = \varepsilon/(2k+2)$, we have

$$\mu-\lambda = \frac{\varepsilon}{2k},\quad \theta-\lambda \ge 1+\varepsilon,\quad \delta = \frac{\mu-\lambda}{\theta-\lambda} \ge \frac{\varepsilon}{2k(1+\varepsilon)}.$$

It follows that for each given $\mathbf{e}\in\mathcal{E}$, the set of $\mathbf{x}$ such that

$$\mathbf{x}\in\mathcal{V}_{i,w}(X),\ \mathbf{x}\text{ satisfies (4.2)},\ H(\mathbf{x}) \ge C_2^{2k/\varepsilon}$$

is contained in a union of $\ll \varepsilon^{-3}(\log\varepsilon^{-1})^2$ proper linear subspaces of K^n. Taking the union over all $\mathbf{e}\in\mathcal{E}$, using (4.1), it follows that the set of $\mathbf{x}$ with

$$\mathbf{x}\in\mathcal{V}_{i,w}(X),\quad H(\mathbf{x}) \ge C_2^{2k/\varepsilon}$$

is contained in a union of

$$\ll (\varepsilon^{-1})^{r+3}(\log\varepsilon^{-1})^2 \tag{4.6}$$

proper linear subspaces of K^k.

Let T be one of these subspaces. By Lemma 3.4, each $\mathbf{x}\in\mathcal{V}_{i,w}(X)$ has $H(\mathbf{x}) \ll X^2$. Now Lemma 3.8 implies that

$$|\mathcal{V}_{i,w}(X)\cap T| \ll (\log X)^{r-1}, \tag{4.7}$$

where the implied constant is independent of T. Lastly, by Lemma 3.7,

$$|\{\mathbf{x}\in\Gamma:\ H(\mathbf{x}) \le C_2^{2k/\varepsilon}\}| \ll \varepsilon^{-r}. \tag{4.8}$$

Combining (4.6), (4.7), (4.8), we infer

$$|\mathcal{V}_{i,w}(X)| \ll (\log X)^{r-1}\cdot(\varepsilon^{-1})^{r+3}(\log\varepsilon^{-1})^2.$$

This holds for all $i = 1,\dots,k$, $w\in S$. It follows that $\mathcal{V}'(X)$ has cardinality

$$|\mathcal{V}'(X)| \ll (\log X)^{r-1}\cdot(\varepsilon^{-1})^{r+3}(\log\varepsilon^{-1})^2. \tag{4.9}$$

We now consider $\mathcal{V}_{\Gamma,\mathbf{a}}(X)\setminus\mathcal{V}'(X)$. Observe that for every $\mathbf{x}$ in this set,

$$\|a_1x_1+\cdots+a_kx_k\|_v \ge \|\mathbf{x}\|_v H(\mathbf{x})^{-\varepsilon}\quad\text{for all } v\in S,$$

while

$$\max(1, \|a_1x_1 + \cdots + a_kx_k\|_v) \geq 1 = \max(1, \|\mathbf{x}\|_v) \quad \text{for } v \in M_K \setminus S.$$

Hence,

$$H(a_1x_1 + \cdots + a_kx_k) \gg H(\mathbf{x})^{1-s\varepsilon} \quad \text{for } \mathbf{x} \in \mathcal{V}_{\Gamma,\mathbf{a}}(X) \setminus \mathcal{V}'(X), \tag{4.10}$$

where $s = |S|$. On the other hand,

$$H(a_1x_1 + \cdots + a_kx_k) \ll H(\mathbf{x}) \quad \text{for } \mathbf{x} \in \Gamma. \tag{4.11}$$

Define

$$\mathcal{H}'_\Gamma(X) := \Big\{\mathbf{x} \in \Gamma : \, H(\mathbf{x}) \leq X, \\ \sum_{i \in I} a_ix_i \neq 0 \text{ for each non-empty } I \subseteq \{1, \ldots, k\}\Big\}. \tag{4.12}$$

Then by Lemmas 3.7 and 3.8,

$$|\mathcal{H}'_\Gamma(X)| = c(\Gamma)(\log X)^r + O((\log X)^{r-1}) \quad \text{as } X \to \infty. \tag{4.13}$$

