Appendix A

Proof of the Bombieri-Vinogradov Theorem

A.1 Key Lemmas

In this chapter, we give a proof of the Bombieri-Vinogradov Theorem, which was stated without proof in Chapter 11. We start by proving the following large sieve type result for characters:

Corollary A.1.1 (Gallagher)**.** *For any integers $M, N, Q \geqslant 1$,*

$$\sum_{q \leqslant Q} \frac{q}{\varphi(q)} \sideset{}{^*}\sum_{\chi \pmod{q}} \left| \sum_{n=M+1}^{M+N} a_n \chi(n) \right|^2 \leqslant (N + Q^2) \sum_{n=M+1}^{M+N} |a_n|^2,$$

where $\sum^*_{\chi \pmod{q}}$ runs over all primitive characters $\pmod{q}$.

Proof. Let

$$S(t) := \sum_{n=M+1}^{M+N} a_n \, \mathrm{e}(nt) \quad \text{and} \quad \tilde{S}(t) := \sum_{n=1}^{N} a_{M+n} \, \mathrm{e}(nt).$$

First note that the value of M is irrelevant in the sense that $|S(t)| = |\tilde{S}(t)|$. Now, let

$$T(\chi) := \sum_{n=M+1}^{M+N} a_n \chi(n).$$

Recall that given a character $\chi \pmod q$, we define the Gauss sum of χ by

$$\tau(\chi) = \sum_{m=1}^{q} \chi(m)\, \mathrm{e}\Big(\frac{m}{q}\Big).$$

By Theorem 3.4.1, we know that

$$\chi(n) = \frac{1}{\tau(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi}(a)\, \mathrm{e}\Big(\frac{an}{q}\Big).$$

Hence,

$$T(\chi) = \frac{1}{\tau(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi}(a) S\Big(\frac{a}{q}\Big).$$

Also recall that if χ is primitive, then by Theorem 3.4.2, $\tau(\overline{\chi}) = q^{\frac{1}{2}}$. Therefore,

$$\begin{aligned} \sideset{}{^*}\sum_{\chi \pmod q} |T(\chi)|^2 &= \frac{1}{q} \sideset{}{^*}\sum_{\chi \pmod q} \Big| \sum_{a=1}^{q} \overline{\chi}(a) S\Big(\frac{a}{q}\Big) \Big|^2 \\ &\leqslant \frac{1}{q} \sum_{\chi \pmod q} \Big| \sum_{a=1}^{q} \overline{\chi}(a) S\Big(\frac{a}{q}\Big) \Big|^2 \\ &= \frac{1}{q} \sum_{a=1}^{q} \sum_{b=1}^{q} S\Big(\frac{a}{q}\Big) \overline{S\Big(\frac{b}{q}\Big)} \sum_{\chi} \overline{\chi}(a)\chi(b) \\ &= \frac{\varphi(q)}{q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \Big| S\Big(\frac{a}{q}\Big) \Big|^2, \qquad \text{(A.1.1)} \end{aligned}$$

where the last equality follows from the orthogonality relations of characters (Theorem 3.2.1). Now, by the large sieve inequality (Theorem 11.2.3), we have

$$\sum_{q=1}^{Q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \Big| S\Big(\frac{a}{q}\Big) \Big|^2 \leqslant (N + Q^2) \sum_{n=M+1}^{M+N} |a_n|^2.$$

This together with equation (A.1.1) completes the proof. □

Now, we proceed to prove two useful lemmas.

Lemma A.1.2. *Let $Q, M, N, u \in \mathbb{Z}_{\geqslant 1}$. Then,*

$$\sum_{q\leqslant Q}\frac{q}{\varphi(q)}\sum_{\chi}{}^{*}\max_{u}\Bigg|\sum_{\substack{1\leqslant m\leqslant M\\ mn\leqslant u}}\sum_{1\leqslant n\leqslant N}a_m b_n\chi(mn)\Bigg|$$
$$\ll (M+Q^2)^{\frac12}(N+Q^2)^{\frac12}\Bigg(\sum_{1\leqslant m\leqslant M}|a_m|^2\Bigg)^{\frac12}\Bigg(\sum_{1\leqslant n\leqslant N}|b_n|^2\Bigg)^{\frac12}\log MN.$$

