SYMMETRIC IMPROVEMENTS OF LIOUVILLE'S INEQUALITY

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ABSOLUTE VALUES AND HEIGHTS

$M_{\mathbb{Q}} = \{\infty\} \cup \{\text{primes}\}$
$= \text{set of places of } \mathbb{Q}$

$|\cdot|_\infty$ = standard absolute value
$|\cdot|_p$ = p-adic absolute value

M_K = set of places of number field K
Define $|\cdot|_v$ $(v \in M_K)$ by

$$|x|_v = |x|_p^{[K_v:\mathbb{Q}_p]} \quad \text{if } v|p, \; p \in M_{\mathbb{Q}}$$

Product formula: $\prod_{v \in M_K} |x|_v = 1$ for $x \in K^*$

Relative height: $H_K(\alpha) := \prod_{v \in M_K} \max(1, |\alpha|_v)$ for $\alpha \in K$.

$H_L(\alpha) = H_K(\alpha)^{[L:K]}$ for $L \supset K$ finite.

LIOUVILLE'S INEQUALITY

$$\begin{tikzcd} & K=K_1K_2 \arrow[dl, dash, "s"'] \arrow[dr, dash, "r"] & \\ K_1 \arrow[dr, dash] & & K_2 \arrow[dl, dash] \\ & \mathbb{Q} & \end{tikzcd}$$

$K=K_1K_2$ (composite)

$[K:K_1]=s$, $[K:K_2]=r$

Let $T \subset M_K$ be finite set of places,
$\mathbb{Q}(\alpha)=K_1$, $\mathbb{Q}(\beta)=K_2$, $\alpha \neq \beta$

Lemma. $\prod_{v\in T} |\alpha-\beta|_v \geq 2^{-[K:\mathbb{Q}]} H_{K_1}(\alpha)^{-s} H_{K_2}(\beta)^{-r}$

Proof.
$$\prod_{v\in T}|\alpha-\beta|_v = \prod_{v\notin T} |\alpha-\beta|_v^{-1}$$
$$\geq 2^{-[K:\mathbb{Q}]} \cdot \prod_{v\notin T}\{\max(1,|\alpha|_v)\cdot\max(1,|\beta|_v)\}^{-1}$$
$$\geq 2^{-[K:\mathbb{Q}]}\{H_K(\alpha)\cdot H_K(\beta)\}^{-1}$$
$$\geq 2^{-[K:\mathbb{Q}]} H_{K_1}(\alpha)^{-s} H_{K_2}(\beta)^{-r}$$
∎

ROTH'S THEOREM

For every fixed $\alpha\in K_1$ and every $\delta>0$, there are only finitely many $\beta\in K_2$ with
$$\prod_{v\in T}|\alpha-\beta|_v \leq H_{K_2}(\beta)^{-2-\delta}$$

For $r=[K:K_2]\geq 3$ this gives a one-sided improvement of Liouville's inequality:

For given α and all but finitely many β it gives an improvement of Liouville's inequality in terms of β.

Can we improve Liouvilles inequality in terms of both α and β?

SYMMETRIC IMPROVEMENTS

A symmetric improvement of Liouville's inequality is a finiteness result for an inequality in two variables,

$$(*) \quad \prod_{v \in T} |\alpha-\beta|_v \leq \psi(H_{K_1}(\alpha), H_{K_2}(\beta))^{-1}$$

in $\alpha \in K_1$, $\beta \in K_2$,

where ψ is a function with

$$\lim_{\max(x,y)\to\infty} \frac{\psi(x,y)}{x^s y^r} = 0$$

Open problem:

What is the infimum of all functions ψ for which $(*)$ has only finitely many solutions (α, β)?

Assumptions throughout lecture:

$$\begin{tikzcd}
 & K = K_1K_2 \arrow[dl, dash, "s"'] \arrow[dr, dash, "r"] & \\
K_1 \arrow[dr, dash, "r"'] & & K_2 \arrow[dl, dash, "s"] \\
 & \mathbb{Q} &
\end{tikzcd}$$

A) $[K_1:\mathbb{Q}] = r$, $[K_2:\mathbb{Q}] = s$

$[K:\mathbb{Q}] = [K_1:\mathbb{Q}] \cdot [K_2:\mathbb{Q}]$
$= rs$

B) $r \geq 3$, $s \geq 3$

Write $T = \bigcup_{p \in S} T_p$,

where $S \subset M_{\mathbb{Q}}$ and where T_p consists of places $v \in M_K$ with $v | p$.

Rough idea:

There exists a symmetric improvement of Liouville's inequality if none of the sets T_p is too large.

MAIN RESULTS

Put $W_T := \max_{p\in S} \sum_{v\in T_p} [K_v : \mathbb{Q}_p]$

Consider

$$(*) \quad \prod_{v\in T} |\alpha-\beta|_v \le \left(H_{K_1}(\alpha)^{-s}\, H_{K_2}(\beta)^{-r}\right)^{1-\varkappa}$$

in α,β with $\mathbb{Q}(\alpha)=K_1$, $\mathbb{Q}(\beta)=K_2$.