Recall that solutions with vanishing subsums have been excluded from $\mathcal{V}_{\Gamma,\mathbf{a}}(X)$. From (4.10), (4.11) it follows that there are positive constants C_1, C_2, depending only on $\Gamma, \mathbf{a}$, such that

$$\mathcal{H}'_\Gamma(C_1X) \subseteq \mathcal{V}_{\Gamma,\mathbf{a}}(X) \subseteq \mathcal{H}'_\Gamma(C_2X^{1/(1-s\varepsilon)}) \cup \mathcal{V}'(X),$$

and using (4.13), (4.9) , we obtain

(4.14)

$$c(\Gamma)(\log X)^r + O\big((\log X)^{r-1}\big) \leq |\mathcal{V}_{\Gamma,\mathbf{a}}(X)| \\ \leq c(\Gamma)(\log X)^r + O\Big(\varepsilon \cdot (\log X)^r + (\log X)^{r-1} \cdot (\varepsilon^{-1})^{r+3}(\log \varepsilon^{-1})^2\Big).$$

We now choose $\varepsilon = \varepsilon(X) = \big((\log\log X)^2/\log X\big)^{1/(r+4)}$. Then (4.9) and (4.14) together give the formula stated in Theorem 2.1,

$$|\mathcal{V}_{\Gamma,\mathbf{a}}(X)| = c(\Gamma)(\log X)^r \left(1 + O\Big(\big(\frac{(\log\log X)^2}{\log X}\big)^{1/(r+4)}\Big)\right) \quad \text{as } X \to \infty.$$

□

5. Proof of Theorem 2.2

Let as before K be a number field, $k \geq 2$ an integer, $a_1, \ldots, a_k \in K[X]$ non-zero polynomials, and $\alpha_1, \ldots, \alpha_k \in K^*$, satisfying (2.11), i.e., α_i/α_j is not a root of unity for each pair $i \neq j \in \{1, \ldots, k\}$. Further, let

$$u_n := \sum_{i=1}^{k} a_i(n)\alpha_i^n.$$

Our purpose is to find an asymptotic formula for $|\mathcal{U}(X)|$, where

$$\mathcal{U}(X) := \{n \in \mathbb{Z}_{\geq 0} :\ H(u_n) \leq X\}.$$

In this section, constants implied by $\ll$, $\gg$ and O-symbols will depend only on $a_1, \ldots, a_k$, $\alpha_1, \ldots, \alpha_k$, and are all effectively computable. Let S be a finite set of places of K, containing all infinite places, such that the coefficients of $a_1, \ldots, a_k$ are all S-integers, while $\alpha_1, \ldots, \alpha_k$ are all S-units. Let $\varepsilon = \varepsilon(X)$ be a positive, decreasing function of X, to be specified later. Put

$$H := H(\alpha_1, \ldots, \alpha_k) = \prod_{v \in M_K} \max(1, \|\alpha_1\|_v, \ldots, \|\alpha_k\|_v).$$

We first estimate from above $|\mathcal{U}'(X)|$, where

$$\begin{aligned}\mathcal{U}'(X) := \Big\{ & n \in \mathbb{Z}_{\geq 0} :\ H(u_n) \leq X, \\ & \exists w \in S \text{ with } \|u_n\|_w \leq \max_{1 \leq i \leq k} \|a_i(n)\alpha_i^n\|_w \cdot H^{-n\varepsilon} \Big\}.\end{aligned}$$

Write $\mathcal{U}'(X) = \bigcup_{w \in S,\, i=1,\ldots,k} \mathcal{U}_{i,w}(X)$, where

$$\begin{aligned}\mathcal{U}_{i,w}(X) := \Big\{ & n \in \mathbb{Z}_{\geq 0} :\ H(u_n) \leq X,\ \|a_i(n)\alpha_i^n\|_w = \max_{1 \leq j \leq k} \|a_j(n)\alpha_j^n\|_w, \\ & \|u_n\|_w \leq \|a_i(n)\alpha_i^n\|_w \cdot H^{-n\varepsilon} \Big\}.\end{aligned}$$

