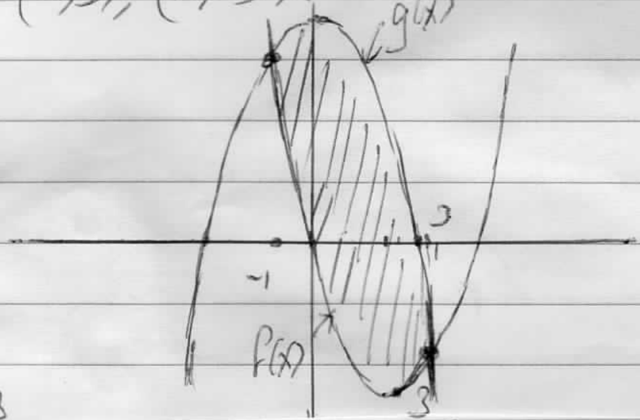
UITWERKING TENTAMEN CONTINUE WISKUNDE 2 27/6/2022

1 a) Snijpunten: $f(x)=g(x) \Leftrightarrow x^2-4x=6-x^2 \Leftrightarrow 2x^2-4x-6=0 \Leftrightarrow$
$x^2-2x-3=0 \Leftrightarrow (x+1)(x-3)=0 \Leftrightarrow x=-1$ of $x=3$
Dit geeft $(-1,5)$, $(3,-3)$.



oppervlakte $\int_{-1}^{3} (g(x)-f(x))dx = \int_{-1}^{3}((6-x^2)-(x^2-4x))\,dx$

$$= \int_{-1}^{3}(6+4x-2x^2)dx = \left[6x+2x^2-\tfrac{2}{3}x^3\right]_{-1}^{3}$$

$$= \left(6\times3+2\times9-\tfrac{2}{3}\times3^3\right)-\left(-6+2+\tfrac{2}{3}\right) = 18-\left(-\tfrac{10}{3}\right) = \boxed{\tfrac{64}{3}}$$

$f(x)=\ln x,\ g'(x)=x^4+1,\ g(x)=\tfrac{1}{5}x^5+x$

b) $\int \ln x\cdot(x^4+1)\,dx = (\ln x)\left(\tfrac{1}{5}x^5+x\right)-\int\left(\tfrac{1}{5}x^5+x\right)\cdot\tfrac{1}{x}\,dx$

$\int f(x)g'(x)dx$
$= f(x)g(x)-\int g(x)f'(x)dx$

$$= (\ln x)\left(\tfrac{1}{5}x^5+x\right)-\int\left(\tfrac{1}{5}x^4+1\right)dx$$

$$= \boxed{(\ln x)\left(\tfrac{1}{5}x^5+x\right)-\tfrac{1}{5}x^5-x+C}$$

$u=x^3+3x-4,\ du=(3x^2+3)dx,\ (x^2+1)dx=\tfrac{1}{3}du$

c) $\int \frac{(x^2+1)\,dx}{\sqrt{x^3+3x-4}} = \int \frac{\frac{1}{3}du}{u^{1/2}} = \frac{1}{3}\cdot 2u^{1/2}+C = \frac{2}{3}(x^3+3x-4)^{1/2}+C$

$$\int_1^2 \frac{(x^2+1)\,dx}{\sqrt{x^3+3x-4}} = \lim_{t\downarrow1}\int_t^2(\ \ )\,dx = \lim_{t\downarrow1}\left[\tfrac{2}{3}(x^3+3x-4)^{1/2}\right]_t^2$$

$$= \lim_{t\downarrow1}\left(\tfrac{2}{3}\sqrt{10}-\tfrac{2}{3}\sqrt{\underbrace{t^3+3t-4}_{\to 0}}\right) = \boxed{\tfrac{2}{3}\sqrt{10}}$$

2 a) $(2x+y)^2+2(x^2-1)^2 = 4x^2+4xy+y^2+2x^4-4x^2+2$
$= 2x^4+4xy+y^2+2$

b) $\frac{\partial f}{\partial x} = 8x^3+4y$, $\frac{\partial f}{\partial y} = 4x+2y$