Proof. We will begin by ignoring the conditions involving u and then work them in later. By Corollary A.1.1, we have

$$\sum_{q\leqslant Q}\frac{q}{\varphi(q)}\sum_{\chi}{}^{*}\Bigg|\sum_{n=M+1}^{M+N}a_n\chi(n)\Bigg|^2 \ll (N+Q^2)\sum_{n=M+1}^{M+N}|a_n|^2.$$

This, together with the Cauchy–Schwarz inequality, shows that

$$\sum_{q\leqslant Q}\frac{q}{\varphi(q)}\sum_{\chi}{}^{*}\Bigg|\sum_{1\leqslant m\leqslant M}\sum_{1\leqslant n\leqslant N}a_m b_n\chi(mn)\Bigg|$$
$$\leqslant\Bigg(\sum_{q\leqslant Q}\frac{q}{\varphi(q)}\sum_{\chi}{}^{*}\Bigg|\sum_{m=1}^{M}a_m\chi(m)\Bigg|^2\Bigg)^{\frac12}\Bigg(\sum_{q\leqslant Q}\frac{q}{\varphi(q)}\sum_{\chi}{}^{*}\Bigg|\sum_{n=1}^{N}b_n\chi(n)\Bigg|^2\Bigg)^{\frac12}$$
$$\ll (M+Q^2)^{\frac12}(N+Q^2)^{\frac12}\Bigg(\sum_{1\leqslant m\leqslant M}|a_m|^2\Bigg)^{\frac12}\Bigg(\sum_{1\leqslant n\leqslant N}|b_n|^2\Bigg)^{\frac12}. \tag{A.1.2}$$

If there were no condition that $mn \leqslant u$ in the statement of our lemma, this proof would be a straightforward application of the large sieve and the Cauchy-Schwarz inequality. To introduce the condition $mn \leqslant u$, we use the following result from complex analysis (for a proof, see Davenport's book):

$$\int_{-T}^{T}\mathrm{e}^{it\alpha}\,\frac{\sin t\beta}{\pi t}\,\mathrm{d}t=\begin{cases}1+O(T^{-1}(\beta-|\alpha|)^{-1}), & \text{if } |\alpha|<\beta\\ \frac12 & \text{if } |\alpha|=\beta\\ O(T^{-1}(\beta-|\alpha|)^{-1}), & \text{if } |\alpha|>\beta.\end{cases}$$

Letting $\beta=\log u$ and $\alpha=-\log mn$, this becomes

$$\int_{-T}^{T}m^{-it}n^{-it}\frac{\sin t\beta}{\pi t}\,\mathrm{d}t=1+O(T^{-1}(\log u-\log mn)^{-1}).$$

Therefore, if we let

$$A(t,\chi) := \sum_{m=1}^{M} a_m \chi(m) m^{-it} \quad \text{and} \quad B(t,\chi) := \sum_{n=1}^{N} b_n \chi(n) n^{-it},$$

then

$$\sum_{m=1}^{M}\sum_{n=1}^{N} a_m b_n \chi(mn) = \int_{-T}^{T} A(t,\chi)B(t,\chi)\frac{\sin t\beta}{\pi t}\,\mathrm{d}t + O\biggl(T^{-1}\sum_{m=1}^{M}\sum_{n=1}^{N}|a_m b_n|\Bigl|\log\frac{mn}{u}\Bigr|^{-1}\biggr).$$

Since $\left|\log\dfrac{1}{x}\right| \geqslant \frac{1}{x}$ for $x \geqslant 2$, then

$$\Bigl|\log\frac{mn}{u}\Bigr| \gg \frac{1}{u} \gg \frac{1}{MN}.$$

Note that

$$\sin(t\log u) \leqslant \min(1, |t|\log MN).$$

Hence,

$$\sum_{m=1}^{M}\sum_{n=1}^{N} a_m b_n \chi(mn) \ll \int_{-T}^{T} A(t,\chi)B(t,\chi)\min\left(\frac{1}{|t|}, \log MN\right)\mathrm{d}t + \frac{MN}{T}\sum_{m=1}^{M}\sum_{n=1}^{N}|a_m b_n|.$$