Fix i, w. To estimate from above $|\mathcal{U}_{i,w}(X)|$, we apply Corollary 3.2, with

$$\mathbf{x} = (x_1, \ldots, x_k) = (a_1(n)\alpha_1^n, \ldots, a_k(n)\alpha_k^n), \quad Q = H^n. \tag{5.1}$$

Define the system of linear forms $\mathcal{L} = \{L_{jv} :\ v \in S,\ j = 1, \ldots, k\}$ by

$$\begin{aligned} & L_{jv} := X_j \quad (v \in S,\ j = 1, \ldots, k,\ (v, j) \neq (w, i)), \\ & L_{iw} := X_1 + \cdots + X_k. \end{aligned}$$

Further, define

$$e_{jv} := \frac{\log \|\alpha_j\|_v}{\log H} \ (v \in S,\ j = 1, \ldots, k),$$
$$d_{jv} := e_{jv} \ (v \in S,\ j = 1, \ldots, k,\ (v,j) \neq (w,i)),\ d_{iw} := e_{iw} - \varepsilon,$$

and let $\mathbf{d} := (d_{jv} : v \in S,\ j = 1, \ldots, k)$.

We first estimate from above $H_{\mathcal{L},\mathbf{d},Q}(\mathbf{x})$, with $n \in \mathcal{U}_{i,w}(X)$ and $\mathbf{x}$, Q as in (5.1). For $v \in S$ we have

$$\max_{1\le j\le k} \|L_{jv}(\mathbf{x})\|_v Q^{-d_{iv}} \le \max_{1\le j\le k} \|x_j\|_w Q^{-e_{iv}} \le \max_{1\le j\le k} \|a_j(n)\|_v,$$

while $\|\mathbf{x}\|_v \le 1$ for $v \in M_K \setminus S$. Then

$$H_{\mathcal{L},\mathbf{d},Q}(\mathbf{x}) \le \prod_{v\in S} \max_{1\le j\le k} \|a_j(n)\|_v \le Q^{\varepsilon/2k},$$

provided that

$$n \gg \varepsilon^{-2}. \tag{5.2}$$

Indeed, $\prod_{v\in S} \max_{1\le j\le k} \|a_j(n)\|_v \ll n^{O(1)}$, and if n satisfies (5.2) with a sufficiently large implied constant, this is smaller than $Q^{\varepsilon/2k} = H^{n\varepsilon/2k}$.

Next, by the product formula we have $\sum_{v\in S}\sum_{j=1}^k e_{jv} = 0$, which implies

$$\frac{1}{k}\sum_{v\in S}\sum_{j=1}^{k} d_{jv} = -\varepsilon/k,$$

while clearly,

$$\sum_{v\in S} \max(d_{1v}, \ldots, d_{kv}) \le \sum_{v\in S} \max(e_{1v}, \ldots, e_{kv}) \le 1.$$

We apply Corollary 3.2 with $\lambda = -\varepsilon/k$, $\mu = -\varepsilon/2k$, $\theta = 1$. By taking the implied constant in (5.2) sufficiently large, we can guarantee that $Q = H^n$ satisfies (3.5). It follows that the set of vectors $\mathbf{x} = (a_1(n)\alpha_1^n, \ldots, a_k(n)\alpha_k^n)$ with $n \in \mathcal{U}_{i,w}(X)$ and with (5.2) lies in a union of

$$\ll \varepsilon^{-3}(\log \varepsilon^{-1})^2$$

proper linear subspaces of K^n. Consider such a subspace T, and take a non-trivial equation $b_1X_1 + \cdots + b_kX_k = 0$ of T. Then if $\mathbf{x} \in T$, we have $\sum_{j=1}^k b_j a_j(n)\alpha_j^n = 0$. By Lemma 3.6, the number of $n \in \mathbb{Z}_{\ge 0}$

satisfying such a relation is $\ll 1$. Further, the number of n violating (5.2) is $\ll \varepsilon^{-2}$. It follows that altogether,

$$|\mathcal{U}_{i,w}(X)| \ll \varepsilon^{-3}(\log \varepsilon^{-1})^2.$$

By taking the union over all w, i, we obtain

$$|\mathcal{U}'(X)| \ll \varepsilon^{-3}(\log \varepsilon^{-1})^2. \tag{5.3}$$

We now consider $\mathcal{U}(X) \setminus \mathcal{U}'(X)$. Observe that for every integer n in this set,

