# Chapter 6

# Tauberian theorems

## 6.1 Introduction

In 1826, Abel proved the following result for real power series. Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series with coefficients $a_n \in \mathbb{R}$ that converges on the real interval $(-1, 1)$. Assume that $\sum_{n=0}^{\infty} a_n$ converges. Then

$$\lim_{x \uparrow 1} f(x) = \sum_{n=0}^{\infty} a_n.$$

In general, the converse is not true, i.e., if $\lim_{x \uparrow 1} f(x)$ exists one can not conclude that $\sum_{n=0}^{\infty} a_n$ converges. For instance, if $f(x) = (1+x)^{-1} = \sum_{n=0}^{\infty} (-1)^n x^n$, then $\lim_{x \uparrow 1} f(x) = \frac{1}{2}$, but $\sum_{n=0}^{\infty} (-1)^n$ diverges.

In 1897, Tauber proved a converse to Abel's Theorem, but under an additional hypothesis. Let again $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series with real coefficients converging on $(-1, 1)$. Assume that

$$\lim_{x \uparrow 1} f(x) =: \alpha \text{ exists,} \tag{6.1}$$

and moreover,

$$\lim_{n \to \infty} n a_n = 0. \tag{6.2}$$

Then

$$\sum_{n=0}^{\infty} a_n \text{ converges and is equal to } \alpha. \tag{6.3}$$

Tauber's result led to various other "Tauberian theorems," which are are all of the following shape:
- suppose one knows something about the behaviour of $f(x)$ as $x \uparrow 1$ (such as (6.1));
- further suppose one knows something about the growth of $a_n$ as $n \to \infty$ (such as (6.2));
- then one can conclude something about the convergence of $\sum_{n=0}^{\infty} a_n$ (such as (6.3)).

There is now a very general "Tauberian theory," which is about Tauberian theorems for functions defined by integrals. These include as special cases Tauberian theorems for power series and Dirichlet series.

We will prove a Tauberian theorem for Dirichlet integrals

$$G(s) := \int_1^{\infty} F(t)t^{-s}dt,$$

where $F : [0, \infty) \to \mathbb{C}$ is a 'decent' function and $s$ is a complex variable. [1] This Tauberian theorem has the following shape.
- Assume that the integral converges for $\operatorname{Re} s > 0$;
- assume that one knows something about the limiting behaviour of $G(s)$ as $\operatorname{Re} s \downarrow 0$;
- assume that one knows something about the growth order of $F$;
- then one can conclude something about the convergence of $\int_1^{\infty} F(t)dt$.

With some modifications, we may view power series as special cases of Dirichlet integrals. Let $g(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series converging for $|z| < 1$. Write $z := e^{-s}$. Then $|z| < 1$ if and only if $\operatorname{Re} s > 0$. Define the function $F(t)$ by

$$F(t) := \frac{a_n}{t} \quad \text{if } e^n \leqslant t < e^{n+1} \quad (n \in \mathbb{Z}_{\geqslant 0}).$$

Then if $\operatorname{Re} s > 0$,

$$\begin{aligned}
\int_1^{\infty} F(t)t^{-s}dt &= \sum_{n=0}^{\infty} \int_{e^n}^{e^{n+1}} F(t)t^{-s}dt = \sum_{n=0}^{\infty} a_n \int_{e^n}^{e^{n+1}} t^{-s-1}dt \\
&= \sum_{n=0}^{\infty} a_n \frac{1}{s}\big(e^{-ns} - e^{-(n+1)s}\big) \\
&= \frac{1 - e^{-s}}{s} \sum_{n=0}^{\infty} a_n e^{-ns}.
\end{aligned}$$

[1] The Dirichlet integral may be viewed as a modification of the *Mellin transform*, which is more established. The Mellin transform of a function $f$ is $\int_0^{\infty} f(t)t^{s-1}dt$.

Hence

$$g(e^{-s}) = \frac{s}{1-e^{-s}} \int_1^\infty F(t)t^{-s}dt \quad \text{if } \operatorname{Re} s > 0.$$

Later, we show how a Dirichlet series can be expressed as a Dirichlet integral.