Now, we apply (A.1.2) to the first term, the Cauchy–Schwarz inequality to the second term, and use the fact that

$$\sum_{q\leqslant Q}\frac{q}{\varphi(q)} \leqslant \sum_{q\leqslant Q} q = \frac{Q(Q+1)}{2} \ll Q^2 \tag{A.1.3}$$

to obtain

$$\sum_{q\leqslant Q}\frac{q}{\varphi(q)}\sideset{}{^*}\sum_{\chi}\max_{u}\Biggl|\sum_{\substack{1\leqslant m\leqslant M\\ mn\leqslant u}}\sum_{1\leqslant n\leqslant N} a_m b_n \chi(mn)\Biggr|$$

$$\ll (M+Q^2)^{\frac{1}{2}}(N+Q^2)^{\frac{1}{2}}\Biggl(\sum_{1\leqslant m\leqslant M}|a_m|^2\Biggr)^{\frac{1}{2}}\Biggl(\sum_{1\leqslant n\leqslant N}|b_n|^2\Biggr)^{\frac{1}{2}}\int_{-T}^{T}\min\left(\frac{1}{|t|}, \log MN\right)\mathrm{d}t$$

$$+\frac{M^{\frac{3}{2}}N^{\frac{3}{2}}Q^2}{T}\Biggl(\sum_{1\leqslant m\leqslant M}|a_m|^2\Biggr)^{\frac{1}{2}}\Biggl(\sum_{1\leqslant n\leqslant N}|b_n|^2\Biggr)^{\frac{1}{2}}.$$

Now,

$$\min\left(\frac{1}{|t|}, \log MN\right) = \begin{cases} \log MN, & \text{if } |t| \leqslant \frac{1}{\log MN} \\ \frac{1}{|t|}, & \text{otherwise,} \end{cases}$$

so that

$$\int_{-T}^{T} \min\left(\frac{1}{|t|}, \log MN\right) \mathrm{d}t = 2(1 + \log T - \log\log MN).$$

Therefore, letting $T = MN$ completes the proof. □

Before proving our next lemma, we will explain the following method due to Vaughan for evaluating sums of the form $\sum_{n\leqslant N} f(n)\Lambda(n)$. For this purpose, let

$$F(s) := \sum_{m\leqslant U} \frac{\Lambda(m)}{m^s} \quad \text{and} \quad G(s) := \sum_{d\leqslant V} \frac{\mu(d)}{d^s},$$

where U and V will be chosen later. We begin by noting that

$$\begin{aligned} -\frac{\zeta'(s)}{\zeta(s)} &= F(s) - \zeta(s)F(s)G(s) - \zeta'(s)G(s) + \Big(-\frac{\zeta'(s)}{\zeta(s)} - F(s)\Big)(1 - \zeta(s)G(s)) \\ &= f_1(s) + f_2(s) + f_3(s) + f_4(s) \end{aligned}$$

This equation is called *Vaughan's identity.* For $j = 1, 2, 3, 4$, let $a_j(n)$ denote the Dirichlet series coefficients of $f_j(s)$. Since

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s},$$

by comparing Dirichlet coefficients, we have

$$\Lambda(n) = a_1(n) + a_2(n) + a_3(n) + a_4(n),$$

where (verify this!):

$$a_1(n) = \begin{cases} \Lambda(n), & \text{if } n \leqslant U \\ 0, & \text{otherwise,} \end{cases}$$
$$a_2(n) = -\sum_{\substack{mdr=n \\ m\leqslant U \\ d\leqslant V}} \Lambda(m)\mu(d),$$
$$a_3(n) = \sum_{\substack{hd=n \\ d\leqslant V}} \mu(d)\log h,$$
$$a_4(n) = -\sum_{\substack{mk=n \\ m>U \\ k>1}} \Lambda(m)\Big(\sum_{\substack{d|k \\ d\leqslant V}} \mu(d)\Big).$$

Therefore,

$$\sum_{n\leqslant N} f(n)\Lambda(n) = S_1 + S_2 + S_3 + S_4,$$

where

$$S_j = \sum_{n\leqslant N} f(n)a_j(n).$$

We write S_2 in the form

$$S_2 = -\sum_{t\leqslant UV}\Big(\sum_{\substack{md=t \\ m\leqslant U \\ d\leqslant V}} \Lambda(m)\mu(d)\Big)\sum_{r\leqslant \frac{N}{t}} f(rt).$$