$$\|u_n\|_v \geq \max_{1\leq i\leq k} \|a_i(n)\alpha_i^n\|_v H^{-n\varepsilon} \quad \text{for } v \in S,$$

hence

$$\max(1, \|u_n\|_v) \geq R_v(n) \max(1, \|\alpha_1\|_v, \dots, \|\alpha_k\|_v)^n H^{-n\varepsilon} \quad \text{for } v \in S,$$

where $R_v(n) := \min(1, \|a_1(n)\|_v, \dots, \|a_k(n)\|_v)$ for $v \in S$. Further, the set S is such that all α_i are S-units. Hence,

$$H(u_n) \geq \Big(\prod_{v\in S} R_v(n)\Big) H^{n(1-s\varepsilon)} \quad \text{for } n \in \mathcal{U}(X) \setminus \mathcal{U}'(X),$$

where $s := |S|$. Let $M_v(n) := \max(1, \|a_1(n)\|_v, \dots, \|a_k(n)\|_v)$ for $v \in M_K$. Assuming $a_1(n)\cdots a_k(n) \neq 0$, we have by the product formula,

$$\begin{aligned}
\prod_{v\in S} R_v(n) \quad &\geq \prod_{v\in S} \frac{\|a_1(n)\cdots a_k(n)\|_v}{M_v(n)^{k+1}} \\
&\geq \prod_{v\in M_K\setminus S} \|a_1(n)\cdots a_k(n)\|_v^{-1} \cdot \prod_{v\in S} M_v(n)^{-k-1} \\
&\geq \prod_{v\in M_K} M_v(n)^{-k-1} \gg n^{O(1)}.
\end{aligned}$$

It follows that

$$H(u_n) \gg n^{O(1)} H^{n(1-s\varepsilon)} \quad \text{for } n \in \mathcal{U}(X) \setminus \mathcal{U}'(X),$$

where we have incorporated those n with $a_1(n)\cdots a_k(n) = 0$ by deminishing the implied constants. Hence,

$$H^n \ll \big(H(u_n)(\log(2 + H(u_n)))^{O(1)}\big)^{1/(1-s\varepsilon)} \quad \text{for } n \in \mathcal{U}(X) \setminus \mathcal{U}'(X). \tag{5.4}$$

On the other hand, for $n \in \mathcal{U}(X)$ we have

$$(5.5)\qquad H(u_n) \ll \prod_{v\in M_K} \max(1, \|a_1(n)\alpha_1^n\|_v, \dots, \|a_k(n)\alpha_k^n\|_v) \ll H(a_1(n),\dots,a_k(n))H^n \ll n^{O(1)}H^n.$$

From (5.5), (5.4), we infer that there are effectively computable positive constants $C_1, C_2, \gamma_1, \gamma_2$, depending only on $a_1,\dots,a_k$, $\alpha_1,\dots,\alpha_k$, such that

$$\{n \in \mathbb{Z}_{\geq 0}:\ H^n \leq C_1 X(\log X)^{-\gamma_1}\} \subseteq \mathcal{U}(X) \subseteq \big\{n \in \mathbb{Z}_{\geq 0}:\ H^n \leq C_2\big(X(\log X)^{\gamma_2}\big)^{1/(1-s\varepsilon)}\big\} \cup \mathcal{U}'(X).$$

By comparing cardinalities, we obtain

$$\frac{\log X}{\log H} + O(\log\log X) \leq |\mathcal{U}(X)| \leq \frac{1}{(1-s\varepsilon)\log H}\big(\log X + O(\log\log X)\big) + O(\varepsilon^{-3}(\log\varepsilon^{-1})^2) \leq \frac{\log X}{\log H} + O\big(\varepsilon\log X + \log\log X + \varepsilon^{-3}(\log\varepsilon^{-1})^2\big).$$

By choosing $\varepsilon = \big((\log\log X)^2/\log X\big)^{1/4}$, we arrive at

$$|\mathcal{U}(X)| = \frac{\log X}{\log H}\left(1 + O\Big(\Big(\frac{(\log\log X)^2}{\log X}\Big)^{1/4}\Big)\right).$$

This completes our proof. □

6. Proofs of Theorems 2.3 and 2.4

As before, K is an algebraic number field and $k \geq 2$ an integer.