THEOREM 1. (E.)

If $W_T < rs-\max(r,s)$,

then for all $\varkappa \le \frac{1}{718(r+s)^2}$

(*) has only finitely many solutions.

THEOREM 2. (E.)

There are sets T with $W_T = rs-\max(r,s)$,

such that for all $\varkappa > 0$,

(*) has infinitely many solutions.

Idea of proof of Theorem 2:

1) Pick arbitrary α_0,β_0 with $\mathbb{Q}(\alpha_0)=K_1$, $\mathbb{Q}(\beta_0)=K_2$

2) Show that there are infinitely many $a,b,c,d\in\mathbb{Z}$ with $ad-bc\neq 0$ such that
$$\alpha = \frac{a\alpha_0+b}{c\alpha_0+d}, \quad \beta = \frac{a\beta_0+b}{c\beta_0+d}$$
is a solution of (*)

This uses Roth's Theorem + geometry of numbers.

The proof of Theorem 1 uses a lower bound for resultants and this follows in turn from Schmidt's Subspace Theorem.

More notation

- $S = \{\infty, p_1, \dots, p_t\} \subset M_{\mathbb{Q}}$

  $O_S = \{x \in \mathbb{Q} : |x|_p \le 1 \text{ for } p \notin S\} = \mathbb{Z}[\frac{1}{p_1 p_2 \cdots p_t}]$

  $SL_2(O_S) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in O_S,\ ad - bc = 1\}$

- For $f(X) = a_0 X^r + a_1 X^{r-1} + \dots + a_r$,

  $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, define

  $f_U(X) := (cX+d)^r f\left(\frac{aX+b}{cX+d}\right)$

- For $f(X) = a_0 X^r + a_1 X^{r-1} + \dots + a_r \in \mathbb{Z}[X]$

  define
  $$H(f) := \frac{\max(|a_0|, \dots, |a_r|)}{\gcd(a_0, \dots, a_r)}$$

Fact: If $f(X) \in \mathbb{Z}[X]$, f irreducible, $f(\alpha) = 0$, $K = \mathbb{Q}(\alpha)$, then

$$(r+1)^{-1/2} H_K(\alpha) \le H(f) \le 2^{r-1} H_K(\alpha).$$

RESULTANTS

For $f(X) = a_0 X^r + a_1 X^{r-1} + \dots + a_r$,
$g(X) = b_0 X^s + b_1 X^{s-1} + \dots + b_s$ define

$$R(f,g) := \begin{vmatrix} a_0 & a_1 & \cdots & a_r & & \\ & \ddots & & & \ddots & \\ & & a_0 & a_1 & \cdots & a_r \\ b_0 & b_1 & \cdots & \cdots & b_s & \\ & \ddots & & & & \ddots \\ & & b_0 & b_1 & \cdots & b_s \end{vmatrix} \begin{array}{l} \left.\vphantom{\begin{matrix} a \\ a \\ a \end{matrix}}\right\} s \\ \left.\vphantom{\begin{matrix} a \\ a \\ a \end{matrix}}\right\} r \end{array}$$

Properties:

a) $f(X) = a_0(X-\alpha_1)\dots(X-\alpha_r)$, $g(X) = b_0(X-\beta_1)\dots(X-\beta_s)$

$\Rightarrow R(f,g) = a_0^s b_0^r \prod_{i=1}^{r} \prod_{j=1}^{s} (\alpha_i - \beta_j)$

b) $R(f_U, g_U) = (\det U)^{rs} R(f,g)$ for $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

c) Suppose $f(X), g(X) \in \mathbb{Z}[X]$, $\gcd(a_0, \dots, a_r) = 1$, $\gcd(b_0, \dots, b_s) = 1$. Then

$\prod_{p \in S} |R(f,g)|_p \le c(r,s)\, H(f)^s H(g)^r$ (Hadamard)

d) $\prod_{p \in S} |R(f,g)|_p \le c(r,s) \cdot \min_{U \in SL_2(O_S)} H(f_U)^s H(g_U)^r$

Lower bound:

THEOREM 3 (E).

Let $f(X), g(X) \in \mathbb{Z}[X]$ be such that

$\deg f = r \geq 3$, $\deg g = s \geq 3$,

$f \cdot g$ has splitting field L

f, g have no multiple roots

f, g have no common roots

Then:

$$\prod_{p \in S} |R(f,g)|_p \geq C(L,S,r,s) \cdot \min_{U \in SL_2(O_S)} \left\{ H(f_U)^s \cdot H(g_U)^r \right\}^{\frac{1}{7+\varepsilon}}$$

Proof. Subspace Theorem

C is ineffective.