Since $\sum_{m|t}\Lambda(m) = \log t \leqslant \log UV$, we have

$$S_2 \ll \log UV \sum_{t\leqslant UV}\sum_{r\leqslant \frac{N}{t}} f(rt). \tag{A.1.4}$$

As for S_3,

$$\begin{aligned} S_3 &= \sum_{d\leqslant V}\mu(d)\sum_{h\leqslant \frac{N}{d}} f(dh)\log h \\ &= \sum_{d\leqslant V}\mu(d)\sum_{h\leqslant \frac{N}{d}} f(dh)\int_1^h \frac{\mathrm{d}w}{w} \\ &= \int_1^N \sum_{d\leqslant V}\mu(d)\sum_{w\leqslant h\leqslant \frac{N}{d}} f(dh)\frac{\mathrm{d}w}{w} \\ &\ll \log N\sum_{d\leqslant V}\max_{w}\Big|\sum_{w\leqslant h\leqslant \frac{N}{d}} f(dh)\Big|, \end{aligned} \tag{A.1.5}$$

where the third equality follows from Exercise A.1. For S_4, since

$$\sum_{\substack{d|k\\ d\leqslant V}}\mu(d) = 0 \quad \text{for} \quad 1 < k \leqslant V,$$

we have

$$S_4 = -\sum_{U\leqslant m\leqslant \frac{N}{V}}\Lambda(m)\sum_{V<k\leqslant \frac{N}{m}}\Big(\sum_{\substack{d|k\\ d\leqslant V}}\mu(d)\Big)f(mk). \tag{A.1.6}$$

Vaughan's identity is a standard tool in analytic number theory, and has many other applications. The point of Vaughan's identity is to introduce more variables (e.g., m, d, r). Why is it useful to introduce more variables? Because, deep down, bounding a double sum (say) amounts to bounding a single sum on average. And bounding something on average is easier than bounding it on its own. More variables also give us more freedom.

We need one last lemma before proving the Bombieri–Vinogradov Theorem:

Lemma A.1.3. *For $x \geqslant 1$, let*

$$\psi(x,\chi) := \sum_{n\leqslant x}\Lambda(n)\chi(n).$$

For all $Q \in \mathbb{Z}_{\geqslant 1}$, we have

$$\sum_{q\leqslant Q}\frac{q}{\varphi(q)}\sideset{}{^*}\sum_{\chi}\max_{y\leqslant x}|\psi(y,\chi)| \ll (x + x^{\frac{5}{6}}Q + x^{\frac{1}{2}}Q^2)(\log Qx)^4.$$

Proof. We consider two cases:

Case 1: $Q^2 > x$.

Letting $M = 1, a_1 = 1, b_n = \Lambda(n), N = x$ in Lemma A.1.2, we see that

$$\begin{aligned}
\sum_{q\leqslant Q}\frac{q}{\varphi(q)}\sum_{\chi}{}^{*}\max_{y\leqslant x}|\psi(y,\chi)| &\ll (1+Q^2)^{\frac{1}{2}}(x+Q^2)^{\frac{1}{2}}\Big(\sum_{1\leqslant n\leqslant x}|\Lambda(n)|^2\Big)^{\frac{1}{2}}\log x \\
&\ll \big((1+Q^2)(2Q^2)\big)^{\frac{1}{2}}\Big(\log x\sum_{1\leqslant n\leqslant x}\Lambda(n)\Big)^{\frac{1}{2}}\log x \\
&\ll (Q+Q^2)(x\log x)^{\frac{1}{2}}\log x \\
&= (x^{\frac{1}{2}}Q + x^{\frac{1}{2}}Q^2)\log^{\frac{3}{2}} x \\
&\ll (x + x^{\frac{5}{6}}Q + x^{\frac{1}{2}}Q^2)(\log Qx)^4,
\end{aligned}$$

where the second bound follows from the fact that $x < Q^2$.

Case 2: $Q^2 \leqslant x$.