Proof of Theorem 2.3. Let S be given by (2.6) and assume that Γ satisfies (2.12). Here and below, the constants implied by $\ll$ and $\gg$ and the O-symbols depend on Γ and $\mathbf{a}$. Put $x_0 := 1$ and take a constant $C_1 > 0$ which is chosen sufficiently large, but depending only on Γ and $\mathbf{a}$. We first estimate from above $|\mathcal{V}'(X)|$, where

$$\mathcal{V}'(X) := \bigcup_{w\in S,\, i\neq j\in\{0,\dots,k\}} \mathcal{V}_{i,j,w}(X),$$

with

$$\mathcal{V}_{i,j,w}(X) := \{\mathbf{x} \in \mathcal{V}_{\Gamma,\mathbf{a}}(X):\ \|x_j\|_w \leq \|x_i\|_w \leq C_1\|x_j\|_w\}$$

(note that we allow i,j to be 0). We estimate from above $|\mathcal{V}_{i,j,w}(X)|$ for all i,j,w with $i\neq j$, and subsequently, $|\mathcal{V}'(X)|$.

Fix i,j,w. Lemma 3.4 implies that for $\mathbf{x}\in\mathcal{V}_{i,j,w}(X)$ we have

$$H(\mathbf{x})\ll H(a_1x_1+\cdots+a_kx_k)^2.$$

Therefore, there is a constant $C_2>0$ depending only on Γ and $\mathbf{a}$, such that

$$\mathcal{V}_{i,j,w}(X)\subseteq\{\mathbf{x}\in\mathcal{H}_\Gamma(C_2X^2):\ \|x_j\|_w\leq\|x_i\|_w\leq C_1\|x_j\|_w\}.$$

By assumption (2.12), there is $\mathbf{x}\in\Gamma$ such that $\|x_i\|_w\neq\|x_j\|_w$. So we can apply Lemma 3.9 to the right-hand side, and obtain

$$|\mathcal{V}_{i,j,w}(X)|\ll(\log X)^{r-1}\ \text{ as } X\to\infty.$$

This holds for all $i\neq j\in\{0,\dots,k\}$ and $w\in S$. Thus, it follows that

$$|\mathcal{V}'(X)|\ll(\log X)^{r-1}\ \text{ as } X\to\infty. \tag{6.1}$$

We now consider $\mathcal{V}_{\Gamma,\mathbf{a}}(X)\setminus\mathcal{V}'(X)$. Notice that for each $\mathbf{x}$ in this set, and for each $v\in S$, there is $i(v)\in\{0,\dots,k\}$ such that

$$\max(1,\|\mathbf{x}\|_v)=\|x_{i(v)}\|_v\geq C_1\max_{j\in\{0,\dots,k\}\setminus\{i(v)\}}\|x_j\|_v.$$

Taking C_1 sufficiently large, we have for $v\in S$ that

$$\max(1,\|a_1x_1+\cdots+a_kx_k\|_v)\gg\max(1,\|\mathbf{x}\|_v)$$

if $i(v)=0$, and

$$\begin{aligned}\max(1,\|a_1x_1+\cdots+a_kx_k\|_v)&\gg\|a_{i(v)}x_{i(v)}\|_v-\sum_{j\neq i(v)}\|a_jx_j\|_v\\&\gg\|x_{i(v)}\|_v=\max(1,\|\mathbf{x}\|_v)\end{aligned}$$

if $i(v)\neq0$. Further, it is clear that for $v\in M_K\setminus S$,

$$\max(1,\|a_1x_1+\cdots+a_kx_k\|_v)\geq1=\max(1,\|\mathbf{x}\|_v).$$

Therefore,

$$H(a_1x_1+\cdots+a_kx_k)\gg H(\mathbf{x})\ \text{ for } \mathbf{x}\in\mathcal{V}_{\Gamma,\mathbf{a}}(X)\setminus\mathcal{V}'(X).$$