$\frac{\partial f}{\partial x}=0, \frac{\partial f}{\partial y}=0 \Leftrightarrow 8x^3+4y=0, 4x+2y=0 \Leftrightarrow y=-2x^3, y=-2x \Rightarrow x^3=x$

$x^3=x \Leftrightarrow x^3-x=0 \Leftrightarrow x(x^2-1)=0 \Leftrightarrow x=0, x=1$ of $x=-1$

$x=0 \Rightarrow y=0$, $x=1 \Rightarrow y=-2$, $x=-1 \Rightarrow y=2$ Dit geeft de stationaire punten $\boxed{(0,0), (1,-2), (-1,2)}$

c) $\frac{\partial^2 f}{\partial x^2} = 24x^2 = A$, $\frac{\partial^2 f}{\partial y \partial x} = 4 = B$, $\frac{\partial^2 f}{\partial y^2} = 2 = C$, $H = AC - B^2$

	A	B	C	H	
(0,0)	0	4	2	$-16<0$	zadelpunt
(1,-2)	24	4	2	$32>0$	minimum
(-1,2)	24	4	2	$32>0$	minimum.

Er geldt: $f(1,-2)=f(-1,2)=0$,
en uit a) volgt: $f(x,y)\geq 0$ voor alle (x,y) (som van kwadraten)
Dus de minima zijn $\boxed{\text{absoluut}}$

3 a) $i^2=-1$, $i^4=(-1)^2=1$, $i^8=1^2=1$, $i^9=i^8 \cdot i = i$

$z = \frac{3+i}{2+i} = \frac{(3+i)(2-i)}{(2+i)(2-i)} = \frac{6-3i+2i-i^2}{2^2+1^2} = \frac{7-i}{5} = \boxed{\frac{7}{5}-\frac{1}{5}i}$

$|\bar{z}^8| = |z|^8 = \left(\sqrt{\left(\frac{7}{5}\right)^2+\left(-\frac{1}{5}\right)^2}\right)^8 = \sqrt{\frac{49}{25}+\frac{1}{25}}^{\,8} = \sqrt{\frac{50}{25}}^{\,8} = \sqrt{2}^{\,8} = 2^4 = \boxed{16}$

b) Gebruik $e^{a+bi} = e^a(\cos b + i \sin b)$

$e^{\ln 5 + 10\pi i/3} = e^{\ln 5}\left(\cos \frac{10\pi}{3} + i \sin \frac{10\pi}{3}\right) = 5\left(\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}\right)$

$= 5\left(-\frac{1}{2}-\frac{1}{2}\sqrt{3}i\right) = \boxed{-\frac{5}{2}-\frac{5}{2}\sqrt{3}i}$

c) $e^z = 3 - 3\sqrt{3}i$

Schrijf $3-3\sqrt{3}i$ in de vorm $r(\cos\varphi + i\sin\varphi)$

$r = |3 - 3\sqrt{3}i| = \sqrt{3^2 + (-3\sqrt{3})^2} = \sqrt{9+9\cdot3} = \sqrt{36} = 6$

$\cos\varphi = \frac{3}{6} = \frac{1}{2}$, $\sin\varphi = -\frac{3}{6}\sqrt{3} = -\frac{1}{2}\sqrt{3} \Rightarrow \varphi = -\frac{1}{3}\pi$

<u>Algemeen</u>: $e^z = r(\cos\varphi + i\sin\varphi)$ heeft als oplossingen

$z_k = \ln r + (\varphi + 2k\pi)i \quad (k \in \mathbb{Z})$

In ons geval: $\boxed{z = \ln 6 + (-\frac{1}{3}\pi + 2k\pi)i} \quad (k \in \mathbb{Z})$

d) $z^{10} - \sqrt{2}.z^5 + 1 = 0$. Stel $w = z^5$. Dit geeft $w^2 - \sqrt{2}.w + 1 = 0$

oplossingen, discriminant $D = (\sqrt{2})^2 - 4.1 = 2 - 4 = -2 < 0$

dus $w_{1,2} = \frac{+\sqrt{2} \pm \sqrt{2}i}{2} = +\frac{1}{2}\sqrt{2} \pm \frac{1}{2}\sqrt{2}i$