Since $\psi(y,\chi) = \sum_{n\leqslant y}\chi(n)\Lambda(n)$, we have

$$\psi(y,\chi) = S_1 + S_2 + S_3 + S_4,$$

where by using (A.1.5) and (A.1.6) we obtain

(A.1.7)
$$\begin{aligned}
S_1 &= \sum_{n\leqslant U}\chi(n)\Lambda(n) \ll U; \\
S_2 &= -\sum_{t\leqslant UV}\Big(\sum_{\substack{md=t\\ m\leqslant U\\ d\leqslant V}}\Lambda(m)\mu(d)\Big)\sum_{r\leqslant\frac{N}{t}}\chi(rt); \\
S_3 &\ll \log N\sum_{d\leqslant V}\max_{w}\Big|\sum_{w\leqslant h\leqslant\frac{N}{d}}\chi(h)\Big|; \\
S_4 &= -\sum_{U\leqslant m\leqslant\frac{N}{V}}\Lambda(m)\sum_{V<k\leqslant\frac{N}{m}}\Big(\sum_{\substack{d|k\\ d\leqslant V}}\mu(d)\Big)\chi(mk).
\end{aligned}$$

Let's begin by bounding S_4: By Lemma A.1.2, we have

$$\sum_{q\leqslant Q}\frac{q}{\varphi(q)}\sum_{\chi}{}^{*}\max_{y\leqslant x}\Big|\sum_{M\leqslant m\leqslant 2M}\Lambda(m)\sum_{V<k\leqslant \frac{y}{m}}\Big(\sum_{\substack{d|k\\ d\leqslant V}}\mu(d)\Big)\chi(mk)\Big|$$
$$\ll (M+Q^2)^{\frac{1}{2}}(\frac{x}{m}+Q^2)^{\frac{1}{2}}\Big(\sum_{M\leqslant m\leqslant 2M}\Lambda(m)^2\Big)^{\frac{1}{2}}\Big(\sum_{k\leqslant \frac{x}{M}}\tau(k)^2\Big)^{\frac{1}{2}}\log x,$$

where $\tau(k)=\sum_{d|k}1$. Now, by the prime number theorem,

$$\sum_{M\leqslant m\leqslant 2M}\Lambda(m)^2 \ll \log 2M\sum_{M\leqslant m\leqslant 2M}\Lambda(m)\ll \log 2M(2M-M)=M\log M.$$

By Exercise A.3,

$$\sum_{k\leqslant z}\tau(k)^2\ll z\log^3 z$$

Hence,

$$\sum_{q\leqslant Q}\frac{q}{\varphi(q)}\sum_{\chi}{}^{*}\max_{y\leqslant x}\Big|\sum_{M\leqslant m\leqslant 2M}\Lambda(m)\sum_{V<k\leqslant \frac{y}{m}}\Big(\sum_{\substack{d|k\\ d\leqslant V}}\mu(d)\Big)\chi(mk)\Big|$$
$$\ll (M+Q^2)^{\frac{1}{2}}(\frac{x}{m}+Q^2)^{\frac{1}{2}}(M\log M)^{\frac{1}{2}}\Big(\frac{x}{M}\log^3\frac{x}{m}\Big)^{\frac{1}{2}}\log x$$
$$\ll (Q^2x^{\frac{1}{2}}+QxM^{-\frac{1}{2}}+Qx^{\frac{1}{2}}M^{\frac{1}{2}}+x)\log^3 x.$$

Now we sum this bound over $M=2^k$ for $\frac{U}{2}\leqslant 2^k\leqslant \frac{x}{V}$ (dyadic decomposition) to obtain

$$\sum_{q\leqslant Q}\frac{q}{\varphi(q)}\sum_{\chi}{}^{*}\max_{y\leqslant x}|S_4|\ll (Q^2x^{\frac{1}{2}}+QxU^{-\frac{1}{2}}+QxV^{-\frac{1}{2}}+x)\log^4 x. \tag{A.1.8}$$

For S_2 we consider two ranges of t:

$$S_2=\sum_{t\leqslant UV}=\sum_{t\leqslant U}+\sum_{U<t\leqslant UV}=S_2'+S_2''.$$

For S_2'' we do exactly the same we did with S_4 to obtain

$$\sum_{q\leqslant Q}\frac{q}{\varphi(q)}\sum_{\chi}{}^{*}\max_{y\leqslant x}|S_2''|\ll (Q^2x^{\frac{1}{2}}+QxU^{-\frac{1}{2}}+Qx^{\frac{1}{2}}U^{\frac{1}{2}}V^{\frac{1}{2}}+x)\log^4 x. \tag{A.1.9}$$