Clearly, we have also

$$H(a_1x_1+\cdots+a_kx_k)\ll H(\mathbf{x})\ \text{ for } \mathbf{x}\in\Gamma.$$

Thus, there are constants $C_3, C_4 > 0$, depending only on Γ, $\mathbf{a}$, such that

$$\mathcal{H}'_\Gamma(C_3X) \subseteq \mathcal{V}_{\Gamma,\mathbf{a}}(X) \subseteq \mathcal{H}'_\Gamma(C_4X) \cup \mathcal{V}'(X).$$

Finally, using (4.13) and (6.1), we arrive at the formula stated in Theorem 2.4,

$$|\mathcal{V}_{\Gamma_1^k,\mathbf{a}}(X)| = c(\Gamma)(\log X)^r + O((\log X)^{r-1}) \text{ as } X \to \infty.$$

□

Proof of Theorem 2.4. Let $\Gamma = \Gamma_1^k$, where Γ_1 has rank $r_1 \geq 1$. So Γ has rank kr_1. Condition (2.6) now becomes

$$S = \{v \in M_K : \text{there is } x \in \Gamma_1 \text{ with } \|x\|_v \neq 1\}.$$

Given $v \in S$, $i \in \{1, \dots, k\}$ we can take $\mathbf{x} = (x_1, \dots, x_k) \in \Gamma_1^k$ with $\|x_i\|_v \neq 1$ and $x_j = 1$ for all $j \neq i$. Thus, we see that $\Gamma = \Gamma_1^k$ satisfies (2.12). Now Theorem 2.4 follows at once from Theorem 2.3 with $r = kr_1$. □

7. Proof of Theorem 2.5

We closely follow the proof of [11, Theorem 2]. We take Theorem 2.4 as a starting point. As before, K is a number field and $k \geq 2$ an integer.

Let $\mathcal{A}$ be a finite set of tuples from $(K^*)^k$. Let $\mathcal{V}_\mathcal{A}(X)$ be the collection of tuples $(\alpha_1, \dots, \alpha_k)$ such that

$(\alpha_1, \dots, \alpha_k) = (a_1x_1, \dots, a_kx_k)$ with $(a_1, \dots, a_k) \in \mathcal{A},\ x_1, \dots, x_k \in \Gamma_1$,
$H(\alpha_1 + \cdots + \alpha_k) \leq X$,
none of the subsums of $\alpha_1 + \cdots + \alpha_k$ vanishes.

In what follows, constants implied by O-symbols and $\ll$-symbols will depend only on $k, \Gamma_1, \mathcal{A}$.

Lemma 7.1. *(i) We have for $X \to \infty$,*

$$|\mathcal{V}_\mathcal{A}(X)| \ll (\log X)^{kr_1}.$$

(ii) If moreover $\mathcal{A}$ satisfies (2.14)*, then for $X \to \infty$,*

$$|\mathcal{V}_\mathcal{A}(X)| = |\mathcal{A}| \cdot c(\Gamma_1^k) \cdot (\log X)^{kr_1} + O((\log X)^{kr_1-1}).$$

Proof. This follows easily from Theorem 2.4. Notice that, trivially, $|\mathcal{V}_{\mathcal{A}}(X)| \le \sum_{\mathbf{a}} |\mathcal{V}_{\mathbf{a}}(X)|$ where the sum is taken over all $\mathbf{a} \in \mathcal{A}$. If $\mathcal{A}$ satisfies (2.14) then we have equality since $\mathcal{V}_{\mathbf{a}}(X)$, $\mathcal{V}_{\mathbf{b}}(X)$ are disjoint for any two distinct $\mathbf{a}, \mathbf{b} \in \mathcal{A}$. □

We assume henceforth that $\mathcal{A}$ satisfies (2.13), (2.14), (2.15). Note that thanks to (2.15), the set $\mathcal{V}_{\mathcal{A}}(X)$ is such that if $(\alpha_1, \ldots, \alpha_k) \in \mathcal{V}_{\mathcal{A}}(X)$, then also $(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(k)}) \in \mathcal{V}_{\mathcal{A}}(X)$ for each $\sigma \in S_k$. Given $(\alpha_1, \ldots, \alpha_k) \in \mathcal{V}_{\mathcal{A}}(X)$, let

$$(\alpha_1, \ldots, \alpha_k)_P := \{(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(k)}) : \sigma \in S_k\}.$$