We moeten de oplossingen bepalen van $z^5 = -\frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2}i$

en van $z^5 = +\frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{2}i$

1) $z^5 = +\frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2}i$ Schrijf $-\frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2}i = r(\cos\varphi + i\sin\varphi)$

$r = (-\frac{1}{2}\sqrt{2})^2 + (\frac{1}{2}\sqrt{2})^2 = \frac{1}{2} + \frac{1}{2} = 1$. $\cos\varphi = \frac{1}{2}\sqrt{2}$, $\sin\varphi = \frac{1}{2}\sqrt{2} \Rightarrow \varphi = \frac{1}{4}\pi$

opl. $\boxed{z_k = \cos(\frac{1}{20}\pi + \frac{2k\pi}{5}) + i\sin(\frac{1}{20}\pi + \frac{2k\pi}{5}) \quad (k = 0, 1, 2, 3, 4)}$

2) $z^5 = +\frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{2}i$. Schrijf $+\frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{2}i = r(\cos\varphi + i\sin\varphi)$

$r = 1$, $\cos\varphi = -\frac{1}{2}\sqrt{2}$, $\sin\varphi = -\frac{1}{2}\sqrt{2} \Rightarrow \varphi = -\frac{1}{4}\pi$

opl $\boxed{z_k' = \cos(-\frac{1}{20}\pi + \frac{2k\pi}{5}) + i\sin(-\frac{1}{20}\pi + \frac{2k\pi}{5}) \quad (k = 0, 1, 2, 3, 4)}$

4 a) In de reeks staat $1{,}01^n$ dus we passen het quotiëntkenmerk toe

$\sum_{n=p}^{\infty} a_n$ convergent als $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|<1$, divergent als $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|>1$.

Neem $a_n = \frac{n+1}{1{,}01^n}$. Dan is $a_{n+1} = \frac{(n+1)+1}{1{,}01^{n+1}} = \frac{n+2}{1{,}01^{n+1}}$, dus

$$\frac{a_{n+1}}{a_n} = \frac{n+2}{1{,}01^{n+1}} \Big/ \frac{n+1}{1{,}01^n} = \frac{n+2}{1{,}01^{n+1}} \cdot \frac{1{,}01^n}{n+1} = \frac{n+2}{1{,}01 \cdot n+1} = \frac{1+2/n}{1{,}01 \cdot 1+1/n}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1+2/n}{1{,}01\cdot(1+1/n)} = \frac{1}{1{,}01} < 1$$

Dus $\sum_{n=1}^{\infty}\frac{n+1}{1{,}01^n}$ $\boxed{\text{convergeert}}$

b) $n+1$ is van orde van grootte n, dus $\frac{\sqrt[3]{n}}{n+1}$ is van orde van grootte $\frac{\sqrt[3]{n}}{n} = n^{-2/3}$

We passen het vergelijkingskenmerk toe met $a_n = \frac{\sqrt[3]{n}}{n+1}$, $b_n = n^{-2/3}$. Er geldt:

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\sqrt[3]{n}}{n+1}\Big/ n^{-2/3} = \lim_{n\to\infty}\frac{\sqrt[3]{n}\, n^{2/3}}{n+1} = \lim_{n\to\infty}\frac{n}{n+1}$$

$$= \lim_{n\to\infty}\frac{1}{1+\frac{1}{n}} = 1$$

De reeks $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} n^{-2/3}$ is divergent

Dus $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\frac{\sqrt[3]{n}}{n+1}$ is $\boxed{\text{divergent}}$