ANALYTIC NUMBER THEORY

Thursday January 22, 14:00-17:00

• Write down your name and student number on each sheet. Take care that these are **VERY WELL READABLE**.
• Indicate whether you are doing Bachelor wiskunde, Master wiskunde, Algant, or any other program and if not from Leiden, from which other university you are coming.
• Formulate the theorems you are using.
• There are three exercises on the back side.
• Grade = #points/4.

---

1. For $\alpha \in \mathbb{R}_{>0}$, the *Hurwitz zeta function* is given by

$$\zeta(s;\alpha) = \sum_{n=0}^{\infty} (n+\alpha)^{-s}.$$

3 *a)* Prove that if a, b are two integers with $a < b$ and $f : [a, b] \to \mathbb{C}$ is a continuously differentiable function, then

$$\sum_{n=a}^{b} f(n) = \int_a^b f(x)dx + f(a) + \int_a^b (x - [x])f'(x)dx.$$

7 *b)* Prove that $\zeta(s;\alpha)$ converges for $s \in \mathbb{C}$ with $\operatorname{Re} s > 1$ and then, using a) that $\zeta(s;\alpha)$ has an analytic continuation to the set $\{s \in \mathbb{C} : \operatorname{Re} s > 0\} \setminus \{1\}$, with a simple pole with residue 1 at $s = 1$.

2. Let $\theta(x) := \sum_{p \le x} \log p$, $\quad \psi(x) := \sum_{p^k \le x} \log p$, where the first sum is over all primes $\le x$ and the second sum over all prime powers $\le x$.

5 *a)* Give an elementary proof that $\theta(x) \le x \log 4$, i.e., $\prod_{p \le x} p \le 4^x$ for $x \ge 2$.

5 *b)* Compute a constant $C > 0$ such that $\psi(x) - \theta(x) \le C\sqrt{x}$ for all $x \ge 2$.

2 **3.** *a)* Formulate a Tauberian theorem for Dirichlet series.

4 *b)* Let Ω be the arithmetic function defined by $\Omega(1) = 0$ and $\Omega(n) = k_1 + \cdots + k_t$ if $n = p_1^{k_1} \cdots p_t^{k_t}$ with distinct primes $p_1, \ldots, p_t$ and positive integers $k_1, \ldots, k_t$. Prove that

$$\sum_{n=1}^{\infty} (-1)^{\Omega(n)} n^{-s} = \frac{\zeta(2s)}{\zeta(s)} \quad \text{for } s \in \mathbb{C} \text{ with } \operatorname{Re} s > 1.$$

4 *c)* Compute $\displaystyle\lim_{x \to \infty} \frac{1}{x} \sum_{n \le x} (-1)^{\Omega(n)}$.

4. Recall that the Gauss sum associated with a character χ modulo q and an integer a is given by $\tau(a, \chi) = \sum_{x=0}^{q-1} \chi(x) e^{2\pi i a x/q}$.

5 *a)* Let p be a prime number and χ a non-principal character mod p. Prove that $\tau(a, \chi) = \overline{\chi(a)}\tau(1, \chi)$ for $a \in \mathbb{Z}$, $|\tau(1, \chi)| = \sqrt{p}$.

5 *b)* Let q be an integer ≥ 2. Prove that $\tau(1, \chi_0^{(q)}) = \mu(q)$, where $\chi_0^{(q)}$ is the principal character modulo q, and μ is the Möbius function.