Lemma 7.2. *Let $\mathcal{V}^*_{\mathcal{A}}(X)$ be the set of $(\alpha_1, \ldots, \alpha_k) \in \mathcal{V}_{\mathcal{A}}(X)$ such that $(\alpha_1, \ldots, \alpha_k)_P$ has precisely $k!$ elements. Then*

$$|\mathcal{V}_{\mathcal{A}}(X) \setminus \mathcal{V}^*_{\mathcal{A}}(X)| \ll (\log X)^{(k-1)r_1} \quad \textit{as } X \to \infty.$$

Proof. Let $(\alpha_1, \ldots, \alpha_k) = (a_1x_1, \ldots, a_kx_k) \in \mathcal{V}_{\mathcal{A}}(X) \setminus \mathcal{V}^*_{\mathcal{A}}(X)$. Then there are distinct $\sigma, \tau \in S_k$ such that $(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(k)}) = (\alpha_{\tau(1)}, \ldots, \alpha_{\tau(k)})$, which implies that there are distinct indices i, j such that $\alpha_i = \alpha_j$. By (2.13) we know that $a_i = a_j$, and thus, $x_i = x_j$. So if we replace α_i, α_j by the single entry $2\alpha_i$, we get a tuple from $\mathcal{V}_{\mathcal{A}'}(X)$, where $\mathcal{A}'$ consists of all $(k-1)$-tuples obtained by taking those tuples from $\mathcal{A}$ having two equal entries, say $a_i = a_j$ for some distinct indices i, j, and replacing a_i, a_j by the single entry $2a_i$. Now use that by the first part of Lemma 7.1 we have $|\mathcal{V}_{\mathcal{A}'}(X)| \ll (\log X)^{(k-1)r_1}$. □

Lemma 7.3. *Let $\mathcal{V}^{**}_{\mathcal{A}}(X)$ be the set of tuples $(\alpha_1, \ldots, \alpha_k) \in \mathcal{V}_{\mathcal{A}}(X)$ such that there is $(\beta_1, \ldots, \beta_k) \in \mathcal{V}_{\mathcal{A}}(X) \setminus (\alpha_1, \ldots, \alpha_k)_P$ with $\sum_{i=1}^k \beta_i = \sum_{i=1}^k \alpha_i$. Then*

$$|\mathcal{V}^{**}_{\mathcal{A}}(X)| \ll (\log X)^{(k-1)r_1} \quad \textit{as } X \to \infty.$$

Proof. Let $(\alpha_1, \ldots, \alpha_k) \in \mathcal{V}^{**}_{\mathcal{A}}(X)$, and choose $(\beta_1, \ldots, \beta_k)$ from the set $\mathcal{V}_{\mathcal{A}}(X) \setminus (\alpha_1, \ldots, \alpha_k)_P$ with $\sum_{i=1}^k \beta_i - \sum_{i=1}^k \alpha_i = 0$. By removing vanishing subsums, we obtain that there are non-empty subsets I, J of $\{1, \ldots, k\}$ such that

$$\sum_{i \in I} \alpha_i - \sum_{j \in J} \beta_j = 0, \tag{7.1}$$

and no proper subsum of this expression vanishes. Here, both I, J are non-empty because of our definition of $\mathcal{V}_{\mathcal{A}}(X)$, and we may choose I

with $|I| \geq 2$, since otherwise $(\beta_1, \ldots, \beta_k) \in (\alpha_1, \ldots, \alpha_k)_P$, which we excluded. Write $\alpha_i = a_i x_i$ for $i = 1, \ldots, k$, where $(a_1, \ldots, a_k) \in \mathcal{A}$ and $x_i \in \Gamma_1$ for $i = 1, \ldots, k$. Pick $h \in I$. Dividing (7.1) by α_h, we get an equation

$$1 + \sum_{i \in I \setminus \{h\}} \frac{a_i}{a_h} \cdot \frac{x_i}{x_h} - \sum_{j \in J} \cdots = 0,$$

of which no proper subsum vanishes. By e.g., Lemma 3.5, there is a finite subset $\mathcal{B}$ of Γ_1, depending only on $k, \Gamma_1, \mathcal{A}$, such that $x_i/x_h \in \mathcal{B}$ for $i \in I$. So altogether, for each $(a_1x_1, \ldots, a_kx_k) \in \mathcal{V}^{**}_{\mathcal{A}}(X)$, there are $\lambda \in \mathcal{B}$ and distinct indices h, i in $\{1, \ldots, k\}$, such that $x_i/x_h = \lambda$.