Around 1930, Wiener developed a general Tauberian theory, which is now part of functional analysis. From this, in 1931, Ikehara deduced a Tauberian theorem for Dirichlet series (now known as the *Wiener-Ikehara Theorem*), with which one can give simple proofs of the Prime Number Theorem and various generalizations thereof. In 1980, Newman published a new method to derive Tauberian theorems, based on a clever contour integration and avoiding any functional analysis. This was developed further by Korevaar.

Using the ideas of Newman and Korevaar, we prove a Tauberian theorem for Dirichlet integrals, and deduce from this a weaker version of the Wiener-Ikehara theorem. This weaker version suffices for a proof of the Prime Number Theorem for arithmetic progressions.

**Literature:**

J. Korevaar, *Tauberian Theory, A century of developments,* Springer Verlag 2004, Grundlehren der mathematischen Wissenschaften, vol. 329.

J. Korevaar, *The Wiener-Ikehara Theorem by complex analysis,* Proceedings of the American Mathematical Society, vol. 134, no. 4 (2005), 1107–1116.

## 6.2 A Tauberian theorem for Dirichlet integrals

**Lemma 6.1.** *Let $F : [1, \infty) \to \mathbb{C}$ be a measurable function. Further, assume there is a constant $M$ such that*

$$|F(t)| \leqslant M/t \quad \textit{for } t \geqslant 1.$$

*Then*

$$G(s) := \int_1^\infty F(t)t^{-s}dt$$

*converges, and defines an analytic function on $\{s \in \mathbb{C} : \operatorname{Re} s > 0\}$.*

*Proof.* We prove that $G(s)$ converges and is analytic on $U_\delta := \{s \in \mathbb{C} : \operatorname{Re} s > \delta\}$ for every $\delta > 0$, and to this end, we apply Theorem 2.29.

We check that the conditions of Theorem 2.29 are satisfied. First, $F(t)t^{-s}$ is measurable on $[1,\infty)\times U_\delta$. Second, for every fixed $t$, the function $s\mapsto F(t)t^{-s}$ is analytic on $U_\delta$. Third, we have

$$|F(t)t^{-s}| \leqslant (M/t)t^{-\operatorname{Re} s} \leqslant M\cdot t^{-1-\delta} \quad \text{for } s\in U_\delta$$

and $\int_1^\infty M\cdot t^{-1-\delta}dt$ converges. So indeed, all conditions of Theorem 2.29 are satisfied and thus, by that Theorem, $G(s)$ is analytic on $U_\delta$. □

We are now ready to state our Tauberian theorem.

**Theorem 6.2.** *Let $F:[1,\infty)\to\mathbb{C}$ be a function with the following properties:*
*(i) $F$ is measurable;*
*(ii) there is $M>0$ such that $|F(t)|\leqslant M/t$ for all $t\geqslant 1$;*
*(iii) there is a function $G(s)$, which is analytic on an open set containing $\{s\in\mathbb{C}:\ \operatorname{Re} s\geqslant 0\}$, such that*

$$\int_1^\infty F(t)t^{-s}dt = G(s) \quad \textit{for } \operatorname{Re} s>0.$$

*Then $\int_1^\infty F(t)dt$ converges, and $\int_1^\infty F(t)dt = G(0)$.*

**Remark.** Theorem 6.2 states that $\lim_{s\to 0,\,\operatorname{Re} s>0}\int_1^\infty = \int_1^\infty \lim_{s\to 0,\operatorname{Re} s>0}$. Although this seems plausible it is everything but trivial. Indeed, it will imply the Prime Number Theorem!

*Proof.* The proof consists of several steps.

**Step 1.** *Reduction to the case $G(0)=0$.*
We assume that Theorem 6.2 has been proved in the special case $G(0)=0$ and deduce from this the general case.