For S_2' we use the Pólya–Vinogradov inequality (Theorem 3.5.1): for $q > 1$, we have

$$\sum_{n=M+1}^{M+N} \chi(n) \ll q^{\frac{1}{2}} \log q.$$

This together with (A.1.4) show that for $q > 1$, we have

$$S_2' \ll \log U \sum_{t \leqslant U} \Big| \sum_{r \leqslant \frac{y}{t}} \chi(r) \Big| \ll q^{\frac{1}{2}} U (\log qU)^2.$$

For $q = 1$, we use the trivial bound

$$S_2' \ll y(\log^2 U).$$

These bounds together with (A.1.3) show that

$$\sum_{q \leqslant Q} \frac{q}{\varphi(q)} \sum_{\chi}{}^{*} \max_{y \leqslant x} |S_2'| \ll (Q^{\frac{5}{2}} U + x)(\log Ux)^2. \tag{A.1.10}$$

We treat S_3 the same way we did with S_2' to find that

$$\sum_{q \leqslant Q} \frac{q}{\varphi(q)} \sum_{\chi}{}^{*} \max_{y \leqslant x} |S_3| \ll (Q^{\frac{5}{2}} V + x)(\log Vx)^2. \tag{A.1.11}$$

Finally. combining (A.1.7), (A.1.10), (A.1.9), (A.1.11) and (A.1.8) we obtain

$$\begin{aligned} &\sum_{q \leqslant Q} \frac{q}{\varphi(q)} \sum_{\chi}{}^{*} \max_{y \leqslant x} |\psi(y, \chi)| \\ &\qquad \ll (Q^2 x^{\frac{1}{2}} + x + QxU^{-\frac{1}{2}} + QxV^{-\frac{1}{2}} + U^{\frac{1}{2}} V^{\frac{1}{2}} Q x^{\frac{1}{2}} + Q^{\frac{5}{2}} U + Q^{\frac{5}{2}} V)(\log xUV)^4. \end{aligned}$$

If we vary U and V in a way such that the product UV is fixed, then it is possible to see that this expression is minimized when $U = V$. Hence,

$$\sum_{q \leqslant Q} \frac{q}{\varphi(q)} \sum_{\chi}{}^{*} \max_{y \leqslant x} |\psi(y, \chi)| \ll (Q^2 x^{\frac{1}{2}} + x + QxU^{-\frac{1}{2}} + UQx^{\frac{1}{2}} + Q^{\frac{5}{2}} U)(\log Qx)^4.$$

Now consider the following cases:

- If $x^{\frac{1}{3}} \leqslant Q \leqslant x^{\frac{1}{2}}$, then the terms involving U are minimized by taking $U = x^{\frac{2}{3}} Q^{-1}$ and
$$QxU^{-\frac{1}{2}} + UQx^{\frac{1}{2}} + Q^{\frac{5}{2}} U \ll Q^{\frac{3}{2}} x^{\frac{2}{3}} \ll Q^2 x^{\frac{1}{2}}.$$

- If $1 \leqslant Q \leqslant x^{\frac{1}{3}}$, then the terms involving U are minimized by letting $U = x^{\frac{1}{3}}$ and
$$QxU^{-\frac{1}{2}} + UQx^{\frac{1}{2}} + Q^{\frac{5}{2}}U \ll x^{\frac{5}{6}}Q.$$

This shows that
$$\sum_{q \leqslant Q} \frac{q}{\varphi(q)} \sum_{\chi}{}^{*} \max_{y \leqslant x} |\psi(y, \chi)| \ll (x + x^{\frac{5}{6}}Q + x^{\frac{1}{2}}Q^2)(\log Qx)^4.$$

□

A.2 Proof of the Bombieri-Vinogradov Theorem

Finally, we are ready to prove the Bombieri-Vinogradov Theorem, which was first presented in Chapter 11. The main idea is to reduce the problem (in a dyadic interval $U < q < 2U$) to Lemma A.1.3. When U is too small for that approach to work, we appeal to the Siegel-Walfisz Theorem (Theorem 11.4.1) instead. We will prove a slightly stronger result than the one presented in Chapter 11. In order to simplify notation, let $E(x; q, a)$ denote the error term in the prime number theorem in arithmetic progressions, i.e.,
$$E(x; q, a) := \psi(x; q, a) - \frac{x}{\varphi(q)}.$$

We also let
$$E(x, q) := \sup_{\substack{a \\ (a,q)=1}} |E(x; q, a)| \quad \text{and} \quad E^*(x, q) := \sup_{y \leqslant x} E(y, q).$$

We are ready for our main theorem:

Theorem 11.3.3 (The Bombieri–Vinogradov Theorem)**.** Let $A > 0$. Then
$$\sum_{q \leqslant Q} E^*(x, q) \ll x^{\frac{1}{2}}Q(\log x)^5$$
for all $x^{\frac{1}{2}}(\log x)^{-A} \leqslant Q \leqslant x^{\frac{1}{2}}$.