Now let $\mathcal{A}''$ be the union of the sets $\mathcal{A}_{h,i,\lambda}$ ($h \neq i \in \{1, \ldots, k\}, \lambda \in \mathcal{B}$), where $\mathcal{A}_{h,i,\lambda}$ is the set of $(k-1)$-tuples obtained by taking the tuples $(a_1, \ldots, a_k)$ from $\mathcal{A}$, and for each of these tuples, replacing the entries a_h, a_i by one single entry $a_h - \lambda a_i$. Then by the first part of Lemma 7.1, we obtain

$$|\mathcal{V}^{**}_{\mathcal{A}}(X)| \leq |\mathcal{V}_{\mathcal{A}''}(X)| \ll (\log X)^{(k-1)r_1}.$$

□

Proof of Theorem 2.5. Let $\mathcal{T}^*_{\mathcal{A}}(X)$ be the set of α satisfying (2.16) but with the additional property that no subsum of $\sum_{i=1}^k a_i x_i$ vanishes. This is the set of α such that there is $(\alpha_1, \ldots, \alpha_k) \in \mathcal{V}_{\mathcal{A}}(X)$ with $\alpha = \alpha_1 + \cdots + \alpha_k$. Lemmas 7.2 and 7.3 imply that with $\ll (\log X)^{(k-1)r_1}$ exceptions, every $\alpha \in \mathcal{T}^*_{\mathcal{A}}(X)$ has precisely $k!$ different representations in the form $\alpha_1 + \cdots + \alpha_k$ with $(\alpha_1, \ldots, \alpha_k) \in \mathcal{V}_{\mathcal{A}}(X)$, all lying in a set $(\alpha_1, \ldots, \alpha_k)_P$. Thus, using the second part of Lemma 7.1,

$$\begin{aligned} (7.2) \quad |\mathcal{T}^*_{\mathcal{A}}(X)| &= (k!)^{-1}|\mathcal{V}_{\mathcal{A}}(X)| + O((\log X)^{(k-1)r_1}) \\ &= \frac{|\mathcal{A}|}{k!} c(\Gamma_1^k)(\log X)^{kr_1} + O((\log X)^{kr_1 - 1}) \quad \text{as } X \to \infty. \end{aligned}$$

It remains to dispose of the non-vanishing subsum condition. Let $\mathcal{T}^{**}_{\mathcal{A}}(X)$ be the set of α satisfying (2.16) but now with the additional condition that every representation $\alpha = \sum_{i=1}^k a_i x_i$ with $(a_1, \ldots, a_k) \in \mathcal{A}$, $x_i \in \Gamma_1$ for $i = 1, \ldots, k$ has a vanishing subsum. Pick $\alpha \in \mathcal{T}^{**}_{\mathcal{A}}$ with $\alpha \neq 0$. Then α is in fact representable as $\sum_{i \in I} a_i x_i$, where $2 \leq |I| < k$ and where no subsum of $\sum_{i \in I} a_i x_i$ vanishes. Letting $\mathcal{A}_I$ consist of all tuples $(a_i : i \in I)$ with $(a_1, \ldots, a_k) \in \mathcal{A}$, and including the case $\alpha = 0$, we deduce from part 1 of Lemma 7.1 that

$$|\mathcal{T}_{\mathcal{A}}^{**}(X)| \leq 1 + \sum_{I \subsetneq \{1,\dots,r\}} |\mathcal{V}_{\mathcal{A}_I}(X)| \ll (\log X)^{(k-1)r_1} \text{ as } X \to \infty.$$

By combining this with (7.2), the proof of Theorem 2.5 is completed. □

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