Assume that $G(0)\neq 0$. Define new functions

$$\widetilde{F}(t) := F(t)-\frac{G(0)}{t^2},\quad \widetilde{G}(s) := G(s)-\frac{G(0)}{s+1}.$$

Then $\widetilde{F}$ satisfies (i),(ii), the function $\widetilde{G}$ is analytic on an open set containing $\{s\in\mathbb{C}:\ \operatorname{Re} s\geqslant 0\}$, we have $\widetilde{G}(0)=0$, and for $\operatorname{Re} s>0$ we have

$$\int_1^\infty \widetilde{F}(t)t^{-s}dt = \int_1^\infty F(t)t^{-s}dt - G(0)\int_1^\infty t^{-s-2}dt = G(s)-\frac{G(0)}{s+1} = \widetilde{G}(s).$$

Hence $\widetilde{F}$ satisfies (iii). Now if we have proved that $\int_1^\infty \widetilde{F}(t)dt = \widetilde{G}(0) = 0$, then it follows that

$$\int_1^\infty F(t)dt = G(0) \int_1^\infty t^{-2}dt = G(0).$$

□

Henceforth we assume, in addition to the conditions (i)–(iii), that $G(0) = 0$.

**Step 2.** *The function $G_T$.*

For $T > 0$, define

$$G_T(s) := \int_1^T F(t)t^{-s}dt.$$

We show that $G_T$ is analytic on $\mathbb{C}$. In fact, it suffices to show that $G_T$ is analytic on $R_A := \{s \in \mathbb{C} : \operatorname{Re} s > -A\}$ for every $A > 0$. We use again Theorem 2.29. First, $F(t)t^{-s}$ is measurable on $[1, T] \times R_A$. Second, for every fixed $t \in [1, T]$, $s \mapsto F(t)t^{-s}$ is analytic on $R_A$; and third,

$$|F(t)t^{-s}| \leqslant (M/t)t^{-Res} \leqslant M \cdot t^{A-1},$$

and clearly, $\int_1^T M \cdot t^{A-1}dt < \infty$. So by Theorem 2.29, $G_T$ is indeed analytic on $R_A$. □

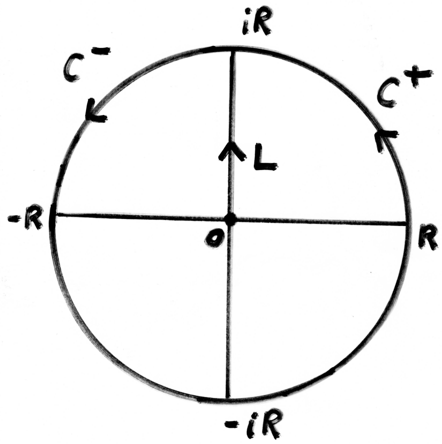
We clearly have

$$G_T(0) = \int_1^T F(t)dt.$$

So we have to prove:

$$\lim_{T\to\infty} G_T(0) = G(0) = 0. \tag{6.4}$$

**Step 3.** *An integral expression for $G_T(0)$.*




We fix a parameter $R > 1$. It will be important in the proof that $R$ can be chosen arbitrarily. Let:

$C^+$ the semi-circle $\{s \in \mathbb{C} : |s| = R, \operatorname{Re} s \geqslant 0\}$, traversed counterclockwise;
$C^-$ the semi-circle $\{s \in \mathbb{C} : |s| = R, \operatorname{Re} s \leqslant 0\}$, traversed counterclockwise;
$L$ the line segment from $-iR$ to $iR$, traversed upwards.

Define the auxiliary function (invented by Newman):

$$J_{R,T}(s) := T^s\Big(1 + \frac{s^2}{R^2}\Big) \cdot \frac{1}{s}.$$

The function $G_T(s) \cdot J_{R,T}(s)$ is analytic for $s \neq 0$, and at $s = 0$ it has a simple pole with residue $G_T(0)$ (or a removable singularity if $G_T(0) = 0$). So by the residue theorem,

(A) $$\frac{1}{2\pi i}\int_{C^+ + C^-} G_T(s) J_{R,T}(s) ds = G_T(0).$$

The function $G(s)$ is analytic on an open set containing $\{\operatorname{Re} s \geqslant 0\}$. Further, $G(s)J_{R,T}(s)$ is analytic on this open set. For it is clearly analytic if $s \neq 0$, and at $s = 0$ the simple pole of $J_{R,T}(s)$ is cancelled by the zero of $G(s)$ at $s = 0$, thanks to our assumption $G(0) = 0$. So by Cauchy's Theorem,