THEOREM 3 $\Rightarrow$ THEOREM 1

Let K_1, K_2, K, r, s, T satisfy the conditions of Theorem 1.

Suppose $\mathbb{Q}(\alpha) = K_1$, $\mathbb{Q}(\beta) = K_2$

Let $f, g \in \mathbb{Z}[X]$ be irreducible polynomials with $f(\alpha) = 0$, $g(\beta) = 0$

Then $H(f) \gg\ll H_{K_1}(\alpha)$, $H(g) \gg\ll H_{K_2}(\beta)$

Lemma. For all $U \in SL_2(O_S)$ we have

$$\prod_{v \in T} |\alpha - \beta|_v \geq c(r,s) \cdot H(f)^{-s} H(g)^{-r} \cdot$$

$$\cdot \left\{ \prod_{p \in S} |R(f,g)|_p \right\} \cdot \max\left(1, \frac{H(f)^{1/s} H(g)^{1/r}}{\{H(f_U) \cdot H(g_U)\}^{r+s}}\right)$$

Proof. Elementary.

THEOREM 3 $\Rightarrow$ THEOREM 1.

We have to prove:

$$\prod_{v\in T}|\alpha-\beta|_v \gg \left(H_{K_1}(\alpha)^{-s} H_{K_2}(\beta)^{-r}\right)^{1-\varkappa}$$

with $\varkappa = \frac{1}{7(8(r+s)^2}$.

Choose $U \in SL_2(\mathcal{O}_S)$ with

$$\prod_{p\in S}|R(f,g)|_p \gg \{H(f_U)^s H(g_U)^r\}^{\frac{1}{7(8}}$$

Then the lemma gives:

$$\prod_{v\in T}|\alpha-\beta|_v \gg H(f)^{-s}H(g)^{-r}\cdot\{H(f_U)^s H(g_U)^r\}^{\frac{1}{7(8}}$$

$$\cdot\max\left(1, \frac{H(f)^{1/r}H(g)^{1/s}}{\{H(f_U)H(g_U)\}^{r+s}}\right)^{\frac{\min(r,s)}{7(8(r+s)}}$$

$$\gg \left(H(f)^{-s}H(g)^{-r}\right)^{1-\varkappa}$$

$$\gg \left(H_{K_1}(\alpha)^{-s}H_{K_2}(\beta)^{-r}\right)^{1-\varkappa}.$$

HOW TO PROVE THEOREM 3?

Write $f(X) = (\gamma_1 X-\delta_1)\cdots(\gamma_r X-\delta_r)$,

$g(X) = (\lambda_1 X-\mu_1)\cdots(\lambda_s X-\mu_s)$

Define

$$\Delta_{ij} = \gamma_i\mu_j - \delta_i\lambda_j,$$
$$\Theta_{ij} = \gamma_i\delta_j - \gamma_j\delta_i,$$
$$E_{ij} = \lambda_i\mu_j - \lambda_j\mu_i$$

We have to estimate from below

$$\prod_{p\in S}|R(f,g)|_p = \prod_{p\in S}\left|\prod_{i=1}^{r}\prod_{j=1}^{s}\Delta_{ij}\right|_p.$$

We have at our disposal identities

$$\begin{vmatrix}\Delta_{ip} & \Delta_{iq} & \Delta_{it}\\ \Delta_{jp} & \Delta_{jq} & \Delta_{jt}\\ \Delta_{kp} & \Delta_{kq} & \Delta_{kt}\end{vmatrix} = \underbrace{\Delta_{ip}\Delta_{jq}\Delta_{kt} - \cdots - \Delta_{kp}\Delta_{jq}\Delta_{it}}_{6 \text{ terms}} = 0$$

$$\Theta_{ij}E_{pq} - \Delta_{ip}\Delta_{jq} + \Delta_{iq}\Delta_{jt} = 0$$

etc.

We

We apply to these identities the following consequence of the Subspace Theorem of Schmidt and Schlickewei.

Let K be a number field.
For $\underline{x}=(x_0,\dots,x_n)\in K^{n+1}$, $v\in M_K$, put
$|\underline{x}|_v := \max(|x_0|_v,\dots,|x_n|_v)$.
Then $H_K(\underline{x}) := \prod_{v\in M_K} |\underline{x}|_v$ defines a height on $\mathbb{P}^n(K)$

Let $T\subset M_K$ be a finite set of places

THEOREM (E., 1984)
For every $\underline{x}=(x_0:\dots:x_n)\in\mathbb{P}^n(K)$ with

$$x_0+x_1+\dots+x_n=0$$
$$\sum_{i\in I} x_i\neq 0 \text{ for each } I\subsetneq\{0,\dots,n\}$$

and every $\delta>0$, we have

$$\prod_{v\in T}\prod_{i=0}^{n}\frac{|x_i|_v}{|\underline{x}|_v} \gg_{n,\delta,T,K} H_K(\underline{x})^{-n-\delta}.$$