Proof. By Exercise A.4,
$$\psi(x; q, a) = \frac{1}{\varphi(q)} \sum_{\chi \pmod q} \overline{\chi}(a)\psi(x, \chi).$$

Now let χ_0 denote the principal character and let

$$\psi'(y,\chi) := \begin{cases} \psi(y,\chi), & \text{if } \chi \neq \chi_0; \\ \psi(y,\chi_0) - y, & \text{if } \chi = \chi_0. \end{cases}$$

Then,

$$\psi(y;q,a) - \frac{y}{\varphi(q)} = \frac{1}{\varphi(q)} \sum_{\chi} \overline{\chi}(a)\psi'(y,\chi).$$

Therefore,

$$|E(y;q,a)| \leqslant \frac{1}{\varphi(q)} \sum_{\chi} |\psi'(y,\chi)|,$$

so that

$$|E(y,q)| \leqslant \frac{1}{\varphi(q)} \sum_{\chi} |\psi'(y,\chi)|,$$

i.e., the bound is independent of a. Now, if $\chi \pmod q$ is induced[1] by $\chi_1 \pmod{q_1}$, then

$$\begin{aligned} \psi'(y,\chi_1) - \psi'(y,\chi) &= \sum_{\substack{n \leqslant y \\ (n,q_1) \neq 1}} \chi_1(n)\Lambda(n) \\ &= \sum_{\substack{p \mid q \\ p \nmid q_1}} \sum_{p^l \leqslant y} \chi_1(p^l) \log p \\ &\ll \sum_{p \mid q} \left\lfloor \frac{\log y}{\log p} \right\rfloor \log p \\ &\ll \log y \sum_{p \mid q} \log p \\ &\ll \log y \log q \\ &\ll \log^2(yq). \end{aligned}$$

[1] Recall that for every character $\chi \pmod q$ there is a unique primitive character $\chi_1 \pmod q$ that induces χ, namely

$$\chi(n) = \begin{cases} \chi_1(n), & \text{if } (n,q) = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Hence

$$E(y,q) \ll \frac{1}{\varphi(q)} \sum_{\chi} |\psi'(y,\chi)| \ll \log^2(yq) + \frac{1}{\varphi(q)} \sum_{\chi_1} |\psi'(y,\chi_1)|,$$

so that

$$\sum_{q\leqslant Q} E^*(x,q) \ll Q\log^2(Qx) + \sum_{q\leqslant Q} \frac{1}{\varphi(q)} \sum_{\chi_1} \max_{y\leqslant x} |\psi'(y,\chi_1)|.$$

Now, we perform a change of variables: since a primitive character (mod q) induces characters (mod q_1), where $q_1 = kq$, we have

$$\sum_{q\leqslant Q} E^*(x,q) \ll Q\log^2(Qx) + \sum_{q\leqslant Q} \sum_{\chi}^{*} \max_{y\leqslant x} |\psi'(y,\chi)| \Big(\sum_{k\leqslant \frac{Q}{q}} \frac{1}{\varphi(kq)} \Big).$$

Note that $\varphi(kq) \geqslant \varphi(k)\varphi(q)$, so that

$$\sum_{k\leqslant z} \frac{1}{\varphi(kq)} \leqslant \frac{1}{\varphi(q)} \sum_{k\leqslant z} \frac{1}{\varphi(k)} \ll \frac{1}{\varphi(q)} \log z,$$

where the last bound follows from Exercise 11.9. This shows that

$$\sum_{q\leqslant Q} E^*(x,q) \ll \log x \sum_{q\leqslant Q} \frac{1}{\varphi(q)} \sum_{\chi}^{*} \max_{y\leqslant x} |\psi'(y,\chi)|.$$