(B) $$\frac{1}{2\pi i}\int_{C^+ + (-L)} G(s) J_{R,T}(s) ds = 0.$$

We subtract (B) from (A). This gives for $G_T(0)$ the expression

$$\begin{aligned} G_T(0) &= \frac{1}{2\pi i}\int_{C^+} G_T(s)J_{R,T}(s)ds + \frac{1}{2\pi i}\int_{C^-} G_T(s)J_{R,T}(s)ds \\ &\qquad -\frac{1}{2\pi i}\int_{C^+} G(s)J_{R,T}(s)ds + \frac{1}{2\pi i}\int_{L} G(s)J_{R,T}(s)ds \\ &= \frac{1}{2\pi i}\int_{C^+} (G_T(s)-G(s))J_{R,T}(s)ds + \frac{1}{2\pi i}\int_{C^-} G_T(s)J_{R,T}(s)ds \\ &\qquad +\frac{1}{2\pi i}\int_{L} G(s)J_{R,T}(s)ds \\ &=: I_1 + I_2 + I_3, \end{aligned}$$

where $I_1, I_2, I_3$ denote the three integrals. To show that $G_T(0) \to 0$ as $T \to \infty$, we have to estimate $|I_1|$, $|I_2|$, $|I_3|$. Here we use $\left|\int_\gamma f(z)dz\right| \leqslant \text{length}(\gamma) \cdot \sup_{z\in\gamma} |f(z)|$.

**Step 4.** *Estimation of* $|I_1|$.

We first estimate $|(G_T(s)-G(s))J_{R,T}(s)|$ for $s \in C^+$. First assume that $s \in C^+$, $\text{Re}\, s > 0$. Using the condition $|F(t)| \leqslant M/t$ for $t \geqslant 1$, we obtain

$$\begin{aligned} |G_T(s) - G(s)| &= \left|\int_T^\infty F(t)t^{-s}dt\right| \leqslant \int_T^\infty |F(t)| \cdot t^{-\text{Re}\, s}dt \\ &\leqslant \int_T^\infty \frac{M}{t} \cdot t^{-\text{Re}\, s}dt = \frac{M}{\text{Re}\, s} \cdot T^{-\text{Re}\, s}. \end{aligned}$$

Further, for $s \in C^+$, we have $s \cdot \overline{s} = |s|^2 = R^2$. Hence

$$|J_{R,T}(s)| = T^{\text{Re}\, s}\left|\left(1 + \frac{s^2}{s\cdot\overline{s}}\right)\frac{1}{s}\right| = T^{\text{Re}\, s}\left|\frac{s+\overline{s}}{s\cdot\overline{s}}\right| = 2T^{\text{Re}\, s} \cdot \frac{\text{Re}\, s}{R^2}.$$

Hence for $s \in C^+$, $\text{Re}\, s > 0$,

$$|(G_T(s)-G(s))J_{R,T}(s)| \leqslant \frac{M}{\text{Re}\, s} \cdot T^{-\text{Re}\, s} \cdot 2T^{\text{Re}\, s} \cdot \frac{\text{Re}\, s}{R^2} \leqslant \frac{2M}{R^2}.$$

By continuity, this is true also if $\text{Re}\, s = 0$. Hence

$$|I_1| \leqslant \frac{1}{2\pi}\text{length}(C^+) \cdot \sup_{s\in C^+} |(G_T(s)-G(s))J_{R,T}(s)| \leqslant \frac{1}{2\pi} \cdot \pi R \cdot \frac{2M}{R^2},$$

i.e.,

$$|I_1| \leqslant \frac{M}{R}.$$

**Step 5.** *Estimation of* $|I_2|$.

The argument is similar to the estimation of $|I_1|$. We start with estimating $|G_T(s)J_{R,T}(s)|$ for $s \in C^-$. First assume that $s \in C^-$, $\operatorname{Re} s < 0$. Then, using again $|F(t)| \leqslant M/t$ for $t \geqslant 1$,

$$\begin{aligned} |G_T(s)| &= \left|\int_1^T F(t)t^{-s}dt\right| \leqslant \int_1^T |F(t)| \cdot t^{-\operatorname{Re} s}dt \\ &\leqslant \int_1^T \frac{M}{t} \cdot t^{-\operatorname{Re} s}dt = \frac{M}{|\operatorname{Re} s|}\big(T^{|\operatorname{Re} s|} - 1\big) \leqslant \frac{M}{|\operatorname{Re} s|} \cdot T^{|\operatorname{Re} s|} \end{aligned}$$

while also

$$|J_{R,T}(s)| = T^{\operatorname{Re} s} \cdot \left|\frac{s+\overline{s}}{s \cdot \overline{s}}\right| = 2T^{-|\operatorname{Re} s|} \cdot \frac{|\operatorname{Re} s|}{R^2}.$$

Hence for $s \in C^-$ with $\operatorname{Re} s < 0$,

$$|G_T(s)J_{R,T}(s)| \leqslant \frac{2M}{R^2}.$$

Again this holds true also if $\operatorname{Re} s = 0$. So

$$|I_2| \leqslant \frac{1}{2\pi}\text{length}(C^-) \cdot \sup_{s \in C^-} |G_T(s)J_{R,T}(s)| \leqslant \frac{1}{2\pi} \cdot \pi R \cdot \frac{2M}{R^2},$$

leading to

$$|I_2| \leqslant \frac{M}{R}.$$

**Step 6.** *Estimation of* $|I_3|$.

We choose for $L$ the parametrization $s = it$, $-R \leqslant t \leqslant R$. Then

$$I_3 = \frac{1}{2\pi i}\int_{-R}^{R} G(it)J_{R,T}(it)d(it) = \frac{1}{2\pi}\int_{-R}^{R} H_R(t)T^{it}dt,$$

where

$$H_R(t) := G(it)\Big(1 - \frac{t^2}{R^2}\Big)\frac{1}{it}.$$

Since by assumption, $G(0) = 0$, the function $G(s)/s$ is analytic on an open set containing $\{s \in \mathbb{C} : \operatorname{Re} s \geqslant 0\}$. Hence $H_R(t)$ is continuously differentiable on $[-R, R]$. Since $H_R$ is independent of $T$, there is a constant $A(R)$ independent of $T$ such that

$$|H_R(t)| \leqslant A(R), \quad |H_R'(t)| \leqslant A(R) \quad \text{for } t \in [-R, R].$$

Using integration by parts, we get

$$\begin{aligned}\int_{-R}^{R} H_R(t) T^{it} dt &= \frac{1}{i \log T} \int_{-R}^{R} H_R(t) dT^{it} \\ &= \frac{1}{i \log T} \left( H_R(R) T^{iR} - H_R(-R) T^{-iR} - \int_{-R}^{R} H_R'(t) T^{it} dt \right).\end{aligned}$$

Since $|T^{it}| = |e^{it \log T}| = 1$, we obtain

$$\begin{aligned}\left| \int_{-R}^{R} H_R(t) T^{it} dt \right| &\leqslant \frac{1}{\log T} \left( A(R) + A(R) + \int_{-R}^{R} |H'(t)| dt \right) \\ &\leqslant \frac{2A(R) + 2RA(R)}{\log T}.\end{aligned}$$

Hence

$$|I_3| \leqslant \frac{C(R)}{\log T},$$

where $C(R)$ depends on $R$, but is independent of $T$.

**Step 7.** *Conclusion of the proof.*

We have to prove that $\lim_{T \to \infty} G_T(0) = G(0) = 0$, in other words, for every $\varepsilon > 0$ there is $T_0$ such that $|G_T(0)| < \varepsilon$ for all $T \geqslant T_0$. Combining steps 3–6, we get, for every choice of $R, T$,

$$|G_T(0)| \leqslant |I_1| + |I_2| + |I_3| \leqslant \frac{2M}{R} + \frac{C(R)}{\log T}.$$

Let $\varepsilon > 0$. Then choose $R$ such that $2M/R < \varepsilon/2$, and subsequently $T_0$ with $C(R)/\log T_0 < \varepsilon/2$. For these choices, it follows that for $T \geqslant T_0$,

$$|G_T(0)| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon.$$

This completes our proof. □

## 6.3 A Tauberian theorem for Dirichlet series

Let $L_f(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$ be a Dirichlet series. Put

$$A(t) := \sum_{n \leqslant t} f(n).$$

We prove the following Tauberian theorem.