Therefore, to complete the proof it suffices to show that

$$\sum_{q\leqslant Q} \frac{1}{\varphi(q)} \sum_{\chi}^{*} \max_{y\leqslant x} |\psi'(y,\chi)| \ll x^{\frac{1}{2}} Q (\log x)^4.$$

By Lemma A.1.3, we have

$$\sum_{U<q<2U} \frac{1}{\varphi(q)} \sum_{\chi}^{*} \max_{y\leqslant x} |\psi(y,\chi)| \ll \Big(\frac{x}{U} + x^{\frac{5}{6}} + x^{\frac{1}{2}} U\Big)(\log Ux)^4.$$

Now, for any $Q_1 \in [1,Q]$, we sum the above inequality over all $U = 2^k$ for integers k such that $\frac{1}{2}Q_1 < 2^k < 2Q$ to obtain

$$\begin{aligned}
\sum_{U<q<2U} \frac{1}{\varphi(q)} \sum_{\chi}^{*} \max_{y\leqslant x} |\psi(y,\chi)| &\leqslant \sum_{\frac{1}{2}Q_1<2^k<2Q} \frac{1}{\varphi(q)} \sum_{2^k<q\leqslant 2^{k+1}} \max_{y\leqslant x} |\psi(y,\chi)| \\
&\ll \sum_{\frac{1}{2}Q_1<2^k<2Q} \Big(\frac{x}{2^k} + x^{\frac{5}{6}} + x^{\frac{1}{2}} 2^k\Big)(\log Qx)^4 \\
&\ll \Big(\frac{x}{Q_1} + x^{\frac{5}{6}} \log Q + x^{\frac{1}{2}} Q\Big)(\log Qx)^4.
\end{aligned}$$

Letting $Q_1 = (\log x)^A$, we have

$$\left(\frac{x}{Q_1} + x^{\frac{5}{6}} \log Q + x^{\frac{1}{2}} Q\right) \ll x^{\frac{1}{2}} Q$$

for all $Q \in [x^{\frac{1}{2}}(\log x)^{-A}, x^{\frac{1}{2}}]$. Therefore,

$$\sum_{(\log x)^A < q < Q} \frac{1}{\varphi(q)} \sum_{\chi}{}^{*} \max_{y \leqslant x} |\psi'(y, \chi)| \ll x^{\frac{1}{2}} Q (\log x)^4.$$

Now we bound the remaining sum over $q \leqslant (\log x)^A$: By Theorem 11.4.1,

$$\max_{y \leqslant x} |\psi(y, \chi)| \ll x \,\mathrm{e}^{-c\sqrt{\log x}} \ll x (\log x)^{-2A}.$$

Summing over all $q \leqslant (\log x)^A$ we have

$$\sum_{1 \leqslant q \leqslant (\log x)^A} \frac{1}{\varphi(q)} \sum_{\chi}{}^{*} \max_{y \leqslant x} |\psi'(y, \chi)| \ll (\log x)^A x (\log x)^{-2A} \leqslant x^{\frac{1}{2}} Q \ll x^{\frac{1}{2}} Q (\log x)^4.$$

This completes the proof. □

A.3 Exercises

Exercise A.1. *Show that if f, g are real or complex valued functions defined on $[1, N]$, then*

$$\sum_{h \leqslant \frac{N}{d}} f(dh) \int_1^h g(w)\,\mathrm{d}w = \int_1^N \sum_{w \leqslant h \leqslant \frac{N}{d}} f(dh) g(w)\,\mathrm{d}w.$$

Exercise A.2. *Let $\tau(k)$ denote the number of divisors of k. Show that*

$$\tau(k)^2 = \sum_{d | k} f(d),$$

where f is the multiplicative function defined by $f(p^a) = 2a + 1$, where p is a prime number and $a \in \mathbb{Z}_{\geqslant 0}$.

Exercise A.3. *Use Exercise A.2 to show that*

$$\sum_{k \leqslant z} \tau(k)^2 \ll z \log^3 z.$$

Exercise A.4. *Show that*

$$\psi(x;q,a) = \frac{1}{\varphi(q)} \sum_{\chi \pmod{q}} \overline{\chi}(a)\psi(x,\chi).$$

Hint: Use the orthogonality relations of characters.

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