**Theorem 6.3.** *Suppose $L_f(s)$ satisfies the following conditions:*
*(i) $f(n) \geqslant 0$ for all $n$;*
*(ii) there is a constant $C$ such that $|A(t)| \leqslant Ct$ for all $t \geqslant 1$;*
*(iii) $L_f(s)$ converges for $s \in \mathbb{C}$ with $\operatorname{Re} s > 1$;*
*(iv) $L_f(s)$ can be continued to a function which is analytic on an open set containing $\{s \in \mathbb{C} : \operatorname{Re} s \geqslant 1\} \setminus \{1\}$ for which $\lim_{s \to 1}(s-1)L_f(s) = \alpha$.*
*Then*

$$\lim_{t \to \infty} \frac{A(t)}{t} = \alpha.$$

**Remarks. 1)** Condition (iii) follows from (ii) (see homework exercise 8a). Further, (iii) implies that $L_f(s)$ is analytic for $\operatorname{Re} s > 1$.
**2)** Condition (iv) means that $L_f(s)$ has a simple pole with residue $\alpha$ at $s = 1$ if $\alpha \neq 0$, and a removable singularity at $s = 1$ if $\alpha = 0$.
**3)** The Wiener-Ikehara Theorem is the same as Theorem 6.3, except that only conditions (i),(iii),(iv) are required and (ii) can be dropped.

We start with some preparations. Notice that condition (iv) of Theorem 6.3 implies that there is an analytic function $g(s)$ on an open set containing $\{s \in \mathbb{C} : \operatorname{Re} s \geqslant 1\}$ such that

$$L_f(s) = \frac{\alpha}{s-1} + g(s) \quad \text{for } s \in \mathbb{C} \text{ with } \operatorname{Re} s > 1. \tag{6.5}$$

Next, we need an expression of $L_f(s)$ as a Dirichlet integral. Let $\operatorname{Re} s > 1$. By partial summation,

$$\sum_{n=1}^{N} f(n)n^{-s} = A(N)N^{-s} + s\int_1^N A(t)t^{-s-1}dt.$$

Since $|A(N)| \leqslant \sum_{n=1}^{N} f(n) \leqslant CN$, we have $A(N)N^{-s} \leqslant CN \cdot N^{-\mathrm{Re}\, s} \to 0$ as $N \to \infty$. Hence

$$L_f(s) = s \int_1^\infty A(t)t^{-s-1}dt. \tag{6.6}$$

Further, we need some lemmas.

**Lemma 6.4.** $\displaystyle\int_1^\infty \frac{A(t) - \alpha t}{t^2} \cdot dt = g(1) - \alpha$ *converges.*

*Proof.* We apply Theorem 6.2 to the function

$$F(t) := \frac{A(t) - \alpha t}{t^2}.$$

We check that this $F$ satisfies conditions (i),(ii),(iii) of Theorem 6.2.

First, $(A(t) - \alpha t)/t^2$ is measurable (e.g., it has only countably many discontinuities). Second, by condition (ii) of Theorem 6.3,

$$\left|\frac{A(t) - \alpha t}{t^2}\right| \leqslant \frac{C + |\alpha|}{t} \quad \text{for } t \geqslant 1.$$

Hence conditions (i),(ii) of Theorem 6.2 are satisfied. As for condition (iii), notice that by (6.6), (6.5) we have for $\mathrm{Re}\, s > 0$,

$$\begin{aligned}\int_1^\infty \frac{A(t) - \alpha t}{t^2} \cdot t^{-s}dt &= \int_1^\infty A(t)t^{-s-2}dt - \alpha \int_1^\infty t^{-s-1}dt \\ &= \frac{1}{s+1}L_f(s+1) - \frac{\alpha}{s} = \frac{1}{s+1}\Big(\frac{\alpha}{s} + g(s+1)\Big) - \frac{\alpha}{s},\end{aligned}$$

implying

$$\int_1^\infty \frac{A(t) - \alpha t}{t^2} \cdot t^{-s}dt = \frac{1}{s+1}\big(g(s+1) - \alpha\big) \quad \text{if } \mathrm{Re}\, s > 0. \tag{6.7}$$

The right-hand side is analytic on an open set containing $\{s \in \mathbb{C} : \mathrm{Re}\, s \geqslant 0\}$, hence (iii) is satisfied as well. So by Theorem 6.2, identity (6.7) extends to $s = 0$, and this gives precisely Lemma 6.4. □

By condition (i), we have $f(n) \geqslant 0$ for all $n$. Hence the function $A(t)$ is non-decreasing. Now Theorem 6.3 follows by combining Lemma 6.4 with the lemma below.

**Lemma 6.5.** *Let $B : [1,\infty) \to \mathbb{R}$ be a non-decreasing function and let $\beta \in \mathbb{R}$. Assume that*

$$\int_1^\infty \frac{B(t)-\beta t}{t^2}\cdot dt \quad \textit{converges.}$$

*Then*

$$\lim_{t\to\infty} \frac{B(t)}{t} \;=\; \beta.$$

*Proof.* We may assume without loss of generality that $\beta > 0$. Indeed, assume that $\beta \leqslant 0$. Choose $\gamma > 0$ such that $\widetilde{\beta} := \beta+\gamma > 0$, and replace $B(t)$ by $\widetilde{B}(t) := B(t)+\gamma t$. Then $\widetilde{B}$ is non-decreasing and $\int_1^\infty(\widetilde{B}(t)-\widetilde{\beta}t)dt/t^2$ converges. If we are able to prove that $\lim_{t\to\infty}\widetilde{B}(t)/t = \widetilde{\beta}$, then $\lim_{x\to\infty} B(t)/t = \beta$ follows.

So assume that $\beta > 0$. Assume that $\lim_{t\to\infty} B(t)/t$ does not exist or is not equal to $\beta$. Then there are two possibilities:

(a) there are $\varepsilon > 0$ and an increasing sequence $\{t_n\}_{n=1}^\infty$ with $t_n \to \infty$ such that $B(t_n)/t_n \geqslant \beta(1+\varepsilon)$ for all $n$;
(b) there are $\varepsilon > 0$ and an increasing sequence $\{t_n\}_{n=1}^\infty$ with $t_n \to \infty$ such that $B(t_n)/t_n \leqslant \beta(1-\varepsilon)$ for all $n$.

We consider only case (a); case (b) can be dealt with in the same manner. So assume (a). Then since $\int_1^\infty(B(t)-\beta t)dt/t^2$ converges, we have

$$\lim_{n\to\infty}\int_{t_n}^{(1+\varepsilon)t_n} \frac{B(t)-\beta t}{t^2}\cdot dt = 0. \tag{6.8}$$

On the other hand, since $B$ is non-decreasing,

$$\begin{aligned}\int_{t_n}^{(1+\varepsilon)t_n} \frac{B(t)-\beta t}{t^2}\cdot dt &\geqslant \int_{t_n}^{(1+\varepsilon)t_n} \frac{B(t_n)-\beta t}{t^2}\cdot dt \\ &\geqslant \int_{t_n}^{(1+\varepsilon)t_n} \frac{(1+\varepsilon)\beta t_n-\beta t}{t^2}\cdot dt = \int_1^{1+\varepsilon}\frac{\beta(1+\varepsilon)-\beta u}{u^2}\cdot du \qquad (u = t/t_n) \\ &= \beta\Big[-\frac{1+\varepsilon}{u} - \log u\Big]_1^{1+\varepsilon} = \beta\big(\varepsilon - \log(1+\varepsilon)\big).\end{aligned}$$

This last number is independent of $n$ and strictly positive, since $\beta > 0$ and $\log(1+\varepsilon) < \varepsilon$. This contradicts (6.8). Hence case (a) is impossible. □